A METHOD OF SOLVING LINEAR PROBLEMS BY USING THE ADJOINT SYSTEM

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ABSTRACT

A usual way of finding the response of a system B, at time t, to some input u, is the following: first, decompose the input into a set of δ-functions; next, find the response of B at t to each individual δ-function, and finally, add up all the contributions.

In this study an alternative method of finding the response of B at t is proposed. By this method we use the adjoint system B^*, and the procedure is the following: apply a δ-function at time t to B^*, and find its response for all t. Then "weigh" this response according to the input u and thus get the response of the original system at time.

A comparison between these two methods shows that while in the usual method a whole set of δ-functions has to be applied to the system B, only one δ-function has to be applied to B^*, namely at time t. However, in the usual method we have to find the response only at one point, while in the proposed method the response at all t is needed.

These concepts are further generalized.

Also some reciprocity relations between B and B^* are derived, relations which are a natural extension of the "reciprocity theorem" in various fields such as circuit theory, electromagnetics, etc.

The application of this adjoint method of solving linear problems is illustrated by several problems in circuit theory and in control system theory.

It is hoped that this study will point out the usefulness of the adjoint system, and show how this system can be applied, in a systematic way, to solve certain linear problems.

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I. INTRODUCTION

Frequently it happens that while solving the general problem of finding the response of a given system we encounter the adjoint system. In this paper we shall demonstrate the following rule: in order to find the response of a system, it may be convenient, and sometimes necessary, to investigate the behavior of the adjoint system rather than that of the original system itself.

The familiar convolution integral is a simple illustration of this rule. By this integral we compute the output $z(t)$ of a linear, single-input, single-output system, as a function of the input $v(t)$, namely,

$$z(t) = \int_{-\infty}^{+\infty} h(t\,;\,\tau)\, v(\tau)\, d\tau \tag{1}$$

Here $h(t\,;\,\tau)$ is the response of the system at time t, as a result of a unit impulse applied at time τ. A consideration of (1) shows that we are more interested in the behavior of $h(t\,;\,\tau)$ as a function of τ, than in the behavior of $h(t\,;\,\tau)$ as a function of t. (τ, and not t, is the variable upon which the integration is performed!) In other words, we are less concerned with the <u>time</u> behavior of the impulsive response $h(t\,;\,\tau)$, than with the behavior of $h(t\,;\,\tau)$ as we vary the <u>time of application</u> τ. But $h(t\,;\,\tau)$ as a function of τ, for a fixed t, happens to be exactly the <u>time behavior of the impulsive response of the adjoint system</u>.

To show in an even more convincing way why we are interested in the adjoint system, let us try to compute, analogically, the output $z(t)$ using the convolution integral. In order to generate $h(t\,;\,\tau)$ as a function of τ we will have to set up the <u>adjoint system</u>, feed it with a unit impulse, multiply its response by $v(t)$ (which is given) and then integrate. This procedure is illustrated in Figure 1.

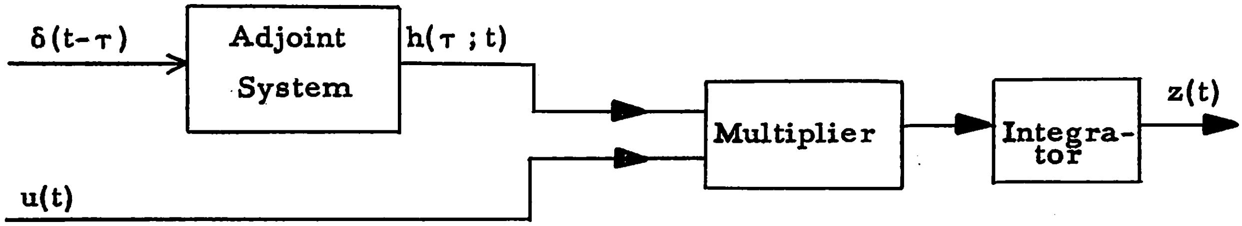

Figure 1.

In this paper we shall show, in a way as general as possible, the reason for the frequent, and in some occasions, unexpected appearance of the adjoint system. Once this reason is understood, a systematic approach to the solution of linear problems, by the use of the adjoint system, can be developed.

In Sections II-XI the theory will be presented in an abstract way, while Sections XII-XIII will be devoted to the application of the theory to specific systems.

II. THE VECTOR SPACE

Let U be a vector space[+] over the field[+] of the complex numbers C. We shall denote the elements of U by u, with or without subscripts, and the elements of C by Greek letters. Thus we write:

$$U = \{u\} \text{ or } U = \{u_i\} \quad , \quad C = \{\alpha, \beta, \dots\}$$

Examples: Although in the following treatment we shall use the terminology of abstract vector spaces, i.e., we shall not specify the nature of the elements u, it might be helpful to have several concrete examples of vector spaces in mind.

(i) All vectors $\underline{z} = (z_1, z_2, \dots, z_n)$ in the n-dimensional Euclidean space form a vector space; the z_i's are fixed complex numbers.

(ii) All continuous functions $f(\cdot)$ defined over the interval (0, 1) form a vector space.

(iii) All n-dimentional time varying vectors $\underline{z}(t) = [z_1(t), z_2(t), \dots, z_n(t)]$, the components of which are in $\mathcal{L}_2[a, b]$[++] form a vector space.

[+] For precise definition see, for example, Reference 1.

[++] $\mathcal{L}_2[a, b]$ is the set of all function f(t) defined for almost all t in [a, b] such that $\int_a^b |f(t)|^2 dt < \infty$.

III. THE INNER PRODUCT

Let us define an inner product over the vector space U, and denote it by $< \cdot , \cdot >_U$.

Definition: An inner product over a vector space U is a mapping of U x U into C, such that $\forall u_i, u_j \in U$ there corresponds one, and only one, $\alpha \in C$ written

$<u_i, u_j>_U = \alpha$

and which satisfies the following three conditions:

(i) $<u_k, \alpha u_i + \beta u_j>_U = \alpha <u_k, u_i>_U + \beta <u_k, u_j>_U \qquad \forall u_i, u_j u_k \in U;$

$\forall \alpha, \beta \in C$

(ii) $<u_i, u_j>_U = \overline{<u_j, u_i>}_U \qquad \forall u_i, u_j \in U$

where the bar stands for "complex conjugate."

(iii) $<u, u>_U > 0 \quad \forall u \neq o \in U$

Examples: We shall present several examples of inner products over different vector spaces.

(i) Let $\underline{v}$ and $\underline{z}$ be any two constant vectors, which belong to the n-dimensional Euclidean vector space, the components of which are complex numbers. We define

$$<\underline{v}, \underline{z}> = \sum_{i=1}^{n} \bar{v}_i \cdot z_i \quad ^{+}$$

where $\bar{v}_i$ is the complex conjugate of the i-th component of $\underline{v}$.

+ To avoid confusion, the four letters u, w, x and y with or without subscript will always denote elements of the sets U, W, X and Y respectively. Other letters, such as v, i, z, with subscript will denote components of the vectors $\underline{v}$, $\underline{i}$, $\underline{z}$.

(ii) Let $v(\cdot)$ and $z(\cdot)$ be any two elements of the vector space which consists of all real valued functions in $\mathcal{L}_2$ over the interval (0, 1). We can define

$$<v(\cdot), z(\cdot)> = \int_0^1 v(t) \cdot z(t)\, dt$$

as an inner product in $\mathcal{L}_2$.

(iii) Let $\underline{E}(x, y, z)$ be the complex amplitude of the electric field. We define:

$$<\underline{E}_a(x, y, z), \underline{E}_b(x, y, z)> =$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sum_{i=i}^{3} \overline{E_{ai}(x, y, z)} \cdot E_{bi}(x, y, z) \cdot dxdydz$$

IV. THE LINEAR OPERATOR

Let also W be a vector space over C (with elements w), and let us define an inner product over W, which will again be denoted by $< \cdot , \cdot >_W$.

Next we define a <u>linear operator</u> L which will map U into W.

<u>Definition</u>: A <u>linear operator</u> with domain U and range in W, is a mapping of U into W such that $\forall u \in U$ there corresponds a unique $w \in W$ written

$$Lu = w \tag{2}$$

and that satisfies the following condition:

$$L(\alpha\, u_i + \beta\, u_j) = \alpha\, L(u_i) + \beta\, L(u_j) \quad \forall u_i, u_j \in U; \ \forall \alpha, \beta \in C$$

To translate the above notions into system terminology: let B be a linear system, characterized by the operator L, u is the input to B. w is the output of B, also referred to as the response of B to u. The different terms are exhibited by the following block-diagram (Figure 2):

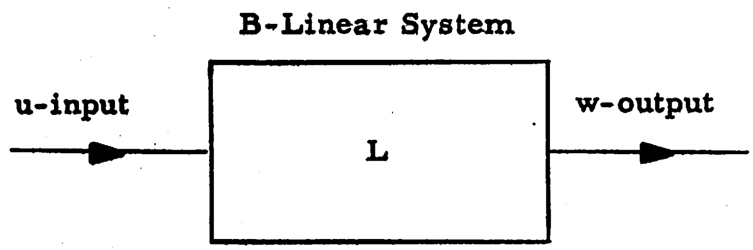

Figure 2.

<u>Examples</u>: Several examples of linear operators will be presented.

(i) B is a linear, single-input, single-output system; $u = v(t)$, $w = z(t)$, and L is an integral operator. Here, Equation (2) reads:

$$z(t) = \int_{-\infty}^{t} h(t\,;\,\tau)\, v(\tau)\, d\tau .$$

(ii) B is an electric circuit consisting of resistors and constant current sources. $u = \underline{I} = (I_1, I_2, \ldots, I_n)$ where I_j is the strength of the current source connected between the j-th node and the datum. $w = \underline{v} = (v_1, v_2, \ldots, v_n)$ where v_j is the voltage between the j-th node and the datum. Here L is the inverse of the admittance matrix of the circuit, and Equation (2) reads

$$\underline{G}^{-1}\underline{i} = \underline{v}$$

(iii) B is a linear system (such as a general RLC circuit), characterized by its state variable equations. $u = \underline{v}(t) = [v_1(t), \ldots, v_p(t)]$ is the forcing vector, $w = z_q(t) = [z_1(t), \ldots, z_q(t)]$ is the state vector and L is a system of first order linear differential equations. Equation (2) reads

$$\dot{\underline{z}}(t) = \underline{A}\,\underline{z}(t) + \underline{D}\,\underline{v}(t)$$

where $\underline{A}$ is a q x q constant matrix and $\underline{D}$ is a q x p constant matrix.

(iv) In electrostatics, let $u = \rho(x, y, z)$ be the charge distribution, and let $w = \underline{E}(x, y, z)$ be the corresponding electric field. Here L is given by

$$L = -\operatorname{grad} \iiint \frac{\cdots}{4\pi\epsilon r}\, du$$

as $\underline{E} = -\operatorname{grad} \phi$ & $\phi = \iiint \frac{\rho(x, y, z)}{4\pi\epsilon r}\, dv.$

V. THE ADJOINT OPERATOR

To each operator L we shall correspond an adjoint operator L^*, which is a mapping of W into U, written

$$L^* w = u \tag{2'}$$

and which is defined as following:

Definition: The adjoint of the operator L, for a specific definition of an inner product over U and over W, is a linear mapping L^* of W into U such that $\forall\, w \in W$ there corresponds a unique $u \in U$, and such that

$$\langle L^* w, u\rangle_U = \langle w, Lu\rangle_W \quad \forall\, u \in U;\ \forall\, w \in W \tag{3}$$

Note that the left hand inner product is defined over U, while the right hand inner product is defined over W.[+]

In the special case that $L = L^*$, the operator L is said to be self-adjoint.

In case that L is a differential operator, the initial (or boundary) conditions will be considered as part of L; thus in finding L^*, the corresponding boundary conditions will have to be found. Also, two operators will not be self-

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[+] To be precise, we remark that a necessary condition for L to have an adjoint is that U be a Hilbert Space.

ajoint unless both operators as well as their corresponding boundary conditions are equal.

Recall that we have denoted by B the system whose operator is L; similarly let us denote by B^* the system whose operator is L^*. B^* will be called the adjoint system. We have denoted by w the response of B to the input u; thus let us denote by w^* the response of B^* to the input u^*. (Note that $w^* \epsilon U$ and $u^* \epsilon W$!) In operational notation we write

$$L^* u^* = w^* \tag{4}$$

It would be instructive to draw the block-diagram of B and B^* in the following way (Figure 3):

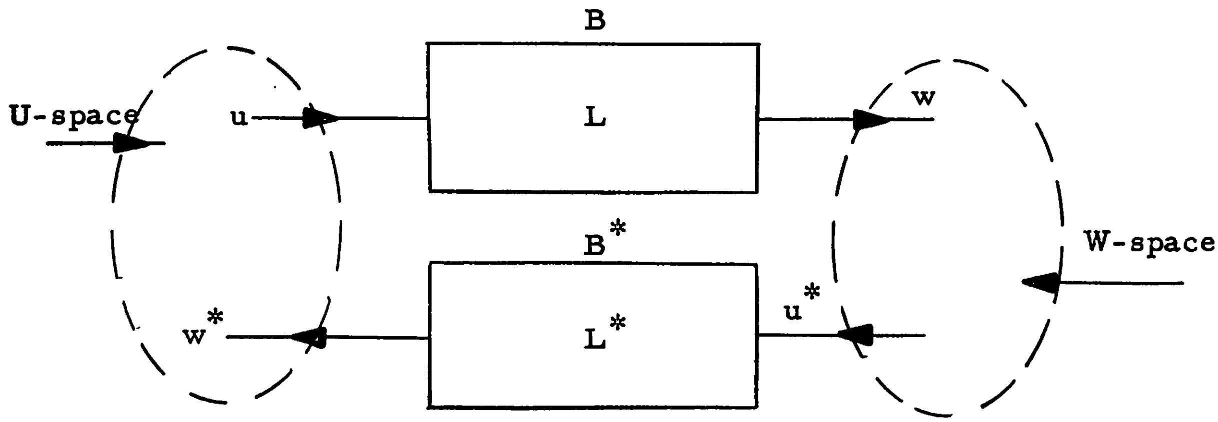


Figure 3.

thus emphasizing the fact that the domain and the range interchange when passing from L to its adjoint L^*.

<u>Example</u>: If L is a matrix whose elements are complex numbers then L^* is the conjugate of the transpose of L, i. e. $L^* = \overline{L}^*$. This statement will be proved in Section XII.

VI. THE RECIPROCITY THEOREM

An immediate result of the definition of the adjoint operator is the following theorem called the reciprocity theorem.

Theorem: For two adjoint systems B and B^*, and two particular definitions of inner products over U and over W,

$$<w^*, u>_U = <u^*, w>_W, \tag{5}$$

where u and w are, respectively, the input and the output of B, and u^* and w^* are, respectively, the input and the output of B^*. (As in Section IV, u^* denotes an element of W, and w^* belongs to the set U. Also note that u^* is not the complex conjugate of u...)

Proof: We start with (3), the definition of the adjoint operator, and replace there w by u^*, thus we get,

$$<L^* u^*, u>_U = <u^*, Lu>_W. \tag{5a}$$

Now from Equations (2) and (4) we have

$$Lu = w \text{ and } L^* u^* = w^* \tag{5b}$$

Inserting (5b) into (5a) we get

$$<w^*, u>_U = <u^*, w>_W$$

and thus the theorem is proved.

Q. E. D.

We remark that it is V. H. Rumsey[2] who introduced the term "reprocity theorem" in a sense similar to that presented here.

VII. THE ELEMENTS OF U AND W AS FUNCTIONS

Until now we have not specified the nature of the elements of the sets U and W. Now let each element of U be a complex valued function, defined over some domain X. In other words, let each element of U be a mapping of the set X (with elements x), into the complex plane C. Thus,

$$u_i : X \longrightarrow C \quad \forall u_i \in U$$

or $\forall u_i \in U$ and $\forall x_j \in X$ there corresponds one, and only one, $\alpha \in C$ written:

$$u_i(x_j) = \alpha .$$

We shall also say that α is the value of u_i at x_j and call x_j the point at which u_i is measured, or the <u>point of observation.</u>

We let also the elements of W be mappings, this time from a set Y (with elements y), into the complex plane C. Thus,

$$w_j : Y \longrightarrow C \quad \forall w_j \in W$$

<u>Examples</u>: We present here several examples of sets of functions U, and sets of points of observation X.

(i) Let U be the set of all real valued continuous functions defined over the real line. Here, $U = \{u(\cdot)\}$ $X = \{-\infty < x < +\infty\}$ and u(x) is the value of some u at some x.

(ii) Let U be the set of n-dimensional vectors $\underline{v}$, and let X be the set which consists of the integers 1, 2, ..., n. Here the value of $\underline{v}$ at i, $1 \leq i \leq n$, is simply the i-th component of $\underline{v}$, i.e., $\underline{v}(i) = v_i$.
Adopting this convention, we can think of picking out a component of a vector $\underline{v}$ as of evaluating the vector at some integer. This is a natural way of extending the terminology of functions of continuous variables to vectors, and it enables us to use the same notation for both scalar and vector functions.

(iii) Refer back to example (iv) of Section IV. There, U is the set of real valued functions of three variables; X is the physical three dimensional space; W is a set of functions of <u>four</u> variables; and Y is a four dimensional space in which three of the variables can vary continuously along the real line, while the fourth variable takes only the values 1, 2 or 3. Note that in this example neither U and W nor X and Y are equal sets.

VIII. THE δ-FUNCTION

We introduce a <u>δ-function</u> over U, denoted by $\delta^U(\cdot|\cdot)$, as a mapping of X x X into the complex plane C; thus $\forall x_i$, $x_j \in X$ there corresponds a unique $\alpha \in C$ and we write:

$$\delta^U(x_i \mid x_j) = \alpha$$

Without specifying the properties of the mapping $\delta^U(\cdot|\cdot)$ it is clear that $\delta^U(\cdot|\, x_j)$, with x_j held constant, is an element of U, since it maps X into C. Similarly, $\delta^U(x_i\,|\cdot)$ with x_i held constant is also an element of U.

We now define the δ-function $\delta^U(\cdot|\cdot)$.

<u>Definition</u>: $\delta^U(\cdot|\cdot)$, called a δ-function over U for a certain inner product, is a function which has the property that

$$< \delta^U(\cdot|\, x_j), \quad u >_U = u(x_j) \quad \forall x_j \in X; \ \forall u \in U \tag{6}$$

We must admit that the introduction of the δ-function is done in a somewhat rough way In many cases $\delta^U(\cdot|\, x_j)$ will not be an ordinary member of the set U. ($\delta^U(\cdot|\, x_j)$ is not always a <u>function</u> but is a <u>distribution</u> in the sense of L. Schwarz.[3]

Nevertheless, it is convenient to consider $\delta^U(\cdot|\, x_j)$ as an ordinary element of U. Moreover, we shall say that $\delta^U(\cdot|\, x_j)$ is "zero" for all values $x_k \neq x_j$. Our justification to this is that when the function $<\delta^U(\cdot|\, x_j), \cdot>_U$ operates upon some u_i, it is only concerned with the values of u_i at x_j, while the values that u_i takes at all other $x_k \neq x_j$ have no influence on the result of the operation.

The first argument of $\delta^U(\cdot|\cdot)$ will be called the <u>point of observation</u>, and the second will be called the <u>point of application</u>. We shall also say that $\delta^U(\cdot|\, x_j)$ is a δ-function over U <u>located</u> at x_j.

<u>Examples</u>: We illustrate some possible forms of the δ-function can take, with two examples.

(i) The δ-function for the set U of continuous functions, and for the following definition of the inner product

$$< u_a(t)\,;\, u_b(t) >_U = \int_{-\infty}^{+\infty} u_a(t)\, u_b(t)\, dt$$

is the familiar Dirac δ-function $\delta(t-t_1)$, since

$$\int_{-\infty}^{+\infty} \delta(t-t_1)\, u(t)\, dt = u(t_1)$$

(ii) Let U be the n-dimensional Euclidean space, and let the inner product be the usual scalar product of two vectors. Then the δ-function is

$$\delta^U(\cdot\,|\, j) = (0, 0, \ldots, 0, \underset{\substack{\uparrow \\ j\text{-th comp.}}}{1}, 0, \ldots 0)$$

IX. CONSEQUENCES OF THE RECIPROCITY THEOREM

Let $G(\cdot\,|\, x_i)$ denote the response of the system B to a δ-function input located at x_i, and let it be called the impulsive response of B. Thus,

$$G(\cdot\,|\, x_i) = L\, \delta^U(\cdot\,|\, x_i) \tag{7}$$

where $\delta^U(\cdot\,|\, x_i) \epsilon\, U$ and $G(\cdot\,|\, x_i) \epsilon\, W$. Here again the first argument of G will be called the point of observation while the second will be called the point of application.

Similarly let $G^*(\cdot\,|\, y_i)$ denote the impulsive response of B^*, i.e.,

$$G^*(\cdot\,|\, y_i) = L^*\, \delta^W(\cdot\,|\, y_i)$$

Here $\delta^W(\cdot\,|\, y_i) \epsilon\, W$ and $G^*(\cdot\,|\, y_i) \epsilon\, U$.

The most important consequence of the reciprocity theorem is the following:

<u>Consequence 1</u>: The output $w(y_i)$ of B at y_i, due to an input u, is given by the inner product of the impulsive response of the adjoint system B^* to an impulse located at y_i, with the input u to B, i.e.,

$$w(y_i) = \langle G^*(\cdot \mid y_i), \quad u(\cdot)\rangle_U \tag{8}$$

and this holds $\forall y_i \in Y$ and $\forall u \in U$.

<u>Proof</u>: In (5) we set $u^* = \delta^U(\cdot \mid y_i)$, thus:

$$\langle w^*, u\rangle_U = \langle \delta^U(\cdot \mid y_i), w\rangle_W \tag{8a}$$

Now by the definition of the δ-function (6), the right side of (8a) is simply $w(y_i)$. Also, w^* in (8a) is the response of B^* to $\delta(\cdot \mid y_i)$; thus by definition $w^* = G^*(\cdot \mid y_i)$. Inserting $w^* = G^*(\cdot \mid y_i)$ in (8a) we get

$$\langle G^*(\cdot \mid y_i), \quad u\rangle_U = w(y_i)$$

which completes the proof.

Q. E. D.

Consequence 1 is a special case of the reciprocity theorem when the input to the adjoint system is a δ-function. The second consequence is a still more limited special case; let also the input to the system B be a δ-function, located at x_j, i. e., set

$$u = \delta^U(\cdot \mid x_j) \tag{9}$$

By substituting (9) into (8) we get the second consequence:

<u>Consequence 2</u>: The response of B at y_i, as a result of a δ-function input located at x_j, is equal to the conjugate of the response of B^* at x_j as a result of a δ-function input located at y_i,

$$G(x_j \mid y_i) = \overline{G^*(y_i \mid x_j)} \tag{10}$$

A similar result to (10) is also given by P. H. Morse and H. Feshbach.[4]

X. EVALUATION OF THE RECIPROCITY THEOREM AND ITS CONSEQUENCES

In this section we shall compare two methods of solving for the response of a system.

From Consequence 1 we have:

$$w(y_i) = \langle G^*(\cdot | \, y_i), \; u(\cdot) \rangle_U \tag{11}$$

We shall call the method of deriving $w(y_i)$ by (11) the adjoint method. Here we make use of the adjoint system B^*, or more specifically: we have to set up the adjoint system and find its response at all points of observation, as caused by a δ-function located at the point y_i.

Using Consequence 2 we can replace $G^*(\cdot | \, y_i)$ by $G(y_i \, | \cdot)$ in (11). Thus we get an alternative expression for $w(y_i)$, i.e.,

$$w(y_i) = \langle G(y_i \, | \cdot), \; u(\cdot) \rangle_U \tag{12}$$

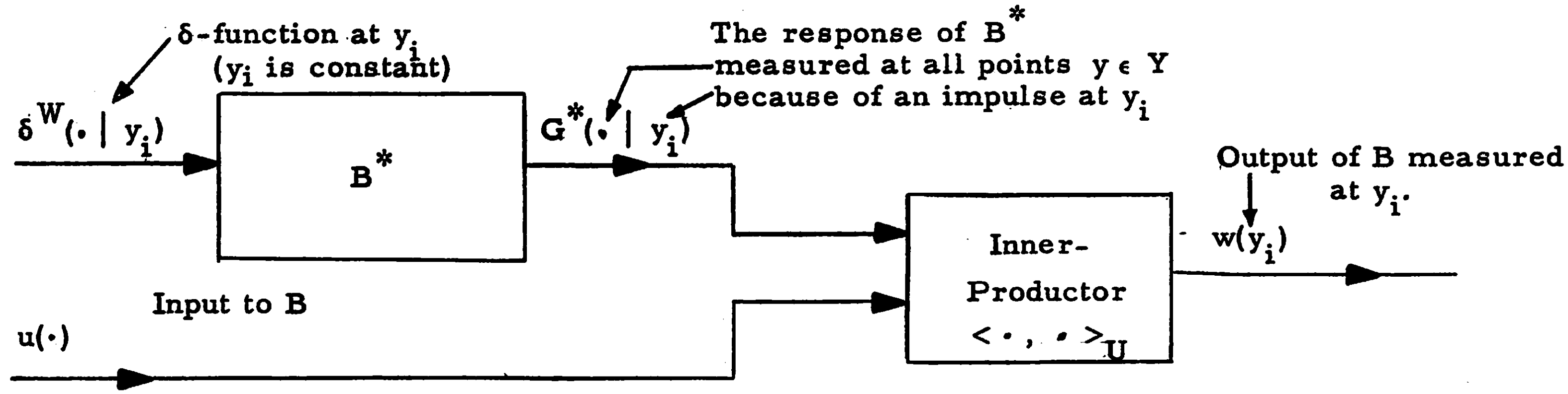
The method of deriving $w(y_i)$ by use of (12) will be called the direct method. With this method we have to find the response of the system B itself at the fixed point y_i, as a result of δ-functions located at all possible points of application.

The difference between these two methods is exhibited by block diagrams in Figure 4.

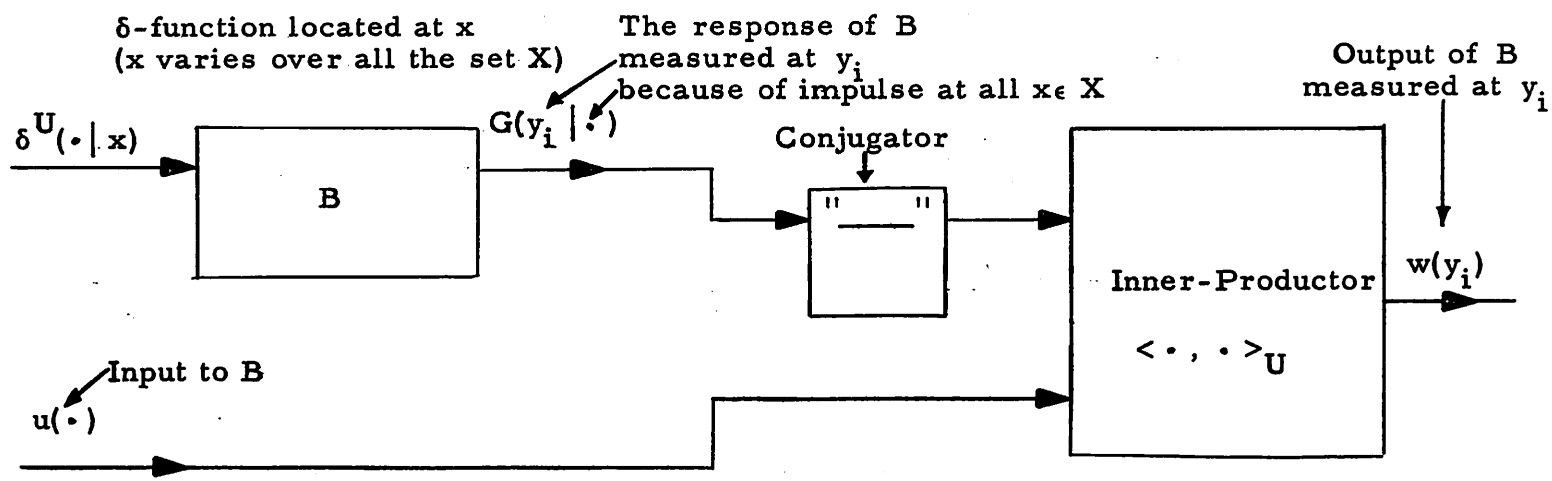
The direct method is, in fact, the "usual" way of finding the response of a system. Namely, first we find the response of the system to δ-functions located at all possible input points, and then we add up all the contributions, weighing each single response according to the actual strength of the input at the corresponding point.

The ajoint method furnishes us with an alternative way of finding the response. This way is less intuitive than the direct method, but seems to be simpler from the experimental point of view, since we have to apply the input at only one point, (the point at which we want to find the response).

Example: We shall compare the direct method with the adjoint method for the case of a single-input, single-output, linear, time-variant system. Here $U = \{u(\cdot)\}$ is the set of all functions $u(\cdot)$ which can serve as an input; for example, we could limit ourselves to all functions in $\mathcal{L}_2$. $W = \{w(\cdot)\}$ is the set of all the corresponding



THE ADJOINT METHOD



THE DIRECT METHOD

Figure 4.

outputs. We define an inner product over U:

$$<u_a(\cdot),\ u_b(\cdot)>_U = \int_{-\infty}^{+\infty} u_a(t)\, u_b(t)\, dt$$

and consequently the δ-function will be the usual Dirac δ-function, i.e.,

$$\delta^U(\cdot|\ t_1) = \delta(t-t_1)$$

Also the inner product and the δ-function in W will be chosen to be the same as in U. Now, let us denote the response of B at τ_2 to an impulse at τ_1 by $h(\tau_2 ; \tau_1)$, and similarly, let $h^*(\tau_2 ; \tau_1)$ denote the response of B^* at τ_2 to an impulse at τ_1. Thus,

$$G(\tau_2|\ \tau_1) = h(\tau_2 ; \tau_1) \text{ and } G^*(\tau_2|\ \tau_1) = h^*(\tau_2 ; \tau_1).$$

Using this notation we get from Equation (11) (the adjoint method)

$$w(\tau) = \int_{-\infty}^{+\infty} h^*(t ; \tau)\, u(t)\, dt \qquad (13)$$

and from Equation (12) (the direct method) we get

$$w(\tau) = \int_{-\infty}^{+\infty} h(\tau ; t)\, u(t)\, dt \qquad (14)$$

In this example it is obviously easier to find $h^*(t ; \tau)$ than $h(\tau ; t)$; $h^*(t ; \tau)$ can be found by measuring for all t the response of B^* to an impulse applied at τ. On the other hand, the direct method would require an infinite number of experiments: to find $h(\tau\,;t)$ we must apply to B an impulse at time t, and observe the response at τ, and repeat this experiment for all times of application t. In Section XII we shall see an example where both the direct and the adjoint methods are practical, and the decision as to which method should be used will depend solely on the technical details of the experiment.

Consequence 2, which is a special case of the reciprocity theorem (Section VI), is a natural extension of what is usually referred to as the "reciprocity theorem" in various fields such as circuit theory, electromagnetics, etc. While the usual "reciprocity theorems" are stated for self-adjoint systems only, Consequence 2 extends this notion to systems which are not necessarily self-adjoint. This point is illustrated in Section XII (Problem 1).

XI. GENERALIZATION OF THE CONSEQUENCES

Let us recall how we derived the two consequences from the reciprocity theorem. We started with Equation (5) (the reciprocity theorem), and replaced u^*, and later on, also u by δ-functions. However, it is not essential to proceed in this way. Any other function could do as well as the δ-function. To state the following generalizations of the consequences in a precise way, we have to define a projection.

Definition: We define the projection of u_i along u_j to be the result of the inner product of u_i and u_j (in this order), i.e.,

projection of u_i along $u_j = \langle u_i, u_j \rangle_U$

Using the term "projection" we can rephrase the reciprocity theorem in the following way: the projection of w along u^* is the same as the projection of w^* along u, where w is the response of B to u and w^* is the

$$\underline{v} = G^{-1}\underline{I}$$

where $\underline{G}$ is the admittance matrix of the circuit, I_j --the current sources and v_j-- the node voltages.

(ii) RCL circuit fed by sinusoidal sources, if we are interested only in the steady-state behavior.

(iii) Laplace-transformed linear time invariant systems.

(iv) Time variant systems in which the state vector $\underline{z}$ satisfies the differential equation

$$\dot{\underline{z}}(t) = \underline{A}(t)\ \underline{z}(t) \qquad \underline{z}(t_o) = \underline{z}_o$$

Here $\underline{z}(t)$ is given in terms of the initial state $\underline{z}_o$ and the state transition matrix, i. e.,

$$\underline{z}(t) = \underline{\Phi}(t\,;\,t_o)\underline{z}_o$$

We define the inner product of two m-dimensional vectors $\underline{v}_a$ and $\underline{v}_b$ as

$$<\underline{v}_a, \underline{v}_b>_V = \sum_{i=1}^{m} \overline{v}_{ai}\, v_{bi}$$

where $\overline{v}_{ai}$ stands for the complex conjugate of the i-th component of $\underline{v}_a$.

Similarly, we define the inner product of two n-dimensional vectors $\underline{z}_a$ and $\underline{z}_b$ as

$$<\underline{z}_a \cdot \underline{z}_b>_Z = \sum_{i=1}^{n} \overline{z}_{ai} \cdot z_{bi}$$

It is easy to verify that these definitions satisfy the conditions of Section III.

Now, let $\underline{v}^*$ --an n-dimensional vector--be the input to the adjoint system B^* and let us denote by $\underline{L}^*$ its operator which is still unknown. In order to find L^* we set, according to (3),

$$<\underline{L}\,\underline{v}\,,\ \underline{v}^*>_Z = <\underline{v}\,,\ \underline{L}^*\,\underline{v}^*>_V$$

Expanding both sides of this equation we get:

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{m}\overline{\ell_{ij}v_j}\right)v_i^* = \sum_{j=1}^{m}\overline{v}_j\left(\sum_{i=1}^{n}\ell_{ij}\,v_i^*\right) \triangleq \sum_{j=1}^{m}\overline{v}_j\left(\underline{L}^*\underline{v}^*\right)_j$$

from which follows that

$$\underline{L}^* = \overline{\underline{L}^t}$$

Thus the adjoint of a matrix operator $\underline{L}$ is the complex conjugate of the transpose of $\underline{L}$. Note that a matrix operator is self-adjoint if, and only if, it is hermitian.

Let $\underline{z}^*$ be the output of B^* as a result of $\underline{v}^*$, i.e.,

$$\underline{z}^* = \underline{L}^*\underline{v}^*$$

then the reciprocity theorem states that

$$<\underline{z}\,,\ \underline{v}^*>_Z = <\underline{v}\,,\ \underline{z}^*>_V$$

or in the expanded form,

$$\sum_{i=1}^{n}\overline{z}_i\,v_i^* = \sum_{i=1}^{m}\overline{v}_i\,z_i^*$$

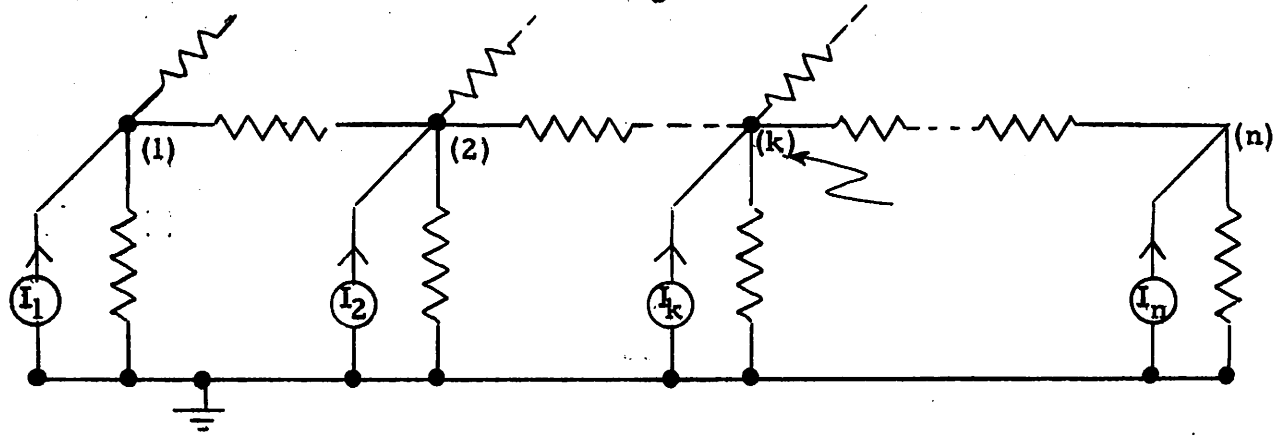
Note also that the δ-function which corresponds to the above definition of the inner product $< ; >_V$ is an m-vector and has the following form:

$$\delta^V(\cdot|\,j) = (0, 0, \ \ldots\ 0, \underset{\substack{\uparrow\\ \text{j-th component}}}{1}, 0, \ \ldots\ 0)$$

Similarly for $< , >_Z$, $\delta^Z(\cdot|\,j)$ is an n-vector with all components zero except for the j-th component which is 1.

To illustrate the procedure of solving problems by the use of the adjoint method let us present the material in terms of concrete systems.

Problem 1: Given a non-reciprocal network comprised of resistors only, n-current sources are connected between the n-nodes of the circuit and the datum. Find the voltage between the k-th node and the datum.



Solution: Let $\underline{I} = (I_1, I_2, \ldots, I_n)$ be the current source vector, $\underline{v} = (v_1, v_2, \ldots, v_n)$ be the voltage response vector and let $\underline{G}$ be the admittance matrix of the circuit. Then

$$\underline{I} = \underline{G}\,\underline{v} \quad \text{or} \quad \underline{v} = \underline{G}^{-1}\underline{I}$$

from which it follows that

$$v_k = (G^{-1})_{k1} I_1 + (G^{-1})_{k2} I_2 + \ldots + (G^{-1})_{kn} I_n \tag{15}$$

We have just seen that if $\underline{G}$ is known and if we can calculate G^{-1} then, of course, the problem is solved. Let us now approach this same problem from an experimental point of view.

a. **The direct method:** To find v_k, the voltage at the k-th node because of the current sources $\underline{I} = (I_1, I_2, \ldots, I_n)$ we first apply a unit source at the first node, while all other sources are removed and measure the resulting voltage at the k-th node. Let us denote this voltage by $v_k^{(1)}$. Next we multiply $v_k^{(1)}$ by the actual strength of the source at the first node, namely by I_1. We repeat this procedure for all nodes and finally add up all the contributions to the voltage at the k-th node to get v_k, i.e.,

$$v_k = v_k^{(1)} I_1 + v_k^{(2)} I_2 + \ldots + v_k^{(n)} I_n. \tag{16}$$

If we compare Equation (16) to (15) we see that $v_k^{(i)} = (G^{-1})_{ki}$, i. e., $(G^{-1})_{ki}$ is the response of the circuit at the k-th node, to a unit source connected at the i-th node.

A different interpretation to (15) is furnished by

b. The adjoint method: We set-up the adjoint system, which is the system whose operator is $\underline{G}^t$. To find v_k, we apply a unit source at the k-th node of this adjoint system. We measure the responses to this unit source at all the nodes of the adjoint system, and denote then by $v_1^{(k)*}, v_2^{(k)*}, \ldots, v_n^{(k)*}$. Finally we add up all contributions; thus,

$$v_k = v_1^{(k)*} I_1 + v_2^{(k)*} I_2 + \ldots + v_n^{(k)*} \cdot I_n \tag{17}$$

(17) follows directly from Consequence 1.

Now comparing (17) with (15) we get a different interpretation for $(G^{-1})_{ki}$, i. e., $(G^{-1})_{ki} = v_i^{(k)*}$. This leads us to Equation (18),

$$v_i^{(k)*} = v_k^{(i)} \tag{18}$$

Equation (18) is a restatement of Consequence 2 (10) for the special case of d. c. circuits; namely, the voltage response at the k-th node of a circuit (as measured by an infinite impedance voltmeter), to a unit current source at the i-th node is the same as the voltage response of the adjoint circuit at the i-th node as a result of a unit current source applied at its k-th node.

Summary: While in the direct method we had to apply a unit source at all the n-nodes, we had to measure the response at the k-th node only. Whereas in the adjoint method we had to apply only one unit source, namely at the k-th node of the adjoint circuit (later to be called a test source) but measure its response at all nodes.

Note 1: Which method is better ? The answer depends purely on the technical details of the experiment: if the measuring instrument is easier to handle than the source, then we would prefer the adjoint method, and vice versa.

Note 2: In the special case where the circuit is reciprocal, the above developed procedure remains the same, except that it is not necessary to set up a new system since the system is self-adjoint. In this case, Equation (18) is what is usually referred to as the reciprocity theorem for reciprocal circuits, i. e., the voltage response at the k-th node to a unit current source at the i-th node remains the same if we interchange the location of the source and the voltmeter.

We shall illustrate with three additional problems the different ways in which the reciprocity theorem can be applied.

Problem 2: Given a reciprocal network comprised of resistors only (i. e., its $\underline{G}$ matrix is real and symmetric) fed at its n nodes by the current sources $\underline{I}_a = (I_{a1}, I_{a2}, \dots I_{an})$. What are the conditions imposed on a source $\underline{I}_b = (I_{b1}, I_{b2}, \dots, I_{bn})$, if it has to produce at the k-th node the same voltage as $\underline{I}_a$?

Solution: Apply a test source $\underline{I}_t$ at the k-th node,

$$\underline{I}_t = (0, 0, \dots, 0, \underset{\substack{\uparrow \\ \text{k-th component}}}{1}, 0, \dots 0)$$

Let v_{ak}, v_{bk} denote the voltage at the k-th node when the circuit is fed by $\underline{I}_a$ and $\underline{I}_b$, respectively.

$$\underline{v}_{ak} = \langle \underline{v}_a, \underline{I}_t \rangle \quad \text{and} \quad v_{bk} = \langle \underline{v}_b, \underline{I}_t \rangle$$

where $\underline{v}_a$, ($\underline{v}_b$) is the set of voltages resulting from $\underline{I}_a$, ($\underline{I}_b$ respectively). As we require $v_{ak} = v_{bk}$, we must set

$$\langle \underline{v}_a, \underline{I}_t \rangle = \langle \underline{v}_b, \underline{I}_t \rangle \tag{19}$$

Applying the reciprocity theorem to both sides of (19) we get

$$\langle \underline{I}_a, \underline{v}_t \rangle = \langle \underline{I}_b, \underline{v}_t \rangle \tag{20}$$

where $\underline{v}_t$ is the voltage at the n nodes as a result of $\underline{I}_t$. Equation (20) is the required condition on $\underline{I}_b$.

Problem 3: Given the same circuit as in Problem 2, also fed by $\underline{I}_a$. What would be the amplitude of a source $\underline{I}_b$, if it acts only at the b-th node, and has to produce the same voltage as $\underline{I}_a$ at the ℓ-th node?

Solution: This problem is a special case of Problem 3. The form of $\underline{I}_b$ is

$$\underline{I}_b = (0, 0, \ldots, 0, \underset{\substack{\uparrow \\ \ell\text{-th component}}}{I_{b\ell}}, 0, \ldots 0)$$

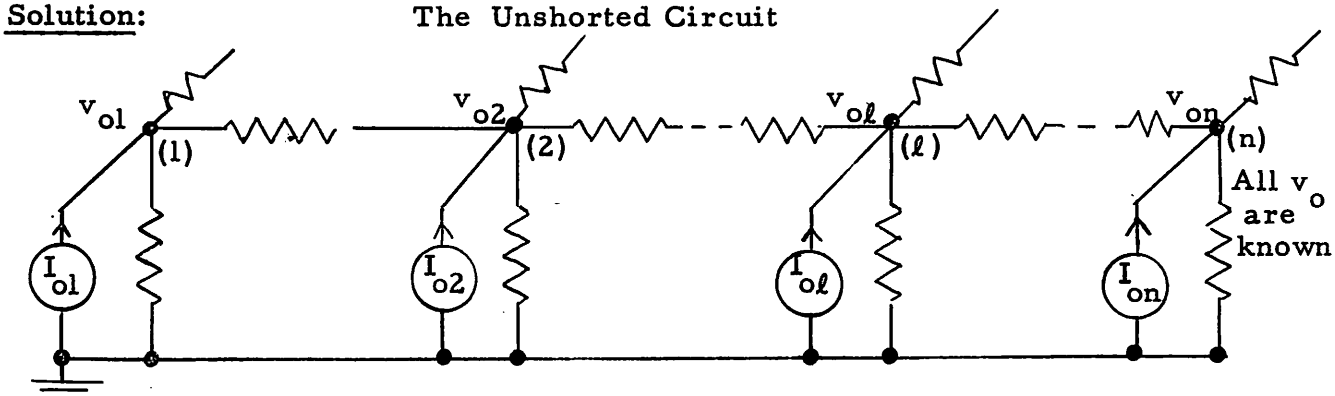
where $I_{b\ell}$ is the unknown. Inserting $\underline{I}_b$ into Equation (20) we get,

$$I_{b\ell} = \frac{<I_a, v_t>}{v_{t\ell}}.$$

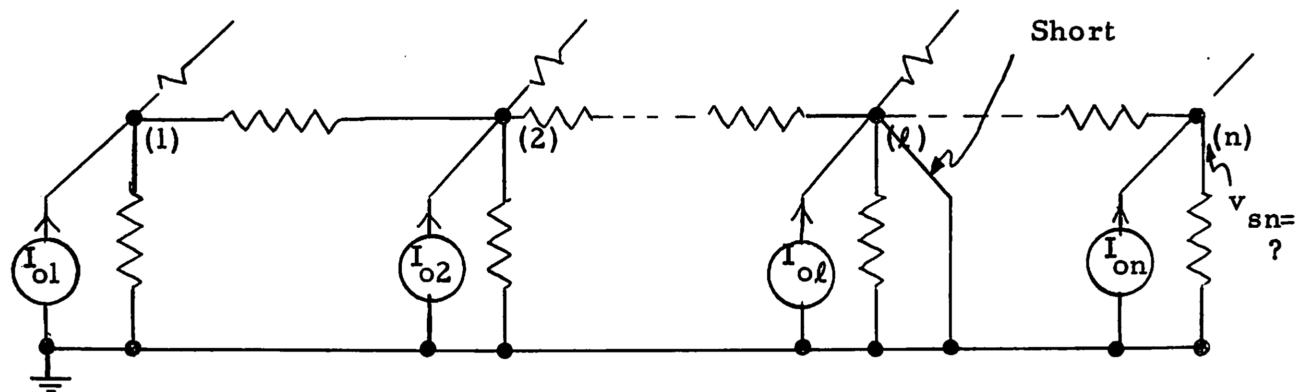
where $v_{t\ell}$ is the voltage at the ℓ-th node as a result of $\underline{I}_t$.

Problem 4: Given a circuit as in Problem 2, fed by $\underline{I}_o$; let all resulting voltages $\underline{v}_o$ be known. Then, short the ℓ-th node to the ground. What will be the new voltage at the n-th node?

Solution: The Unshorted Circuit



The Shorted Circuit



Let $\underline{v}_o$ be the voltage of the unshorted circuit; let $\underline{v}_s$ be the voltage after the short is applied. We introduce a source $\underline{I}_i$,

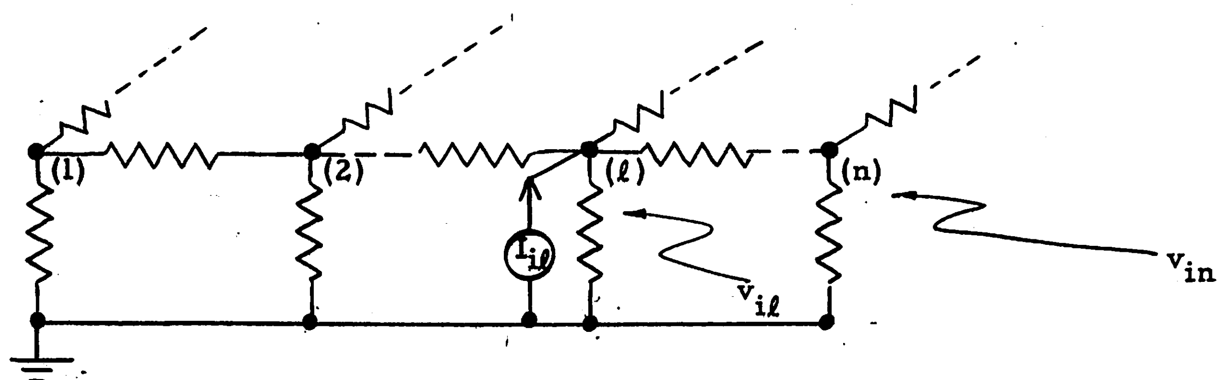
$$\underline{I}_i = (0, 0, \ldots, 0, I_{i\ell}, 0, \ldots 0)$$

ℓ-th component

where the amplitude of $\underline{I}_i$ is such that

$$-v_{o\ell} = v_{i\ell}$$

i.e., $v_{i\ell}$, the voltage resulting from $\underline{I}_i$ at the ℓ-th node of the unshorted circuit, is equal and opposite to $v_{o\ell}$.



Clearly, if $\underline{I}_o$ and $\underline{I}_i$ are applied simultaneously to the unshorted circuit the resulting voltage at the ℓ-th node is zero, hence,

$$v_{sn} = v_{on} + v_{in}$$

To find v_{in} we must first compute $I_{i\ell}$; from Problem 3 it follows that

$$I_{i\ell} = -\frac{\langle \underline{I}_o \, \underline{v}_t^{(\ell)} \rangle}{v_{t\ell}^{(\ell)}}$$

where $\underline{v}_t^{(\ell)}$ is the response to the source $I_t^{(\ell)}$

$$\underline{I}_t^{(\ell)} = (0, 0, \ldots, 0, 1, 0, \ldots 0)$$

ℓ-th component

and $v_{t\ell}^{(\ell)}$ is the ℓ-th component of $\underline{v}_t^{(\ell)}$.

Now to find v_{in} we introduce another test source $\underline{I}_t^{(n)}$,

$$\underline{I}_t^{(n)} = (0, 0, \ldots, 0, 0, \underset{\substack{\uparrow \\ \text{n-th component}}}{1})$$

From the reciprocity theorem it follows that

$$<\underline{I}_t^{(n)}, \underline{v}_i> = <\underline{v}_t^{(n)}, \underline{I}_i>$$

which, when simplified is simple

$$v_{in} = I_{i\ell} \cdot v_{t\ell}^{(n)}$$

Thus, finally:

$$v_{sn} = v_{on} - \frac{v_{t\ell}^{(n)} <\underline{v}_t^{(\ell)} I_o>}{v_{t\ell}^{(\ell)}}$$

We conclude this section with three remarks.

(a) Note that the same method of solution was used for all these four problems: a test source was applied at the point the voltage of which was to be found.

This idea follows directly from Consequence 1.

(b) If the circuits of Problems 2-4 were not reciprocal, i.e., the corresponding operator self-adjoint, the method of solution would remain the same, only that the test source should have been applied to the adjoint circuit.

(c) Note that although these four problems were stated in terms of d. c. circuits, the method of solution would remain the same for similar problems provided the operator is represented by a matrix.

XIII. APPLICATION OF THE ADJOINT METHOD TO SYSTEMS CHARACTERIZED BY THEIR STATE EQUATIONS

Let the system B be described by the state equations

$$\dot{\underline{z}}(t) = \underline{A}(t)\ \underline{z}(t) + \underline{v}(t) \qquad (21)$$

where $\underline{z}(t)$ is the state vector, $\underline{v}(t)$ is the forcing vector, both n-dimensional, and $\underline{A}(t)$ is an n x n matrix, whose elements $a_{ij}(t)$, are continuous functions of time.

Without any loss of generality we shall assume that the initial conditions of this system B are zero, as we can incorporate the initial conditions into the forcing vector by means of δ-functions. Hence we set

$$\underline{z}(t_o) = \underline{0}$$

We define an inner product of two n-dimensional vectors $\underline{z}_a(t)$ and $\underline{z}_b(t)$ as

$$<\underline{z}_a(t), \underline{z}_b(t)> = \int_{t_o}^{t_1} \sum_{i=1}^{n} z_{ai}(t)\, z_{bi}(t)\, dt,$$

where t_o is the time at which $\underline{z}(t_o) = \underline{0}$ and t_1 is the time at which the process terminates.

To find the adjoint system B^* we write (21) in the following form

$$L\underline{z} = \underline{v}.$$

For the system B^* we write

$$L^*\underline{z}^* = \underline{v}^*$$

where L^* is still unknown.

By definition (3),

$$<L\underline{z}, \underline{z}^*> = <\underline{z}, L^*\underline{z}^*>$$

Expanding the right side we get:

$$<L\,\underline{z}, \underline{z}^*> = \int_{t_o}^{t_1} \sum_{i=1}^{n} \dot{z}_i\, z_i^*\, dt - \int_{t_o}^{t_1} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} z_j \right) z_i^*\, dt$$

$$= \sum_{i=1}^{n} z_i z_i^* \Big|_{t_o}^{t_1} - \int_{t_o}^{t_1} \sum_{i=1}^{n} \dot{z}_i^* z_i\, dt - \int_{t_o}^{t_1} \sum_{j=1}^{n} z_j \left(\sum_{i=1}^{n} a_{ij} z_i^* \right) dt \qquad (22)$$

If we set in (22)

$$\underline{z}^*(t_1) = \underline{0}$$

then,

$$\sum_{i=1}^{n} z_i z_i^* \Big|_{t_o}^{t_1} = 0$$

and we get,

$$< L\underline{z},\ \underline{z}^* > = < \underline{z},\ -\dot{\underline{z}}^* - A^t \underline{z}^* >.$$

This shows that the state equations of B^* are

$$\dot{\underline{z}}^*(t) = -A^t(t)\,\underline{z}^*(t) - \underline{v}^*(t)$$

with the initial condition $\underline{z}^*(t_1) = \underline{0}$.

Since $\underline{z}(t_o) = \underline{0}$ and $\underline{z}^*(t_1) = \underline{0}$, B is non-anticipative, while B^* is completely anticipative: roughly speaking, B^* is said to run backwards from $\underline{z}^*(t_1)$ to $\underline{z}^*(t)$ $(t < t_1)$ where state vector, $\underline{z}^*(t)$, of B^* at time t $(t < t_1)$ depends on the input to B^* over the interval $[t, t_1]$.

The reciprocity theorem states, in this case, that,

$$< \underline{v},\ \underline{z}^* > = < \underline{z}\ ,\ \underline{v}^* > \tag{23}$$

In the case that the systems B and B^* have non-zero initial states, we shall, as we have already mentioned, incorporate the initial states into the forcing vectors as δ-functions. More specifically, assume that the initial states of B and B^* are respectively $\underline{z}(t_o) = \underline{z}_o$ and $\underline{z}^*(t_1) = \underline{z}_1^*$, and that, as before, $\underline{v}$ and $\underline{v}^*$ are the respective forcing vectors of B and B^*. This situation is equivalent to the case that both B and B^* are initially at rest, i.e., $\underline{z}(t_o) = \underline{0}$ and $\underline{z}^*(t_1^*) = \underline{0}$ whereas the forcing vectors are changed into $\underline{v} + \underline{z}_o\ \delta(t-t_o)$, and $\underline{v}^* + \underline{z}_1^* \cdot \delta(t-t_1)$. The reciprocity theorem for this case is+

$$< \underline{v} + \underline{z}_o\ \delta(t-t_o),\ \underline{z}^* > = < \underline{z},\ \underline{v}^* + \underline{z}_1^*\,\delta(t-t_1) > \tag{24}$$

Further, in the case that $\underline{v} = \underline{0}$ and $\underline{v}^* = \underline{0}$, which corresponds to considering the free mtoion of the system, the respective forcing vectors are $\underline{z}_o\ \delta(t-t_o)$ and $\underline{z}_1^*\delta(t-t_1)$ and the reciprocity theorem is simply

+ G. A. Bliss[5] obtained a similar result to (24) by direct computation.

$$\sum_{i=1}^{n} z_i^*(t_1) \cdot z_{oi} = \sum_{i=1}^{n} z_i(t_1) \cdot z_{1i}^* \tag{25}$$

We turn now to the Consequences 1 and 2.

To get Consequence 1 we set:

$$\underline{v}^* = (0, 0, \ldots, 0, \underset{\substack{\uparrow \\ \text{i-th component}}}{\delta(t-t_1)}, 0, \ldots 0) \tag{26}$$

Inserting (26) into (23) we get:

$$z_i(t_1) = \int_{t_o}^{t_1} \sum_{j=1}^{n} z_j^*(t; t_1, i)\, v_j(t)\, dt \quad 1 \leq i \leq n \tag{27}$$

where $\underline{z}^*(t; t_1, i)$ is the response of B^* at t to the $\underline{v}^*$ given by (26).

If we compare this with the super

$$\underline{z}(t_1) = \int_{t_o}^{t_1} \underline{\underline{\Phi}}(t_1; t)\, \underline{v}(t)\, dt$$

where $\underline{\underline{\Phi}}(t_1; t)$ is the state transition matrix, it is clear that actually $\underline{\underline{\Phi}}(t_1; t)$, as a function of t, is the impulsive response of the adjoint system.

Now Consequence 2 will tell us that

$$z_i(t_1; t_o, j) = z_j^*(t_o; t_1, i) \tag{28}$$

where $\underline{z}(t_1; t_o, j)$ is the response of B at t_1 to a forcing function

$$\underline{v} = (0, 0, \ldots, 0, \underset{\substack{\uparrow \\ \text{j-th component}}}{\delta(t-t_o)}, 0, \ldots, 0)$$

Also in this case we have the two methods of solving for the response of a system: (27) is the adjoint method, and by substituting (28) into (27) we get the direct method, i. e.,

$$z_i(t_1) = \int_{t_o}^{t_1} \sum_{j=1}^{n} z_i(t_1; t, j)\, v_j(t)\, dt \tag{29}$$

It is easy to rephrase the four problems presented in Section I so that they should apply to systems described by their state equations. The method of solution will remain the same.

Three additional problems will be presented, illustrating possible ways of applying formulae (24) and (25) to problems in control.

Problem 5: Given a system $\dot{\underline{z}}(t) = \underline{f}[\underline{z}(t), \underline{v}(t), t]$ $\underline{z}(t_o) = \underline{z}_o$, where $\underline{z}(t)$, an n-vector, is the state vector, $\underline{v}(t)$, an m-vector, is the input vector, and $\underline{f}$ is an n-vector defined over $R^n \times R^m \times \underline{t}$ with components differentiable with respect to all the variables. A "normal" input $\overset{o}{\underline{v}}(t)$ $t_o \leq t \leq t_e$ and its corresponding trajectory $\overset{o}{\underline{z}}(t)$ $t_o \leq t \leq t_e$ with end point $\overset{o}{\underline{z}}(t_e) = \underline{z}_e$ are also given. At time t_1, $t_o \leq t_1 \leq t_e$, the actual state is measured, and is found, because of some disturbances, to differ slightly from $\overset{o}{\underline{z}}(t_1)$. By what constant factor should the amplitude of $\overset{o}{\underline{v}}(t)$ be multiplied, during the time interval $[t_1, t_e]$, so as to get, for some given $\underline{\alpha} \neq \underline{0}$,

$[\underline{z}(t_e) - \overset{o}{\underline{z}}_e] \cdot \underline{\alpha} = 0$ (30). $\underline{z}(t)$ is the actual trajectory and $\underline{z}(t_e)$ is its end point. Assume that in the interval $[t_1, t_e]$ the system is free from disturbances. See Figure 5.

Before starting with the solution let us remark that 1) the length of the given $\underline{\alpha}$ is immaterial, and that 2) in the special case that $\alpha = (1, 0, 0, \ldots, 0)$ the condition $[\underline{z}(t_e) - \overset{o}{\underline{z}}_e] \cdot \underline{\alpha} = 0$ is simply $z_1(t_e) = \overset{o}{z}_{e1}$.

Solution: In the interval $[t_1, t_e]$, $\overset{o}{\underline{z}}(t)$ satisfies the equation

$$\dot{\overset{o}{\underline{z}}}(t) = \underline{f}\,[\overset{o}{\underline{z}}(t), \overset{o}{\underline{v}}(t), t] \tag{31}$$

where $\overset{o}{\underline{z}}(t_1)$ is the initial state, and $\underline{z}(t)$ satisfies the equation

$$\dot{\underline{z}}(t) = \underline{f}[\underline{z}(t), (1+\beta)\overset{o}{\underline{v}}(t), t] \tag{32}$$

Illustration to Problem V.

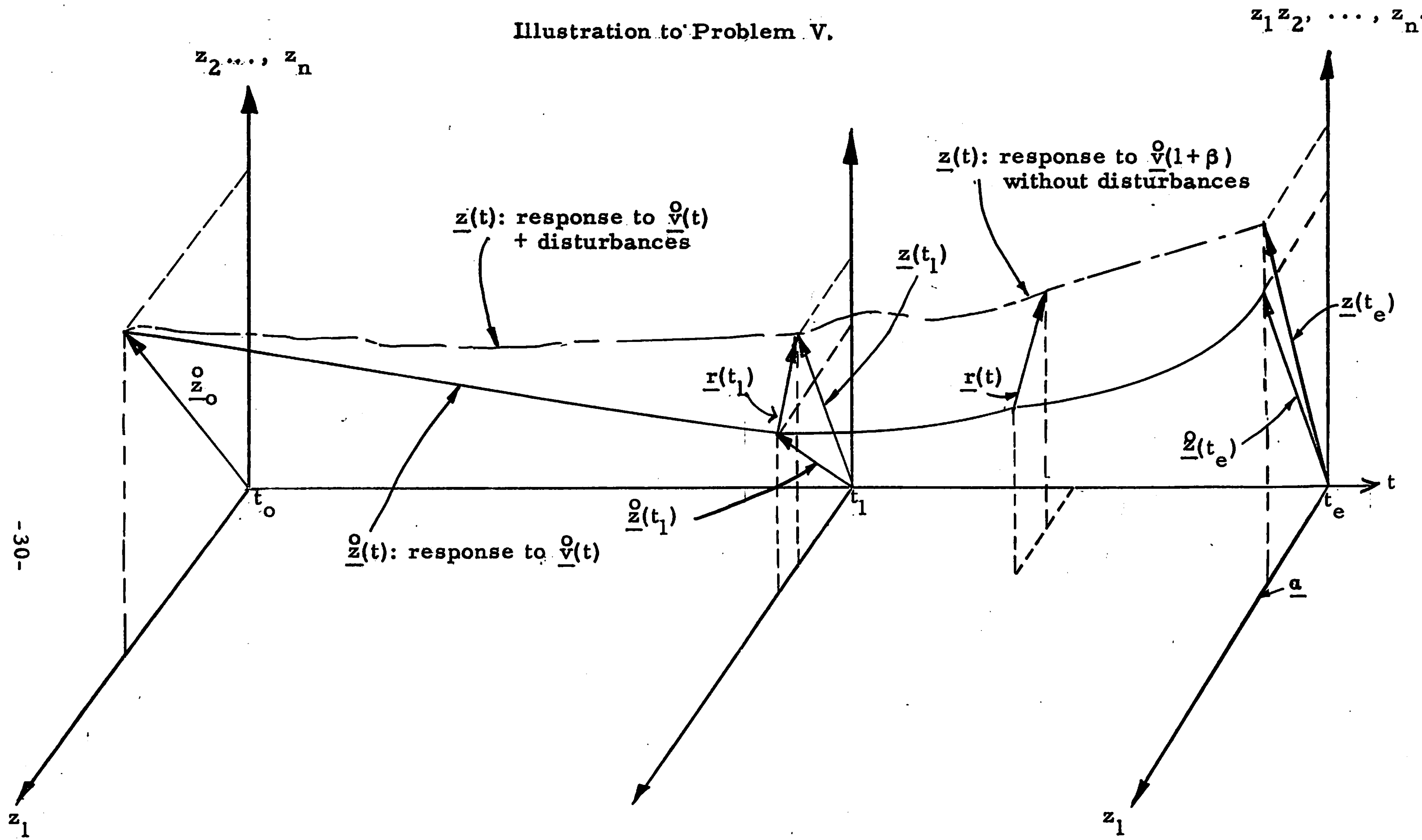


Figure 5.

where $\underline{z}(t_1)$ is the initial state, and $1+\beta$ is the factor with which we are going to correct the input $\overset{o}{\underline{v}}(t)$, so as to satisfy the end condition (30). Now we set $\underline{z}(t) = \underline{r}(t) + \overset{o}{\underline{z}}(t)$ and substruct (31) from (32) and thus get the variational equation of the system

$$\dot{\underline{r}}(t) = \frac{\partial f}{\partial \underline{z}}\Bigg|_{\substack{\overset{o}{\underline{z}}(t)\\ \overset{o}{\underline{v}}(t)}} \cdot \underline{r}(t) + \frac{\partial f}{\partial \underline{v}}\Bigg|_{\substack{\overset{o}{\underline{z}}(t)\\ \overset{o}{\underline{v}}(t)}} \beta \cdot \overset{o}{\underline{v}}(t)\,;\; \underline{r}(t_1) = \underline{z}(t_1) - \overset{o}{\underline{z}}(t_1) \qquad (33)$$

which is correct to the first order. Here we used the notation,

$$\frac{\partial f}{\partial \underline{z}} = \begin{bmatrix} \frac{\partial f_1}{\partial z_1}[\underline{z}(t), \underline{v}(t), t] & \cdots\cdots & \frac{\partial f_1}{\partial z_n}[\underline{z}(t), \underline{v}(t), t] \\ \vdots & & \\ \frac{\partial f_n}{\partial z}[\underline{z}(t), \underline{v}(t), t] & \cdots\cdots & \frac{\partial f_n}{\partial z_n}[\underline{z}(t), \underline{v}(t), t] \end{bmatrix}_{\substack{\underline{z}(t) = \overset{o}{\underline{z}}(t)\\ \underline{v}(t) = \overset{o}{\underline{v}}(t)}}$$

Our problem is to find the value of β so that $\underline{r}(t_e) \cdot \underline{a} = 0$. We introduce now the adjoint system: let

$$\dot{\underline{r}}^*(t) = -\left(\frac{\partial f}{\partial \underline{z}}\right)^t \underline{r}(t) \qquad \underline{r}^*(t_e) = \underline{a}$$

then by (24),

$$<\beta\left(\frac{\partial f}{\partial \underline{v}}\right)\overset{o}{\underline{v}} + \underline{r}(t_1)\ \delta(t-t_1),\ \underline{r}^*(t)> = <\underline{r}(t),\ \underline{a}\,\delta(t-t_e)> \qquad (34)$$

Expanding (34) we get,

$$\underline{r}^*(t_1)\,\underline{r}(t_1) + \int_{t_1}^{t_e} \beta\left(\frac{\partial f}{\partial \underline{v}}\right)\overset{o}{\underline{v}}\,\underline{r}^*(t)\,dt = \underline{r}(t_e) \cdot \underline{a}$$

thus finally, if we set

$$\beta = \frac{-\underline{r}^*(t_1)\cdot\underline{r}(t_1)}{\int_{t_1}^{t_e} \frac{\partial f}{\partial \underline{v}}\ \underline{\overset{o}{v}}\cdot\underline{r}^*(t)\,dt} \tag{35}$$

we get $\underline{r}(t_e)\ \underline{\alpha} = 0$.

So, to find β (35), we have to generate $\underline{r}^*(t)$, which is the response of the adjoint of the variational system to the input $\underline{\alpha}$. Note that this can be done before the actual process started. Also $\underline{\overset{o}{v}}$ and $\left(\frac{\partial f}{\partial \underline{v}}\right)$ are known in advance, thus $\underline{r}(t_1)$ is the only quantity that has to be measured during the process. Note also that β can be computed continuously, thus allowing us to adjust β, continuously, during the process.

Problem 6: Given a system $\underline{\dot{z}}(t) = \underline{f}[\underline{z}(t), \underline{v}(t), t]$ $\underline{z}(t_o) = \underline{z}_o$ as in Problem 5, whose input $\underline{\overset{o}{v}}_{[t_o, t_e]}$ is given. At some given time t_1 $t_o < t_1 < t_e$, the input is shut off, and the resulting trajectory $\underline{\overset{o}{z}}(t)$, with $\underline{\overset{o}{z}}(t_e) = \underline{z}_e$ has already been calculated. At t_1, "the normal shut off time," the system is inspected and found to have a slight deviation $\underline{r}(t_1)$, from its normal position $\underline{\overset{o}{z}}(t_1)$. For how long should we delay the shut off so as to get $[\underline{z}(t_e) - \underline{z}_e]\cdot \underline{\alpha} = 0$, where $\underline{z}(t)$ denotes the actual trajectory and $\underline{z}(t_e)$ is its end point. Again it is assumed that in the interval $[t_1, t_e]$, the system is free from disturbances.

Solution: Let t_2 be the actual shut off time. As t_2-t_1 is small, the input will remain constant (to the first order), during this time. Let us denote this constant input by $\underline{\overset{o}{v}}(t_1)$.

Also let us define a function $G(t)$, where

$$G(t) = \begin{Bmatrix} 1 & t_1 \leq t \leq t_2 \\ 0 & \text{otherwise} \end{Bmatrix}$$

By using (24), letting, as in Problem 5, $\underline{\alpha}$ be the input to the adjoint of the variational system, and using the same notation

$$r(t_e)\cdot\underline{\alpha} = r(t_1)\cdot r^*(t_1) + \int_{t_1}^{t_e} \underline{r}^*(t)\cdot G(t)\left(\frac{\partial f}{\partial \underline{v}}\right)\ \underline{\overset{o}{v}}(t_1)\,dt \tag{36}$$

Now

$$\int_{t_1}^{t_e} \underline{r}^*(t) \cdot G(t) \left(\frac{\partial f}{\partial v}\right) \quad \underline{v}(t)\, dt = (t_2 - t_1)\, \underline{r}^*(t_1) \left(\frac{\partial f}{\partial v}\right)_{t_1} \overset{o}{\underline{v}}(t_1) \tag{37}$$

Thus finally (inserting (37) into (36)),

$$t_2 - t_1 = -\frac{\underline{r}(t_1) \cdot \underline{r}^*(t_1)}{\underline{r}^*(t_1) \cdot \left(\frac{\partial f}{\partial \underline{v}}\right)_{t_1} \overset{o}{v}(t_1)} \tag{38}$$

is the correct value for t_2, so as to get $\underline{a} \circ \underline{r}(t_e) = 0$. (If $t_2 - t_1) < 0$ then we have to reverse the sign of the input, during the time interval $[t_1, t_2]$, if it is only physically possible.)

To illustrate a possible application of (25) in the case that the end condition rather than the initial conditions are known, we present the following problem.

Problem 7: Given a system the state vector of which satisfies the following equation

$\dot{\underline{z}}(t) = \underline{A}(t)\ \underline{z}(t)$

No input is applied to the system. Find the initial conditions $\underline{z}(t_o)$ such that

$\underline{z}(t_1) = \underline{z}_1$ where $\underline{z}_1$ is given.

Solution: Apply

$$\underline{z}_i^*(t_1) = (0, 0, \ldots 0, \underset{\substack{\uparrow \\ \text{i-th component}}}{1}, 0, \ldots, 0)$$

to B^*, and measure the corresponding response $\underline{z}_i^*(t_o)$; repeat this experiment for all $i = 1, 2, \ldots, n$. Inserting the results into (25), we get n equation for the n unknowns $z_i(t_o)$, and thus find $\underline{z}(t_o)$.

We conclude by mentioning that works by H. S. Tsien,[6] Z. H. Lanning and R. H. Battin,[7] and W. J. Welch[8] treat some problems in a way related to this work.

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