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EQUIVALENT CIRCUITS OF OBSTACLES IN WAVEGUIDES USING THE VARIATIONAL METHOD

by

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TABLE OF CONTENTS

I. INTRODUCTION

When using waveguides in microwave transmission, discontinuities in the form of bends, obstacles, variations of cross-sectional area etc., are apt to show up. These discontinuities exist because they either cannot be avoided due to space limitations, energy feeding and energy absorbing devices, physical constraints, or are employed intentionally for several uses such as matching, phase shift, attenutation of undesired propagating modes, and others.

Upon encountering a discontinuity, a propagating electromagnetic wave in a uniform waveguide will divide its energy into three parts. The first goes to the further propagation of waves whose propagation the waveguide supports. The second goes to a reflected wave. The third goes to excite an infinite number of higher normal modes at the discontinuity. The presence of the infinite number of these higher normal modes serves to satisfy the new boundary conditions imposed by the presence of the discontinuity. Usually waveguide dimensions are selected so as not to support the propagation of the higher modes, and hence will be attenuated very rapidly as they travel away from the discontinuity.

The presence of the higher normal modes at the discontinuity signifies stored energy, so that the discontinuity could be represented by an energy storing element in the characteristic equivalent circuit of the waveguide. This energy storing element takes the shape of a shunt inductance or a shunt capacitance depending on whether the energy is stored in the form of magnetic or electric energy, respectively.

Figure 1

Considering a discontinuity made of a thin plate of metal that partially blocks the guide and is perpendicular to the axis of a waveguide where the window opening extends completely across the guide and is symmetrically located with the sides parallel to the electric field (Fig. 1), the equivalent circuit is an inductive reactance shunted across the guide. This is due to the fact that the metal diaphragm is parallel to the electric field, thus making it vanish everywhere at the discontinuity except in the window opening; while the magnetic field does not vanish, thus leaving most of the energy to be stored in the form of magnetic energy. In circuits this is represented by energy stored in an inductance.

To consider the other form of representation, we will take the case of a metal diaphragm which leaves a window opening extending completely across the waveguide with the sides perpendicular to the electric field (Fig. 2). The equivalent circuit parameter is a shunted capacitance. Here the electric field does not go to zero on the metal because it is perpendicular to it; and hence, energy is stored mainly in the electric form, which in circuits is represented by the energy stored in a capacitor.

Figure 2

The above equivalent circuit representation of the metal diaphragm by a single shunt element applies when the diaphragm is very thin compared to its transverse dimensions. If the diaphragm has a finite thickness, the equivalent circuit is usually a more complex T or Π network, which approaches a simple shunting reactance as the thickness approaches zero.

If there is more than one propagating mode in the waveguide, then the equivalent circuit parameter of the discontinuity has no meaning, since for each propagating mode we see a different equivalent circuit parameter. Thus, the waveguide to be considered in this project will have dimensions which support the propagation of the dominant mode only.

The problem of finding the equivalent circuit parameter of the discontinuity boils down to finding the electric and magnetic field distribution at the discontinuity, since it is sufficient to determine the dominant mode magnetic and electric fields at the terminals of the discontinuity to obtain a complete description of its reflection and transmission characteristics.[5] But it is practically impossible to find the fields at the discontinuity by solving Maxwell's equations. The only other alternative is to assume the form of the electric and magnetic field distribution at the discontinuity and then use approximate means to find the equivalent circuit parameter of the discontinuity.

II. VARIATIONAL METHOD

The behaviour of physical objects in nature is observed to be directed towards minimizing or maximizing some aspect of their state of condition. Water running downhill will take the shortest possible path to the bottom of the hill. Water will distribute itself in a water supply system, so that least power is lost due to friction in the pipes. "The statement that a physical system so acts that some function of its behaviour is least (or greatest) is often both the starting point for theoretical investigation and the ultimate distillation of all the relationships between facts in a large segment of physics."*

"The mathematical formulation of the superlative is usually that the integral of some function, typical of the system has a smaller (or else larger) value for the actual performance of the system than it would have for any other imagined performance subject to the same very general requirements which serve to particularize the system under study."*

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*See Ref. 7, page 275.

Thus if a function of the independent variables of the system and of the derivatives of these variables with respect to the parameters of integration, is integrated between the limits we are interested in, then the integral obtained would give a minimum or a maximum value when the exact values of the variables and their derivatives are inserted in the integrand. This process is called the variational method.

In the problem of this project, the variational method is used. The admittance of the discontinuity in the waveguide is expressed in terms of an integral which contains the unknown electric field distribution at the discontinuity. The expression would be variational, if upon inserting the mathematical expression of the exact electric field distribution at the discontinuity, the value of the calculated discontinuity equivalent admittance would be a relative maximum or minimum.

The use of the variational method to calculate the admittance of a discontinuity in a waveguide has many advantages over other classical methods. The advantages include the fact that the value of the equivalent admittance does not depend on the amplitude of the electric field, but only on its functional form. Thus, the variational method permits the exploitation of any information bearing on the problem such as might be available from purely intuitional considerations. However, the main advantage of the variational method is that when the exact electric field distribution is impossible to find (as is usually the case), then a first degree approximation of the electric field distribution at the discontinuity, yields a second degree approximation for the equivalent admittance of the discontinuity. This is clearly a property of great importance, since we are able to obtain accurate estimates of the equivalent admittance of the discontinuity by employing fairly crude trial functions, and thus do not have to obtain a complete solution of Maxwell's equations. The variational method enables us also to calculate upper and lower bounds for the equivalent parameters and hence we have a knowledge of the maximum error involved in the values of the calculated equivalent parameters.

In solving the problems of this project we are going to find the variational expression for the equivalent admittance of the discontinuity and then apply the trial fields in the variational expression. The trial fields used will be approximately obtained by calculating a corresponding static field distribution.

III. CIRCULAR APERTURE IN A RECTANGULAR WAVEGUIDE

Our first problem is to find the equivalent circuit parameter of a centered circular aperture in a transverse metallic diaphragm of zero thickness in a rectangular waveguide in which a TE_{10} mode is propagating

Marcuvitz[4] gives a variational expression for the susceptance of the equivalent circuit parameter of a discontinuity between two parallel plates of infinite width and length as:

$$j\,\frac{B}{2} = \sum_{n=1}^{\infty} Y_n \left[\frac{\int_{ap} E(y)\, h_n(y)\, dy}{\int_{ap} E(y)\, h(y)\, dy} \right]^2$$

where

Y_n is the input admittance of a uniform waveguide (no discontinuity) for the nth mode

$E(y)$ is the trial field

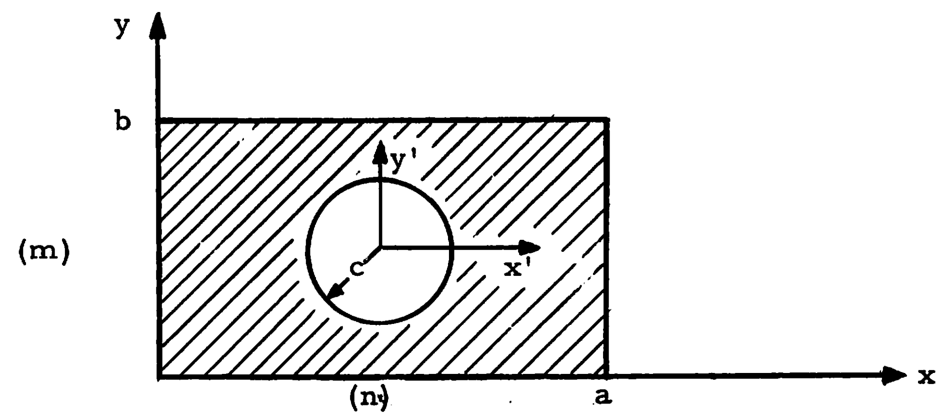
$h_n(y)$ is mode function of nth mode

$h(y)$ is mode function of propagating mode and

$$h(y) = - \int_{ap} G(y, y')\, E(y')\, dy$$

$$G(y, y') = \sum_{1}^{\infty} Y_n\, h_n(y)\, h_n(y')$$

The prime denotes the exact field. The proof that the above expression is variational is given in Appendix I of this paper.



The electric field distribution in the aperture has the static field form

$$\overline{E}_t = (c^2 - x'^2 - y'^2)^{1/2}\, a_z \times \overline{H}_t$$

where

$$\overline{H}_t = I_1\sqrt{\frac{\epsilon_1}{a}}\, \sin\frac{\pi x}{a} = I_1\sqrt{\frac{\epsilon_1}{a}}\, \cos\frac{\pi x'}{a}\, a_x \quad \text{for } TE_{10} \text{ mode}$$

and

$$h_{nm} = \sqrt{\frac{\epsilon_{nm}}{(ab)^{1/2}}}\left[\cos\frac{m\pi y}{b}\sin\frac{n\pi x}{a}\, a_x + \sin\frac{m\pi y}{b}\cos\frac{n\pi x}{a}\, a_y\right].$$

Adapting the variational expression for a rectangular waveguide, we get for the equivalent susceptance

$$j\frac{B}{2} = \sum_{n=1,3}^{\infty}\sum_{m=0,2,4}^{\infty} Y_{nm}\left[\frac{\iint_{hole} \overline{E}_t(r)\times \overline{h}_{nm}\cdot a_z\,dxdy}{\iint_{hole} \overline{E}_t(r)\times h_{10}\cdot a_z\,dxdy}\right]^2$$

excluding combination $n=1$, $m=0$

The substitution and mathematical work is done in Appendix II and the result is

$$\frac{B}{Y_o} = \left(\frac{8\lambda a^{5/2} c^3}{b^{1/2}\pi^2}\right)\left(\frac{\epsilon_{mn}}{\epsilon_1}\right)\left(\frac{1}{\frac{4c^3}{3} + \frac{a^3}{2\pi^3}\sin\frac{2\pi c}{a} - \frac{a^2 c}{\pi^2}\cos\frac{2\pi c}{a}}\right)^2 \cdot$$

$$\left(\sum_{n=1,3}^{\infty}\sum_{m=0,2}^{\infty}\sqrt{n^2 + (\frac{ma}{b})^2 - (\frac{2a}{\lambda})^2}\right) \cdot$$

excluding
n=1, m=0

$$\left[\frac{J_{3/2}\left(\frac{c\pi}{a}\sqrt{(n-1)^2 + (\frac{ma}{b})^2}\right)}{\left[(n-1)^2 + (\frac{ma}{b})^2\right]^{3/4}} + \frac{J_{3/2}\left(\frac{c\pi}{a}\sqrt{(n+1)^2 + (\frac{ma}{b})^2}\right)}{\left[(n+1)^2 + (\frac{ma}{b})^2\right]^{3/4}}\right]^2$$

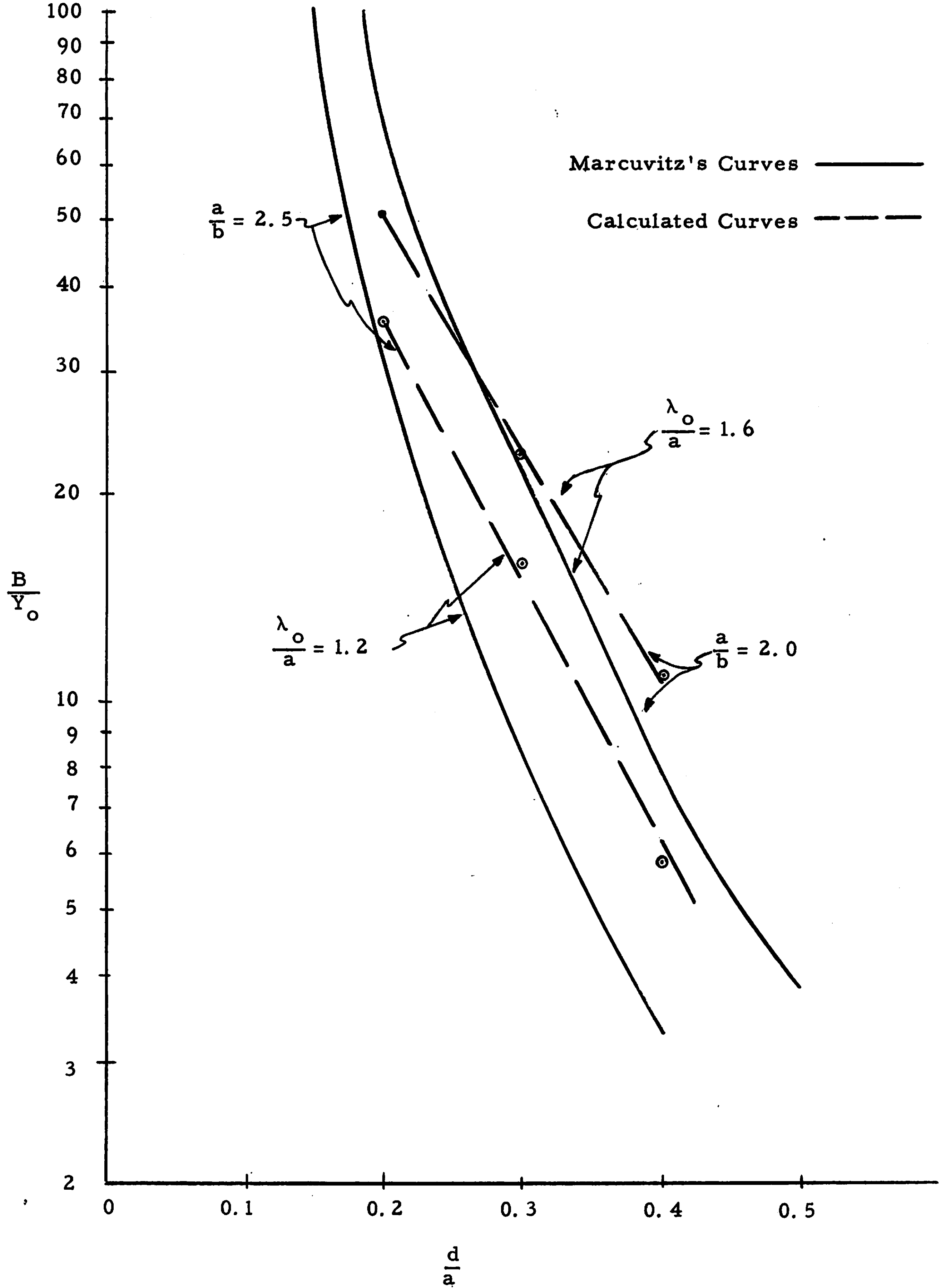
The values of $\frac{B}{Y_o}$ were calculated for 3 values of $\frac{c}{a}$ for two combinations of $\frac{a}{b}$, $\frac{\lambda}{a}$ (Fig. 3) and compared with those of Marcuvitz.* The curves by the above series calculations are very nearly linear while those of Marcuvitz tend to curve at the start and end. The large difference between the two sets of curves may be due to the fact that in our calculations we took only the first five terms of the series and left the other terms due to the lengthy calculations involved.

IV. WINDOW FORMED BY ONE OBSTACLE

We will consider here a window formed by a zero thickness metallic diaphragm uniformly placed in the x-axis direction in a rectangular wave-guide in which a TM_{10} mode is propagating.

The equivalent circuit of such an obstacle is of use in all electron tubes which employ electrostatic accelerating anodes which are at

*See Ref. 4, page 240.



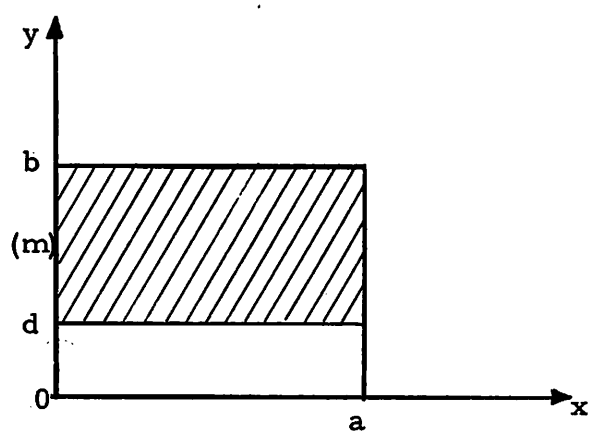
Relative Susceptance of Centered Circular Apertures

Figure 3

different potentials with respect to each other. It is of importance also in velocity modulating electron tubes. This is because of the longitudinal electric field that exists between each anode and the others.

Such an obstacle, with the TM_{10} mode impinging on it, will excite an infinite number of higher TM and TE modes. These would all be attenuated as they travel away from the obstacle except the TM_{10} and TE_{10} modes, because the dimensions of the waveguide are designed to support only these two waves. The TE_{10} mode will propagate in a waveguide designed to support TM_{10} mode.

The trial electric field distribution employed in the variational expression is that of the static field, and its correctness will be judged by the obtained results.



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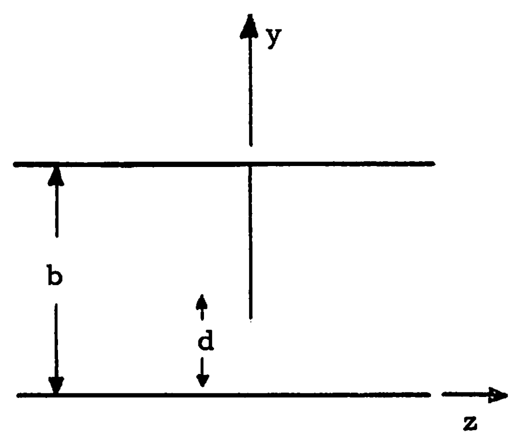


Propagating in the waveguide is a TM_{10} mode with the electric field polarized in the y-direction. In the slot the electric field distribution is

$$\vec{E} = I_1\sqrt{\frac{\epsilon_1}{a}}\ \sin\frac{\pi x}{a}\cdot\frac{1}{\sqrt{d^2 - y^2}}\ a_y$$

The variational expression for the susceptance of the equivalent circuit parameter of the iris is

$$j\frac{B}{2} = \sum_n \sum_m Y_{nm} \left[\frac{\iint_{slot} (\vec{E} \times h_{nm}) \cdot a_z \, dx\, dy}{\iint_{slot} (E \times h_{10}) \cdot a_z \, dx\, dy} \right]^2$$

where Y_{nm} is input admittance at obstacle for the mn-th mode and

$$(h_{nm})_x = \sqrt{\frac{\epsilon_{mn}}{(ab)^{1/2}}} \left[\cos\frac{m\pi y}{b} \sin\frac{n\pi x}{a} \right] a_x \quad \text{for TE and TM}$$

and

$$h_{10} = \sqrt{\frac{\epsilon_1}{a}} \sin\frac{\pi x}{a} a_x \qquad \text{for } TM_{10} \text{ only.}$$

The normalizing factor

$$\epsilon_{mn} = 4\ (n \neq 0,\ m \neq 0) \quad = \quad 2\ (m = 0 \text{ or } n = 0) \quad = \quad 1\ (\text{if } m = n = 0)$$

Hence

$$j\frac{B}{2} = \sum_n \sum_m Y_{nm} \left[\frac{\iint_{slot} I_1 \sqrt{\frac{\epsilon_1 \epsilon_{mn}}{a(ab)^{1/2}}} \cos\frac{m\pi y}{b} \sin\frac{n\pi x}{a} \sin\frac{\pi x}{a} \cdot \frac{1}{\sqrt{d^2 - y^2}} \, dx\, dy}{\iint_{slot} I_1 \frac{\epsilon_1}{a} \sin\frac{\pi x}{a} \sin\frac{\pi x}{a} \cdot \frac{1}{\sqrt{d^2 - y^2}} \, dx\, dy} \right]^2$$

$$= \sum_n \sum_m Y_{nm} \left(\frac{a}{b}\right)^{1/2} \frac{\epsilon_{mn}}{\epsilon_1} \left[\frac{\iint_{slot} \cos\frac{m\pi y}{b} \sin\frac{n\pi x}{a} \sin\frac{\pi x}{a} \cdot \frac{1}{\sqrt{d^2 - y^2}} \, dx\, dy}{\iint_{slot} \sin^2 \frac{\pi x}{a} \cdot \frac{1}{\sqrt{d^2 - y^2}} \, dx\, dy} \right]^2$$

Numerator

$$\int\int_{slot} \frac{1}{\sqrt{d^2-y^2}} \cos\frac{m\pi y}{b} \sin\frac{n\pi x}{a} \sin\frac{\pi x}{a}\, dx\, dy$$

$$= \frac{1}{2}\int_0^d \int_0^a \frac{1}{\sqrt{d^2-y^2}} \cos\frac{m\pi y}{b} \left[\cos\frac{(n-1)\pi x}{a} - \cos\frac{(n+1)\pi x}{a} \right] dy\, dx \,.$$

Upon integrating with respect to x, the above integrand will vanish except for n = 1. It equals

$$\frac{1}{2}\int_0^d \frac{1}{d^2-y^2} \cos\frac{m\pi y}{b}\, dy \left[x - \frac{\sin\frac{2\pi x}{a}}{\frac{2\pi}{a}} \right] \Bigg|_0^a$$

$$= \frac{a}{2}\int_0^d \frac{1}{\sqrt{d^2-y^2}} \cos\frac{m\pi y}{b}\, dy$$

From Morse and Feshbach, page 1323

$$J_o(z) = \frac{2}{\pi}\int_0^1 \frac{\cos(z\mu)}{\sqrt{1-u^2}}\, du$$

let $z = \frac{m\pi d}{b}$ and $u = \frac{y}{d}$ $du = \frac{1}{d}dy$.

Therefore, we get

$$J_o\left(\frac{m\pi d}{b}\right) = \frac{2}{\pi}\int_0^d \frac{\cos\left(\frac{m\pi y}{b}\right)}{\sqrt{d^2-y^2}}\, dy$$

Hence the numerator becomes

$$\frac{\pi a}{4} J_o\left(\frac{m\pi d}{b}\right)$$

Denominator

$$\int_0^d \int_0^a \frac{1}{\sqrt{d^2 - y^2}} \sin^2 \frac{\pi x}{a} \, dy \, dx = \frac{1}{2} \int_0^d \int_0^a \frac{1}{\sqrt{d^2 - y^2}} dy \left(1 - \cos \frac{2\pi x}{a}\right) dx$$

$$= \frac{a}{2} \int_0^d \frac{1}{\sqrt{d^2 - y^2}} dy = \frac{\pi a}{4}$$

Susceptance

Susceptance B exists for n = 1 only. Therefore

$$j\frac{B}{2} = \sum_{m=1}^{\infty} Y_{1m} \left(\frac{a}{b}\right)^{1/2} \frac{\epsilon_{1m}}{\epsilon_1} \left[\frac{\frac{\pi a}{4} J_o\left(\frac{m\pi d}{b}\right)}{\frac{\pi a}{4}}\right]^2$$

$$= \sum_{m=1}^{\infty} Y_{1m} \left(\frac{a}{b}\right)^{1/2} \frac{\epsilon_{1m}}{\epsilon_1} J_o^2\left(\frac{m\pi d}{b}\right)$$

Now to determine Y_{1m}

TM

$$Y_{1m} = \frac{j\omega\epsilon}{k_c^2 - k^2} = \frac{j\omega\epsilon}{\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 - \left(\frac{2\pi}{\lambda}\right)^2}} = \frac{j\omega\epsilon}{\frac{\pi}{a}\sqrt{1 + \left(\frac{ma}{b}\right)^2 - \left(\frac{2a}{\lambda}\right)^2}}$$

and for propagating TM_{10} mode

$$Y_{10} = \frac{j\omega\epsilon}{\frac{\pi}{a}\sqrt{1 - \left(\frac{2a}{\lambda}\right)^2}} = \frac{\omega\epsilon}{2\pi\sqrt{\left(\frac{1}{\lambda}\right)^2 - \left(\frac{1}{2a}\right)^2}}$$

Now for $\lambda << 2a = \lambda_c$, usually then

$$Y_{10} = \frac{\omega \epsilon \lambda}{2\pi}$$

and therefore

$$Y_{1m} = j \frac{2a \, Y_{10TM}}{\lambda} \frac{1}{\sqrt{1 + (\frac{ma}{b})^2 - (\frac{2a}{\lambda})^2}}$$

TE

The waveguide for propagating $Y_{10\,TE}$ is

$$Y_{10_{TE}} = \frac{-2\pi}{\omega \mu \lambda} \qquad \text{where } \lambda << 2a = \lambda_c$$

and for attenuated TE modes

$$Y_{1m} = -j \frac{\lambda \, Y_{10\,TE}}{2a} \sqrt{1 + (\frac{ma}{b})^2 - (\frac{2a}{\lambda})^2}$$

Hence the obstacle susceptance becomes

$$j\frac{B}{2} = \underbrace{\sum_{m=1}^{\infty} j \frac{2a \, Y_{10\,TM}}{\lambda} \frac{1}{\sqrt{1 + (\frac{ma}{b})^2 - (\frac{2a}{\lambda})^2}} \left(\frac{a}{b}\right)^{1/2} \frac{\epsilon_{1m}}{\epsilon_1} J_o^2 (\frac{m\pi d}{b})}_{TM}$$

$$+ \underbrace{\sum_{m=0}^{\infty} - j \frac{\lambda}{2a} Y_{10_{TE}} \sqrt{1 + (\frac{ma}{b})^2 - (\frac{2a}{\lambda})^2} \left(\frac{a}{b}\right)^{1/2} \frac{\epsilon_{1m}}{\epsilon_1} J_o^2 (\frac{m\pi d}{b})}_{TE}$$

Now the asymptotic expression for $J_o(z)$ is

$$J_o(z) = \frac{2}{\pi z} \cos (z - \frac{\pi}{4})$$

Hence it is clear that the series which represents contributions from TM modes will converge, but the series that represents contributions from TE model will diverge.

Hence this representation of the equivalent circuit of the obstacle is not real. The reason is that the choice of the electric field distribution we employed in the variational expression was not correct.

The problem is the to find a better approximation for the electric field distribution and calculate again the equivalent circuit for the obstacle. This would not be carried out in this project because of lack of time.

APPENDIX I

$$j\frac{B}{2} = \sum_{1}^{\infty} Y_n \left[\frac{\int_{ap} E(y)h_n(y)\,dy}{\int_{ap} E(y)h(y)\,dy} \right]^2$$

The proof that the above expression is variational consists of trying to insert the exact field distribution, and seeing what the minimum value $j\frac{B}{2}$ takes.

Proof:

$$\int_{ap} E(y)h(y)\,dy = \iint_{ap} E(y)\,G(y,y')\,E(y')\,dy\,dy'$$

where

$$h(y) = \int_{ap} G(y,y')\,E(y')\,dy'$$

and

E(y') is the exact field.

Sustituting in expression for $j\frac{B}{2}$

$$j\frac{B}{2} = \sum_{1}^{\infty} Y_n \left[\frac{\int_{ap} E(y)h_n(y)\,dy}{\iint_{ap} E(y)G(y,y')\,E(y')\,dy\,dy'} \right]^2$$

Upon taking a small variation $\delta(j\frac{B}{2})$ and setting it equal to zero, we will get a variation in the trial field E(y), but none in E(y') since it is the exact field.

$$2\cdot\delta\left(j\frac{B}{2}\right)=0=\sum_n Y_n\left[\frac{\int_{ap} E(y)\,h_n(y)\,dy}{\iint_{ap} E(y)\,E(y')\,G(y,y')\,dy\,dy'}\right]\cdot$$

$$\left[\iint E(y)\,G(y,y')\,E(y')\,dy'\,dy\cdot\int\delta E(y'')\,h_n(y'')dy''\right.$$

$$\left.-\int E(y)\,h_n(y)\,dy\cdot\iint\delta E(y'')\,G(y'',y')\,E(y')\,dy''\,dy'\right]$$

We have $y\longrightarrow y''$ in some integrals to avoid confusion of terms. Therefore

$$0=\sum_n Y_n\left\{\left[\int E(y)\,h_n(y)\,dy\cdot\int\delta E(y'')\,h_n(y'')\,dy''\right]\right.$$

$$\left.-\left[\int E(y)\,h_n(y)\,dy\right]^2\left[\frac{\iint\delta E(y'')\,G(y'',y')\,E(y')\,dy''\,dy'}{\iint E(y)\,G(y,y')\,E(y')\,dy\,dy'}\right]\right\}$$

$$0=\sum_n Y_n\left[\int E(y)\,h_n(y)\,dy\cdot\int\delta E(y'')\,h_n(y'')\,dy''\right]$$

$$-j\frac{B}{2}\left[\iint E(y)\,G(y,y')\,E(y')\,dy\,dy'\cdot\iint\delta E(y'')G(y'',y')E(y')\,dy\,dy'\right]$$

$$0=\left[\int\delta E(y'')\,dy''\right]\left\{\left[\sum_n Y_n\int E(y)\,h_n(y)\,h_n(y'')\,dy\right]\right.$$

$$\left.-j\frac{B}{2}\left[\iint E(y)\,G(y,y')\,E(y')\,dy\,dy'\cdot\int G(y'',y')\,E(y')\,dy'\right]\right\}$$

Now since $\delta E(y'')$ is arbitrary,

$$\sum_n Y_n \int E(y) h_n(y'') h_n(y)\, dy - j\frac{B}{2}\left[\iint E(y) G(y, y') E(y')\, dy\, dy' \cdot \int G(y'', y) E(y')\, dy'\right] = 0$$

Multiplying by $E(y'')\,dy''$ and integrating, and noting that

$$\int E(y'') h_n(y'')\, dy'' = \int E(y) h_n(y)\, dy$$

we get

$$\sum_n Y_n \left[\int E(y) h_n(y)\, dy\right]^2 - j\frac{B}{2}\left[\iint E(y) G(y, y') E(y')\, dy\, dy'\right]^2 = 0$$

and

$$j\frac{B}{2} = \sum_n Y_n \left[\frac{\int E(y) h_n(y)\, dy}{\iint E(y) G(y, y') E(y')\, dy\, dy'}\right]^2$$

$$= \sum_n Y_n \left[\frac{\int E(y) h_n(y)\, dy}{\int E(y) h(y)\, dy}\right]^2$$

Hence the above expression is variational because for this value of $j\frac{B}{2}$ it gives a stationary characteristic with respect to changing electric field distribution.

APPENDIX II

Considering the static field (electric) distribution in the hole to be

$$\overline{E}_t = (c^2 - x'^2 - y'^2)^{1/2}\, a_z \times \overline{H}_t$$

where

$$\overline{H}_t = I_1\sqrt{\frac{\epsilon_1}{a}}\ \underbrace{\sin\frac{\pi x}{a}}_{\cos\frac{\pi x'}{a}}\ a_x \quad \text{for } TE_{10} \text{ mode}$$

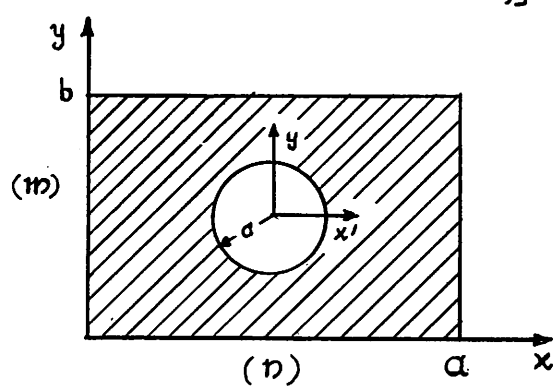
and

$$h_{nm} = \sqrt{\frac{\epsilon_{mn}}{(ab)^{1/2}}}\left[\cos\frac{m\pi y}{b}\sin\frac{n\pi x}{a}\, a_x + \sin\frac{m\pi y}{b}\cos\frac{n\pi x}{a}\, a_y\right]$$

shifting the origin of the axes

$$= \sqrt{\frac{\epsilon_{mn}}{(ab)^{1/2}}}\left[(-1)^{n+m}\cos\frac{m\pi y'}{b}\cos\frac{n\pi x'}{a}\, a_x + \sin\frac{m\pi y'}{b}\sin\frac{n\pi x'}{a}\, a_y\right]$$

$$n = \pm 1,\ \pm 3$$
$$m = 0,\ \pm 2,\ \pm 4$$



and normalizing factor

$$\epsilon_{mn} = \begin{cases} 4 & \text{if } m \neq 0,\ n \neq 0 \\ 2 & \text{if } m \text{ or } n = 0 \\ 1 & \text{if } m = 0 = n \end{cases}$$

The variational expression for the shunt susceptance of the obstacle is

$$j\frac{B}{2} = \sum_{n=1,3}\ \sum_{m=0,2,4} Y_{nm}\left[\frac{\iint_{hole} \overline{E}_t(r) \times h_{nm}\cdot a_z\, dx\, dy}{\iint_{hole} \overline{E}_t(r) \times h_{10}\cdot a_z\, dx\, dy}\right]^2$$

excluding $n=1 \quad m=0$

Now Y_n is the characteristic admittance of the waveguide for all the excited waves (modes) excluding the propagating mode. For a TE wave

$$Y_{nm} = \frac{\gamma_z}{j\omega\mu} = -\frac{\sqrt{K^2 - B^2}}{j\omega\mu} = -\frac{\sqrt{(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2 - (\frac{2\pi}{\lambda})^2}}{j\omega\mu}$$

$$= \frac{-\frac{\pi}{a}\sqrt{n^2 + (\frac{ma}{b})^2 - (\frac{2a}{\lambda})^2}}{j\omega\mu}$$

The characteristic admittance for the propagating mode (TE_{10})

$$Y_o = \frac{-\sqrt{(\frac{\pi}{a})^2 - (\frac{2\pi}{\lambda})^2}}{j\omega\mu} = \frac{-2\pi\sqrt{(\frac{1}{\lambda})^2 - (\frac{1}{2a})^2}}{\omega\mu} \cong \frac{-2\pi}{\omega\mu\lambda} \quad \text{where } \lambda \ll 2a = \lambda_c$$

Therefore

$$\frac{Y_{nm}}{Y_o} = \frac{+\lambda}{j2a}\sqrt{n^2 + (\frac{ma}{b})^2 - (\frac{2a}{\lambda})^2}$$

hence

$$Y_{nm} = \frac{+\lambda Y_o}{j2a}\sqrt{n^2 + (\frac{ma}{b})^2 - (\frac{2a}{\lambda})^2}$$

Then upon substituting back in the variational expression and re-arranging terms we get

$$\frac{B}{Y_o} = \frac{\lambda}{a}\sum_{n=1,3}\ \sum_{m=0,2,4}^{\text{excluding } n=1\ m=0} \sqrt{n^2 + (\frac{ma}{b})^2 - (\frac{2a}{\lambda})^2}\left[\frac{\iint_{hole} \bar{E}_t(r) \times h_{nm} \cdot a_z\, dx\, dy}{\iint_{hole} \bar{E}_t(r) \times h_{10} \cdot a_z\, dx\, dy}\right]^2$$

To evaluate the integral we first take the numerator

$$\iint_{hole} \left[(c^2 - x'^2 - y'^2)^{1/2} a_z \times \left(I_1 \sqrt{\frac{\epsilon_1}{a}} \cos\frac{\pi x'}{a} a_x \right)\right]$$

$$\times \left[\left(\cos\frac{m\pi y'}{b}\cos\frac{n\pi x'}{a} a_x \quad \sin\frac{m\pi y'}{b}\sin\frac{n\pi x'}{a}\right)\sqrt{\frac{\epsilon_{mn}}{(ab)^{1/2}}}\,\right]\cdot a_z\, dx'\, dy'$$

$$= I_1\sqrt{\frac{\epsilon_1 \epsilon_{mn}}{a^{3/2} b^{1/2}}} \iint (c^2 - x'^2 - y'^2)^{1/2} \cos\frac{\pi x'}{a} \cos\frac{m\pi y'}{b} \cos\frac{n\pi x'}{a}\, dx'\, dy'$$

$$= \frac{I_1}{2}\sqrt{\frac{\epsilon_1 \epsilon_{mn}}{a^{3/2} b^{1/2}}} \iint (c^2 - x'^2 - y'^2)^{1/2} \cos\frac{m\pi y'}{b} \left[\cos\frac{(n-1)\pi x'}{a} + \cos\frac{(n+1)\pi x'}{a}\right] dx'\, dy'$$

Consider for the time being the $\cos\frac{(n-1)\pi x'}{a}$ term and forget about $\cos\frac{(n+1)\pi x'}{a}$, change to cylindrical coordinates $x' = \rho\cos\theta$, $y' = \rho\sin\theta$, $dx'\,dy' = \rho\, d\rho\, d\theta$. Then we get

$$\frac{I_1}{2}\sqrt{\frac{\epsilon_1 \epsilon_{mn}}{a^{3/2} b^{1/2}}} \int_\theta \int_\rho \cos\left(\frac{m\pi}{b}\rho\sin\theta\right) \cos\left(\frac{(n-1)\pi}{a}\rho\cos\theta\right) (c^2 - \rho^2)^{1/2} \rho\, d\rho\, d\theta$$

Now

$$\cos\left(\frac{m\pi}{b}\rho\sin\theta\right) = \sum_{\ell=0,2} J_o\left(\frac{m\pi}{b}\rho\right) + 2J_2\left(\frac{m\pi}{b}\rho\right)\cos 2\theta + \ldots$$

$$\ldots + \ell J_\ell\left(\frac{m\pi}{b}\rho\right)\cos\ell\theta + \ldots$$

and

$$\cos\left(\frac{(n-1)\pi}{a}\rho\cos\theta\right) = \sum_{\ell=0,2} J_o\left(\frac{(n-1)\pi}{a}\rho\right) - 2J_2\left(\frac{(n-1)\pi}{a}\rho\right)\cos 2\theta + \ldots$$

Plugging the expansions into the integrand and integrating with respect to θ, we note that due to orthogonality of the sinusoidal f ions of the series

$$\int_0^{2\pi} \cos^2 \rho\,\theta\, d\theta = \pi$$

while the product of the first two terms equals

$$\int_0^{2\pi} J_o(\frac{m\pi}{b}\rho)\, J_o\left[\frac{(n-1)\pi}{a}\rho\right] d\theta = 2\pi\, J_o(\frac{m\pi}{b}\rho)\, J_o\left[\frac{(n-1)\pi}{a}\rho\right]$$

Hence the integral becomes (keeping in mind that we will consider the $\cos\frac{(n+1)\pi}{a}x'$ term later)

$$\frac{I_1}{2}\sqrt{\frac{\epsilon_1 \epsilon_{mn}}{a^{3/2} b^{1/2}}} \int_0^c \sum_{\ell=0,2} \left\{ \left[J_o(\frac{m\pi}{b}\rho)\, J_o\, \frac{(n-1)\pi}{a}\rho\right] 2\pi - \left[4J_2(\frac{m\pi}{b}\rho) J_2\, \frac{(n-1)\pi}{a}\rho\right] \pi + \ldots\ldots\ldots \right\} (c^2-\rho^2)^{1/2}\, \rho\, d\rho$$

From Morse and Feshbach (page 621) we find the following identity

$$J_1\left(\sqrt{x^2+y^2-2xy\cos\phi}\right) \equiv \sum_{-\infty}^{+\infty} J_\ell(x)\, J_\ell(y) \cos\ell\phi$$

Since we are interested only in even values of ℓ, i. e., $\ell = \ldots -2, 0, 2, \ldots$ we choose $\phi = \frac{\pi}{2}$, thus only even values exist and they alternate sign.

Putting $\frac{m\pi}{b}\rho = x$ and $y = \frac{(n-1)\pi}{a}\rho$

(Note that, though our sum is over $\ell = 0, 2, 4, \ldots$, we know that $J_{2K} = J_{-2K}$ (even f$\frac{\text{ion}}{}$) so that

$$\sum_{\ell=0,2,\ldots} \quad = \frac{1}{2}\sum_{\substack{-\infty \\ \text{even}}}^{\infty} \quad)$$

the integral becomes

$$\frac{2\pi I_1}{2}\sqrt{\frac{\epsilon_1 \epsilon_{mn}}{a^{3/2}b^{1/2}}}\int_{\rho=0}^{c} J_o(\sqrt{x^2+y^2})(c^2-\rho^2)^{1/2}\rho d\rho$$

$$= \pi I_1\sqrt{\frac{\epsilon_1 \epsilon_{mn}}{ab^{1/2}}}\int_0^c \left[J_o\left(\rho\sqrt{\left[\frac{(n-1)\pi}{a}\right]^2 + \left(\frac{m\pi}{b}\right)^2}\right)\right](c^2-\rho^2)^{1/2}\rho d\rho$$

Again from Morse and Feshbach (page 1325), the following relationship is found

$$\int_0^{\frac{\pi}{2}} J_m(z\sin\phi)\sin^{m+1}\phi\cos^{2n+1}\phi\, d\phi = \frac{2^n\Gamma(n+1)}{z^{n+1}} J_{m+n+1}(z)$$

Let $\rho = c\sin\phi$ and $z = c\sqrt{\left[\frac{(n-1)\pi}{a}\right]^2 + \left(\frac{m\pi}{b}\right)^2}$

Therefore

$d\rho = c\cos\phi\, d\phi$ and as ρ varies from 0 to c, ϕ goes from 0 to $\frac{\pi}{2}$.

After these substitutions our integral becomes

$$\pi I_1\sqrt{\frac{\epsilon_1 \epsilon_{mn}}{a^{3/2}b^{1/2}}}\int_0^{\frac{\pi}{2}} \left[J_o(z\sin\phi)\right](c^2-c^2\sin^2\phi)^{1/2}c\sin\phi\cdot c\cos\phi\, d\phi$$

$$= c^3\pi I_1\sqrt{\frac{\epsilon_1 \epsilon_{mn}}{a^{3/2}b^{1/2}}}\int_0^{\frac{\pi}{2}} \left[J_o(z\sin\phi)\sin\phi\right]\cos^2\phi\, d\phi$$

After applying the above mentioned identity from Morse and Feshbach we get that it equals

$$\pi c^3 I_1\sqrt{\frac{\epsilon_1 \epsilon_{mn}}{a^{3/2}b^{1/2}}}\left[\frac{2^{1/2}\Gamma(\frac{1}{2}+1)J_{3/2}\left(c\sqrt{\left[\frac{(n-1)\pi}{a}\right]^2 + \left(\frac{m\pi}{b}\right)^2}\right)}{\left(c\sqrt{\left[\frac{(n-1)\pi}{a}\right]^2 + \left(\frac{m\pi}{b}\right)^2}\right)^{3/2}}\right]$$

Using the same procedure for the cos $\frac{(n+1)\pi}{a}$ x' term we would end up with a value

$$\pi c^3 I_1 \sqrt{\frac{\epsilon_1 \epsilon_{mn}}{a^{3/2} b^{1/2}}} \left[\frac{2^{1/2} \Gamma(\frac{3}{2}) J_{3/2}\left(c\sqrt{\left[\frac{(n+1)\pi}{a}\right]^2 + (\frac{m\pi}{b})^2}\right)}{\left(c\sqrt{\left[\frac{(n+1)\pi}{a}\right]^2 + (\frac{m\pi}{b})^2}\right)^{3/2}} \right]$$

Hence the numerator becomes the sum of two expressions

$$\sqrt{\frac{\epsilon_1 \epsilon_{mn}}{a^{3/2} b^{1/2}}}\, \pi c^3 I_1 2^{1/2} \Gamma(\tfrac{3}{2}) \left[\frac{J_{3/2}\left(c\sqrt{\left[\frac{(n-1)\pi}{a}\right]^2 + (\frac{m\pi}{b})^2}\right)}{\left(c\sqrt{\left[\frac{(n-1)\pi}{a}\right]^2 + (\frac{m\pi}{b})^2}\right)^{3/2}} + \frac{J_{3/2}\left(c\sqrt{\left[\frac{(n+1)\pi}{a}\right]^2 + (\frac{m\pi}{b})^2}\right)}{\left(c\sqrt{\left[\frac{(n+1)\pi}{a}\right]^2 + (\frac{m\pi}{b})^2}\right)^{3/2}} \right]$$

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\sqrt{\pi}$$

$$= (\pi c^3)^{3/2} I_1 \sqrt{\frac{\epsilon_{mn} \epsilon_1}{2a^{3/2} b^{1/2}}} \left[\frac{J_{3/2}\left(c\sqrt{\left[\frac{(n-1)\pi}{a}\right]^2 + (\frac{m\pi}{b})^2}\right)}{\left[\left[\frac{(n-1)\pi}{a}\right]^2 + (\frac{m\pi}{b})^2\right]^{3/4}} + \frac{J_{3/2}\left(c\sqrt{\left[\frac{(n+1)\pi}{a}\right]^2 + (\frac{m\pi}{b})^2}\right)}{\left[\left[\frac{(n+1)\pi}{a}\right]^2 + (\frac{m\pi}{b})^2\right]^{3/4}} \right]$$

For the <u>denominator</u> of the integral

$$\iint_{\text{hole}} \left[\overline{E}_t(r) \times h_{10}\right] \cdot a_z \, dx\, dy = \frac{I_1 \epsilon_1}{a} \iint_{\text{hole}} (c^2 - x'^2 - y'^2)^{1/2} \cos^2 \frac{\pi x'}{a} \, dx'\, dy'$$

Integrating with respect to y'

$$\frac{I_1\epsilon_1}{a}\int_{-c}^{c}\cos^2\frac{\pi x'}{a}dx' \cdot \frac{1}{2}\left[y'\sqrt{c^2-x'^2-y'^2}+(c^2-x'^2)\sin^{-1}\frac{y'}{(c^2-x'^2)^{1/2}}\right]_{y'=-(c^2-x'^2)^{1/2}}^{y'=(c^2-x'^2)^{1/2}}$$

$$=\frac{\pi\epsilon_1 I_1}{2a}\int_{-c}^{c}(c^2-x'^2)\cos^2\frac{\pi x'}{a}dx' = \frac{\pi\epsilon_1 I_1}{4a}\int_{-c}^{c}(c^2-x'^2)(1+\cos\frac{2\pi x'}{a})dx'$$

$$=\frac{\pi\epsilon_1 I_1}{4a}\int_{-c}^{c}\left[c^2+c^2\cos\frac{2\pi x'}{a}-x'^2-x'^2\cos\frac{2\pi x'}{a}\right]dx'$$

$$=\frac{\pi\epsilon_1 I_1}{4a}\left[c^2x'-\frac{x'^3}{3}+\frac{c^2a}{2\pi}\sin\frac{2\pi x'}{a}-\frac{ax'^2}{2\pi}\sin\frac{2\pi x'}{a}-\frac{2a^2x'}{4\pi^2}\cos\frac{2\pi x}{a}\right.$$

$$\left.+\frac{2a^3}{8\pi^3}\sin\frac{2\pi x}{a}\right]_{x'=-c}^{x'=c}$$

$$=\frac{\pi\epsilon_1 I_1}{4a}\left[\frac{4c^3}{3}+\frac{a^3}{2\pi^3}\sin\frac{2\pi c}{a}-\frac{a^2c}{\pi^2}\cos\frac{2\pi c}{a}\right]$$

Hence the value of the shunt susceptance due to the obstacle is:

$$\frac{B}{Y_o}=\frac{\lambda}{a}\sum_{n=1,3}\sum_{m=0,2,4}\sqrt{n^2+(\frac{ma}{b})^2-(\frac{2a}{\lambda})^2}\left[\frac{\iint_{hole}E_t(r)\times h_{nm}\cdot a_z\,dx\,dy}{\iint_{hole}E_t(r)\times h_{10}\cdot a_z\,dx\,dy}\right]^2$$

$$\frac{B}{Y_o} = \frac{\lambda}{a} \sum_{n=1,3} \sum_{m=0,2,4} \sqrt{n^2 + \left(\frac{ma}{b}\right)^2 - \left(\frac{2a}{\lambda}\right)^2}$$

$$\frac{\pi^3 c^3 I_1^2 \epsilon_{mn} \epsilon_1 \left[\frac{J_{3/2}\left(\frac{c\pi}{a}\sqrt{(n-1)^2 + (\frac{ma}{b})^2}\right)}{\left[(n-1)^2 + (\frac{ma}{b})^2\right]^{3/4}} + \frac{J_{3/2}\left(\frac{c\pi}{a}\sqrt{(n+1)^2 + (\frac{ma}{b})^2}\right)}{\left[(n+1)^2 + (\frac{ma}{b})^2\right]^{3/4}} \right]^2}{(2a^{3/2} b^{1/2})\left(\frac{\pi^2 \epsilon_1^2 I_1^2}{16a^2}\right)\left[\frac{4c^3}{3} + \frac{a^3}{2\pi^3}\sin\frac{2\pi c}{a} - \frac{a^2 c}{\pi^2}\cos\frac{2\pi c}{a}\right]^2}$$

$$= \frac{8\pi\lambda c^3}{b^{1/2} a^{1/2}} \frac{\epsilon_{mn}}{\epsilon_1} \frac{1}{\left[\frac{4c^3}{3} + \frac{a^3}{2\pi^3}\sin\frac{2\pi c}{a} - \frac{a^2 c}{\pi^2}\cos\frac{2\pi c}{a}\right]^2} \sum_{n=1,3} \sum_{m=0,2,4} \sqrt{n^2 + \left(\frac{ma}{b}\right)^2 - \left(\frac{2a}{\lambda}\right)^2}$$

$$\left(\frac{a}{\pi}\right)^2 \left[\frac{J_{3/2}\left(\frac{c\pi}{a}\sqrt{(n-1)^2 + (\frac{ma}{b})^2}\right)}{\left[(n-1)^2 + (\frac{ma}{b})^2\right]^{3/4}} + \frac{J_{3/2}\left(\frac{c\pi}{a}\sqrt{(n+1)^2 + (\frac{ma}{b})^2}\right)}{\left[(n+1)^2 + (\frac{ma}{b})^2\right]^{3/4}} \right]^2$$

and hence

$$\frac{B}{Y_o} = \frac{8\lambda a^{5/2} c^3}{b^{1/2} \pi^2} \frac{\epsilon_{mn}}{\epsilon_1} \frac{1}{\left[\frac{4c^3}{3} + \frac{a^3}{2\pi^3}\sin\frac{2\pi c}{a} - \frac{a^2 c}{\pi^2}\cos\frac{2\pi c}{a}\right]^2} \sum_{\substack{n=1,3 \\ \text{excluding} \\ n=1,\ m=0}} \sum_{m=0,2,4} \sqrt{n^2 + \left(\frac{ma}{b}\right)^2 - \left(\frac{2a}{\lambda}\right)^2}$$

$$\left[\frac{J_{3/2}\left(\frac{c\pi}{a}\sqrt{(n-1)^2 + (\frac{ma}{b})^2}\right)}{\left[(n-1)^2 + (\frac{ma}{b})^2\right]^{3/4}} + \frac{J_{3/2}\left(\frac{c\pi}{a}\sqrt{(n+1)^2 + (\frac{ma}{b})^2}\right)}{\left[(n+1)^2 + (\frac{ma}{b})^2\right]^{3/4}} \right]^2$$

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