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THE DETERMINATION OF LYAPUNOV FUNCTIONS WHICH VERIFY AIZERMAN'S CONJECTURE

by

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I. INTRODUCTION

I.1 Statement of the Problem

An important problem in the analysis of control systems is that of stability. In the case of linear time-invariant systems, the differential equations describing the dynamic behavior of the system can be analysed very simply to determine the stability properties of the system. In particular, the Routh-Hurwitz tests can be applied to the characteristic equation associated with the system to determine if the eigenvalues of the system have negative real parts. If this is the case, the system is asymptotically stable (a. s.). Since the stability of linear systems is independent of the initial disturbance, the system is also asymptotically stable in the large (a. s. i. l.).

In the case of the nonlinear systems, the problem is not so easily solved. Methods have been developed to determine the a. s. of nonlinear systems by considering a linearized system formed by replacing a nonlinear function by its incremental slope at the equilibrium point. However, these methods do not give any information regarding the allowable initial disturbances for which the system will be a. s., which is essential in practical stability problems.

A problem then of great interest and importance in the analysis of nonlinear control systems is that of determining the region of a. s. and, in particular, that of determining if the system is a. s. i. l. This problem would, of course, be greatly simplified if a linearized model of the nonlinear system could be used to determine the stability properties of the system.

In 1949, M. A. Aizerman[1] made the following conjecture concerning the stability of the zero equilibrium point of the system of equations,

$$\underline{\dot{x}} = \underline{\underline{A}}\underline{x} + \underline{b}\, f(x_1) \tag{1}$$

where $\underline{x}$ is a column vector whose elements $(x_1, x_2, \ldots, x_n)$ are the state variables of the system, $\underline{\dot{x}}$ is the derivative of $\underline{x}$ with respect to time, $\underline{\underline{A}}$ is an nxn constant matrix, $\underline{b}$ is a column vector * and $f(x_1)$ is a continuous single-valued scalar function with $f(0)$ equal to zero.

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* Aizerman assumed that the state variables were chosen in such a way that the nonlinearity only appeared in one of the differential equations, that is, $\underline{b}$ has only one non-zero element. Since it is sometimes convenient not to choose the state variables in this manner, this assumption will not be made here.

Aizerman's Conjecture

If the nonlinearity is replaced by a linear gain kx_1 to form the linearized system,

$$\underline{\dot{x}} = \underline{A}\,\underline{x} + \underline{b}\,kx_1 = \underline{G}\,(k)\,\underline{x} \qquad (2)$$

and it is determined that this linear system is asymptotically stable for any k in a certain open interval (k_1, k_2), then the nonlinear system (1) is a. s. i. l. for any $f(x_1)$ satisfying the inequality,

$$k < \frac{f(x_1)}{x_1} < k_2$$

If this conjecture were true, a. s. i. l. could be insured for a nonlinear system simply by restricting the nonlinear function to be bounded within certain limits determined from a linear stability analysis. However, this conjecture is not true in general.* The objective of this report is to present a procedure which can be used to verify this conjecture for certain nonlinear control systems.

For some cases it is not possible to verify the conjecture above, but it is possible to verify the slightly modified version which follows:

A Modified Conjecture

If the linear system (2) is asymptotically stable for any k in a certain open interval (k_1, k_2), then the nonlinear system (1) is a. s. i. l. for any $f(x_1)$ satisfying the inequality

$$k_1 + \epsilon < \frac{f(x_1)}{x_1} < k_2$$

where ϵ is an arbitrarily small positive constant.

In this report, nonlinear control systems of the type shown in Figure 1 will be considered.

* Some known counterexamples are discussed in Appendix I.

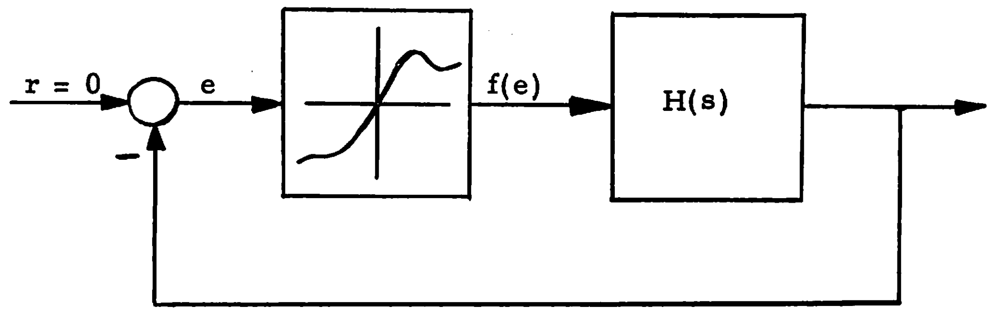

Figure 1

The system is assumed to have unity feedback, but since it is also assumed that the input r is equal to zero, more general cases can be reduced to this form. The transfer function H(s) will be assumed to have a number of poles at least one greater than the number of zeros. Furthermore, it will be assumed that H(s) has real coefficients in its numerator and denominator polynomials which is always the case if H(s) represents a transfer function between two real valued functions of time.

If the error e, which is the input to the nonlinearity, is chosen as the first state variable, the dynamic behavior of this system can be described by a system of differential equations in the form of system (1). In order to simplify the algebra of this investigation, it will be assumed that $\underline{A}$, $\underline{b}$ and $\underline{x}$ have real elements. This is always possible if the above restrictions on H(s) are satisfied.

The second method of Lyapunov provides a rigorous basis for the investigation of nonlinear system stability and can be used to verify Aizerman's conjecture when it is true if a proper Lyapunov function can be found. The problem to be considered by this investigation is that of determining Lyapunov functions of the Lur'e type which will verify Aizerman's conjecture for a particular system or class of systems.

I. 2. Review of Earlier Work

Since the publication of Aizerman's conjecture, several Russian authors have verified the conjecture for certain classes of second and third order systems using the second method of Lyapunov. Malkin[2] and Erugin[3] investigated the second order case and proved that the modified conjecture was true for second order systems like that of Figure 1 with H(s) having at least one more pole than the number of zeros.

V. A. Pliss[4] made a detailed study of a class of third order systems in which the transfer function H(s) had two zeros and three poles. He found the conjecture to be true for some combinations of the system parameters and to be false for other combinations.

E. A. Barbashin[5] verified the modified conjecture for the class of systems in which H(s) has one zero at the origin and three poles.

The above authors all used a Lyapunov function of a form first proposed by Lur'e and Postnikov[6] consisting of a quadratic form of the variables of the system plus an integral of the nonlinearity. However, their aim was to verify the conjecture for particular classes of systems and not to present methods for the determination of Lyapunov functions. Consequently, they do not describe the process by which their respective Lyapunov functions were found.

Lur'e[7] developed a method for the determination of Lyapunov functions for nonlinear systems which consisted of using a special type of quadratic form, the parameters of which must be determined by solving a set of quadratic equations. These equations become very difficult to solve for third and higher order systems. Furthermore, Lur'e was only concerned with systems which would be a. s. i. l. for any nonlinearity in the first and third quadrants.*

Rekasius and Gibson[8] have extended the methods of Lur'e to cover wider classes of systems by using certain transformations of variables which modify the nonlinear function. This allows systems to be considered which are stable only for nonlinearities in a sector of the first and third quadrants. The difficulties in solving a set of quadratic equations for the parameters of the Lyapunov function are, however, still present.

Ingwerson[9] developed a method for the generation of Lyapunov functions which he was able to use to verify Aizerman's conjecture for the class of systems in which H(s) has three poles and no zeros. Ingwerson's method is not direct, however, in that it involves intuition or considerable trial and error in selecting the form for a certain matrix involved in Ingwerson's procedure.

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* Russian authors refer to these nonlinearities as "Class A" nonlinearities.

Schultz and Gibson[10] developed a method for the generation of Lyapunov functions which they termed the variable gradient method. This method reduces the amount of trial and error necessary in generating a suitable Lyapunov function but does not eliminate it entirely. In this case, a form for the derivative of the Lyapunov function is found which must be constrained to be at least negative semi-definite. For third and fourth order systems the general constraints are usually not sufficient to completely determine the Lyapunov function. Thus, further assumptions must be made on a trial and error or intuitive basis. This makes the method very difficult to use for generating Lyapunov functions which will verify Aizerman's conjecture for third order systems and virtually impossible to use for fourth order cases due to the very broad stability statement the Lyapunov function must verify.

Recently, there has been much interest in a method for the generation of Lyapunov functions developed by Zubov[11, 12] who has shown that a Lyapunov function must satisfy a certain partial differential equation. Szego[13, 14] and Margolis and Vogt[15] have successfully applied this method to certain second order systems to determine quantitative results regarding regions of asymptotic stability and the location of limit-cycles. Margolis and Vogt have also shown that in some cases it is possible to construct an approximate series solution for the Lyapunov function which can be programmed on a digital computer. However, because of computational difficulties, the Zubov method does not readily lend itself to the determination of Lyapunov functions for a system of order greater than two.

A result equivalent to that of Ingwerson's was proven recently by Bergen and Williams[16] using a physical argument to help determine the Lyapunov function. This physical argument formed the basis for the development of the procedure presented in this report.

I. 3. Summary of Results

A direct procedure for the determination of Lyapunov functions of the Lur'e type which verify Aizerman's conjecture for a particular nonlinear system is developed. The problem of determining the Lyapunov function for the nonlinear system is reduced to that of determining a common Lyapunov function for a family of linear systems. It is shown that the derivative of this function must satisfy certain constraints in order to be a suitable Lyapunov function for this family of linear systems. These constraints require the derivative of the Lyapunov function to be zero along specified vectors determined from the dynamics of the linearized system. The derivative of the Lyapunov function is constructed incorporating these constraints which considerably reduces the number of parameters to be determined. The elements of the Lyapunov function itself are then found by solving a set of linear algebraic equations.

This procedure is a direct process by which the Lyapunov function can be found if it exists without resort to trial and error methods and consequently is well suited for application to third and higher order systems. Several third and fourth order examples are presented to demonstrate this procedure.

II. THE SECOND METHOD OF LYAPUNOV

II. 1. Theorems on Stability

The second method of Lyapunov constitutes a set of theorems on the stability of differential systems of the type

$$\underline{\dot{x}} = \underline{F}(\underline{x}) \tag{3}$$

with $\underline{F}(\underline{x}) = \underline{0}$ only if $\underline{x} = \underline{0}$. These theorems allow the stability of the system to be investigated by considering a scalar function of the variables of the system, $V(\underline{x})$. The properties of this function and its derivative formed in the following manner,

$$\dot{V}(\underline{x}) = \underline{\Delta V}'\,\underline{\dot{x}} = \underline{\Delta V}'\,\underline{F}(\underline{x})$$

where $\underline{\Delta V}'$ represents the transpose of the vector $\underline{\Delta V}$, can be examined without any knowledge of the solution to the differential equations of the system.

Two theorems which will be of use in this investigation are stated below.

Theorem 1

If $V(\underline{x})$ is a scalar function with continuous first partial derivatives for all $\underline{x}$ such that

(1) $V(\underline{x}) > 0$ for all $\underline{x} \neq \underline{0}$ and $V(0) = 0$

(2) $\dot{V}(\underline{x}) < 0$ for all $\underline{x} \neq \underline{0}$ and $\dot{V}(0) = 0$

and (3) $V(\underline{x}) \to \infty$ as $\|\underline{x}\| \to \infty$

where $\|\underline{x}\|$ is any norm of the vector $\underline{x}$,

then system (3) is a. s. i. l.

Theorem 2

If condition (2) of Theorem 1 is replaced by

$$\dot{V}(\underline{x}) \leq 0$$

and the origin is the largest invariant set* for which $\dot{V}(\underline{x}) = 0$, the theorem is still valid.

For detailed proofs of these theorems, the reader is referred to the book by LaSalle and Lefschetz[17]. Intuitively, the Lyapunov function $V(\underline{x})$ may be thought of as a distance or potential function which is always positive except at the origin where it is zero. If this function is always decreasing by virtue of the system of differential equations (3), then it seems apparent that the system will be a. s. i. l. . In the case of Theorem 2, $\dot{V}(\underline{x})$ may be zero at certain points in the state space. But if the system point cannot remain within a set of points for which $\dot{V}(\underline{x}) = 0$, the function $V(\underline{x})$ must then, necessarily, decrease to the zero value at the origin.

II. 2. Lyapunov Functions

The simplest positive definite function which can be used as a Lyapunov function is a quadratic form

$$V(\underline{x}) = \underline{x}' \, \underline{P} \, \underline{x}$$

where $\underline{P}$ is a real symmetric positive definite matrix. For linear systems which are a. s. i. l., a function of this type may always be found which will satisfy the conditions of Theorem 1.

For nonlinear systems, this type of function is useful to some extent. Aizerman[18] pointed out that a function of this type could be used to determine a sub-interval of the interval (k_1, k_2) such that if the nonlinear function $\frac{f(x_1)}{x_1}$ was bounded within this sub-interval, the system would be a. s. i. l.

Lur'e found that if a term incorporating an integral of the nonlinearity were added to the quadratic form in the following manner $V(\underline{x}) = \underline{x}' \, \underline{Px} + \gamma \int_0^{x_1} f(u)du$

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* A set of points in the state space is said to be invariant if the system point $\underline{x}(t)$ remains in the set for all time if the initial point $\underline{x}(t_o)$ is in the set.

where γ is a scalar constant, a more useful function was obtained for non-linear systems. This term allows the Lyapunov function to compensate somewhat for changes in the characteristics of the nonlinearity. Unfortunately, this compensation only affects the x_1^2 term of the quadratic form as can be seen by replacing $f(x_1)$ by kx_1 to obtain

$$V_L(\underline{x}) = \underline{x}' \, \underline{P} \, \underline{x} + \frac{\gamma}{2} k x_1^2$$

However, attempts at using the nonlinearity in other forms in the Lyapunov function such as $x_2 f(x_1)$ or $\frac{f(x_1)}{x_1} x_2^2$ lead to taking a derivative of the nonlinear function when $\dot{V}(\underline{x})$ is calculated. This, then, usually leads to making restraints upon the derivative of $f(x_1)$ as well as upon $f(x_1)$ itself in order to insure stability. The term used by Lur'e, however, possesses the advantage of being differentiable without taking the derivative of $f(x_1)$.

As pointed out earlier, previous authors have unanimously used a function of this type to verify the conjecture even though they may have derived the function by a method of generating Lyapunov functions of a more general nature. Pliss was able to prove that the conjecture was not true for all the third order systems of the type he investigated for which a function of the Lur'e type did not exist which would verify the conjecture. This then is a good indication of the usefulness of the Lur'e type of Lyapunov function in verifying Aizerman's conjecture.

III. REDUCTION TO A LINEAR PROBLEM

III. 1. Introductory Remarks

The problem under consideration is that of determining a Lyapunov function

$$V(\underline{x}) = \underline{x}' \, \underline{P} \, \underline{x} + \gamma \int_0^{x_1} f(u) \, du$$

for a nonlinear system (1) such that it satisfies all the conditions of Theorem 1 or 2 for any $f(x_1)$ satisfying

$$k_1 < \frac{f(x_1)}{x_1} < k_2$$

where k_1 and k_2 are the limits of stability for the linearized system (2).

If $f(x_1)$ is replaced by kx_1, the following linearized Lyapunov function is obtained.

$$V_L(\underline{x}) = \underline{x}'\,\underline{P}\,\underline{x} + \frac{\gamma}{2}\,k x_1^2$$

By letting

$$x_1 = \underline{a}'\underline{x} \quad \text{where} \quad \underline{a} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$V_L(\underline{x}) = \underline{x}'\underline{P}\,\underline{x} + \frac{\gamma}{2}\,k\,\underline{x}'\,\underline{a}\,\underline{a}'\underline{x}$$

or $$V_L(\underline{x}) = \underline{x}'\underline{P}\,(k)\underline{x} \quad \text{where} \quad \underline{P}\,(k) = \underline{P} + \frac{\gamma k}{2}\,\underline{a}\,\underline{a}'$$

Computing the derivative of $V(\underline{x})$ using system (1) and the derivative of $V_L(\underline{x})$ using system (2), it is found that

$$\dot{V}(\underline{x}) = \underline{x}'\left[\underline{P}\,\underline{A} + \underline{A}'\underline{P}\right]\underline{x} + \frac{f(x_1)}{x_1}\,\underline{x}'\left[\underline{P}\,\underline{b}\,\underline{a}' + \underline{a}\,\underline{b}'\,\underline{P} + \frac{\gamma}{2}\left(\underline{a}\,\underline{a}'\,\underline{A} + A'\,\underline{aa}'\right)\right]\underline{x}$$
$$+ \frac{\gamma}{2}\left[\frac{f(x_1)}{x_1}\right]^2 \underline{x}'\left[\underline{a}\,\underline{a}'\,\underline{b}\,\underline{a}' + \underline{a}\,\underline{b}'\,\underline{a}\,\underline{a}'\right]\underline{x}$$

and $$\dot{V}_L(\underline{x}) = \underline{x}'\left[\underline{PA} + \underline{A}'\underline{P}\right]\underline{x} + k\,\underline{x}'\left[\underline{Pba}' + \underline{a}\,\underline{b}'\underline{P} + \frac{\gamma}{2}\left(\underline{a}\,\underline{a}'\underline{A} + \underline{A}'\underline{a}\,\underline{a}'\right)\right]\underline{x}$$
$$+ \frac{\gamma}{2}\,k^2\,\underline{x}'\left[\underline{a}\,\underline{a}'\underline{b}\,\underline{a}' + \underline{ab}'\,\underline{a}\,\underline{a}'\right]\underline{x}$$

or $\dot{V}_L(\underline{x}) = -\underline{x}'\,\underline{Q}\,(k)\underline{x}$ which defines $\underline{Q}(k)$. Note that the linearized function is the same function of k as the original function is of $\frac{f(x_1)}{x_1}$.

That is,

$$\dot{V}_L(\underline{x}, k) = \dot{V}\left[\underline{x}, \frac{f(x_1)}{x_1}\right].$$

This fact will enable the problem of determining the function $V(\underline{x})$ for the nonlinear system (1) to be reduced to that of determining the function $V_L(\underline{x})$ for the linearized system (2). Pliss originated this result in his study of third order systems.

The function $V_L(\underline{x})$ is positive definite for a certain value of k if $\underline{P}(k)$ is a positive definite matrix. Similarly $\dot{V}_L(\underline{x})$ is negative semi-definite for a certain value of k if $\underline{Q}(k)$ is positive semi-definite. The tests for positive definiteness or positive semi-definiteness of a matrix are presented in Appendix II.

III. 2. Theorems on Equivalence

The following theorem will now be proven.

Theorem 3

If a Lyapunov function $V_L(\underline{x})$ can be found for the linearized system (2) such that for all $k_1 \leq k \leq k_2$,

$$V_L > 0 \text{ for all } \underline{x} \neq 0$$

$$\dot{V}_L \leq 0 \text{ for all } \underline{x}$$

and $\underline{x} = \underline{0}$ is the only invariant set by virtue of system (1) for which $\dot{V}(\underline{x})$ equals zero if $k_1 < \frac{f(x_1)}{x_1} < k_2$, then the nonlinear system (1) is a. s. i. l. for all $f(x_1)$ satisfying

$$k_1 < \frac{f(x_1)}{x_1} < k_2.$$

Proof

Assume that $\gamma > 0$ which can always be accomplished by reversing the sign of $f(x_1)$ if necessary and that $k_1 < \frac{f(x_1)}{x_1} < k_2$. Then

$$\gamma \int_0 f(u)\, du > \gamma \frac{k_1}{2} x_1^2 \text{ for all } x_1$$

and $\quad V(\underline{x}) > V_L(\underline{x}, k_1) > 0$ for all x_1

Furthermore, since $V_L(\underline{x}, k_1)$ is a positive definite quadratic form

$$V_L(\underline{x}, k_1) \rightarrow \infty \text{ as } \|\underline{x}\| \rightarrow \infty$$

Consequently,

$$V(\underline{x}) \rightarrow \infty \text{ as } \|\underline{x}\| \rightarrow \infty$$

Since $\dot{V}_L(\underline{x}, k)$ is equivalent to $\dot{V}\left[\underline{x}, \frac{f(x_1)}{x_1}\right]$,

$$\dot{V}(\underline{x}) \leq 0 \text{ if } k_1 \leq \frac{f(x_1)}{x_1} \leq k_2.$$

Then if $\underline{x} = \underline{0}$ is the only invariant set for which $\dot{V}(\underline{x}) = 0$, the conditions of Theorem 2 are satisfied and the nonlinear system (1) is a. s. i. l.

This theorem can be used to verify Aizerman's conjecture if the linearized Lyapunov function $V_L(\underline{x})$ can be found. The following theorem will be of use in verifying the modified conjecture.

Theorem 4

If a Lyapunov function $V_L(\underline{x})$ can be found for the linearized system (2) such that for

$$k_1 < k \leq k_2,$$

$$V_L > 0 \text{ for all } \underline{x} \neq 0$$

$$\dot{V}_L \leq 0 \text{ for all } \underline{x}$$

and $\underline{x} = \underline{0}$ is the only invariant set by virtue of system (1) for which $\dot{V}(\underline{x})$ equals zero if $k_1 < \frac{f(x_1)}{x_1} < k_2$, then the nonlinear system (1) is a. s. i. l. for all $f(x_1)$ satisfying

$$k_1 + \epsilon < \frac{f(x_1)}{x_1} < k_2$$

Proof

Again assume that $\gamma > 0$ and $k_1 + \epsilon < \frac{f(x_1)}{x_1} < k_2$.

Then

$$V_L(\underline{x}) > 0 \text{ for } k > k_1,$$

$$V_L(\underline{x}, k_1 + \epsilon) > 0$$

and $$V(\underline{x}, \frac{f(x_1)}{x_1}) > V_L(\underline{x}, k_1 + \epsilon) > 0 .$$

The completion of the proof follows as for Theorem 3.

This theorem allows the cases to be treated for which $V_L(\underline{x}, k_1)$ is not positive definite but rather positive semi-definite.

IV. RESTRICTIONS ON Q(k)

IV. 1. Introductory Remarks

Consider the matrix equation relating $\underline{Q}(k)$ to $\underline{P}(k)$ and $\underline{G}(k)$:

$$\underline{P}(k)\,\underline{G}(k) + \underline{G}'(k)\,\underline{P}(k) = -\,\underline{Q}(k) \tag{4}$$

The objective now is to find a solution to this equation such that $\underline{P}(k)$ is positive definite and $\underline{Q}(k)$ is positive semi-definite for all k in the open interval (k_1, k_2). In order to accomplish this result, the general form of this equation will be investigated. Equations of this form have been investigated by Bellman[19] using Kronecker products* to write the equations

* See Appendix III for a definition of Kronecker products, their properties and a demonstration of this procedure.

for the coefficients of the matrices. For the general matrix equation

$$\underline{Y}\,\underline{C} + \underline{C}'\,\underline{Y} = -\,\underline{D}, \tag{5}$$

the equations for the coefficients are given by

$$\left[\underline{I} \otimes \underline{C}' + \underline{C}' \otimes \underline{I}\right]\underline{y} = -\,\underline{d} \tag{6}$$

where $\otimes$ designates the Kronecker product and

$$\underline{y} = \begin{bmatrix} Y_{11} \\ \vdots \\ Y_{1n} \\ Y_{21} \\ \vdots \\ Y_{2n} \\ \vdots \\ Y_{nn} \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{first row of } \underline{Y} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{second row of } \underline{Y} \\ \dots \text{ etc.} \\ \\ \end{array} \qquad \underline{d} = \begin{bmatrix} D_{11} \\ \vdots \\ D_{1n} \\ D_{21} \\ \vdots \\ D_{2n} \\ \vdots \\ D_{nn} \end{bmatrix}$$

Let $\underline{K}$ equal the $n^2 x n^2$ matrix defined by the sum of Kronecker products,

$$\underline{K} = \underline{I} \otimes \underline{C}' + \underline{C}' \otimes \underline{I}\ .$$

Bellman has shown that the matrix equation (5) has a solution for $\underline{Y}$ for an arbitrary matrix $\underline{D}$ if all the eigenvalues of $\underline{C}$ have negative real parts. This was accomplished by showing that under this condition, the matrix $\underline{K}$ is non-singular and consequently,

$$\underline{y} = \underline{K}^{-1}\,\underline{d}\ .$$

This does not place any restriction upon the matrix $\underline{D}$ in order for a solution to exist for $\underline{Y}$. For the matrix equation (4), it is known that the eigenvalues of $\underline{G}(k)$ have negative real parts for all k in the open interval (k_1, k_2). However, $\underline{G}(k_1)$ and $\underline{G}(k_2)$* must have eigenvalues with zero real parts if (k_1, k_2) is the maximum open interval for which $\underline{G}(k)$ has eigenvalues with negative real parts. Now if the matrix $\underline{C}$ of the general equation (5) has eigenvalues with zero real parts, then $\underline{K}$ will have eigenvalues equal to zero. If $\underline{K}$ is singular, it is known[20] that in order for a solution to exist, the vector $\underline{d}$ must be orthogonal to all vectors $\underline{z}$ which are eigenvectors for the eigenvalues of zero of the matrix $\underline{K}$. That is,

$$\underline{z}'\,\underline{d} = 0$$

for all $\underline{z}$ such that

$$\underline{z}'\,\underline{K} = 0\,.$$

This fact can be used to place restrictions upon the matrix $\underline{D}$. In the next section, a theorem will be proven which relates these restrictions

* In some cases k_2 may be infinite in which $\underline{G}(k)$ may have eigenvalues with negative real parts for arbitrarily large k.

to certain eigenvectors of $\underline{C}$. It will be assumed that $\underline{D}$ is positive semi-definite since this is a necessary restriction on $\underline{Q}(k)$ for the purposes of this report.

IV. 2. Necessary Conditions

In order to develop the restrictions on $\underline{Q}(k_1)$ or $\underline{Q}(k_2)$ to satisfy the necessary conditions for a solution to exist for $\underline{P}(K_1)$ or $\underline{P}(k_2)$, the following two lemmas will be proven.

LEMMA 1

If $\underline{u}_i$ and $\underline{u}_j$ are eigenvectors of the matrix $\underline{C}$ such that

$$\underline{C}\,\underline{u}_i = \lambda_i\,\underline{u}_i$$

$$\underline{C}\,\underline{u}_j = \lambda_j\,\underline{u}_j$$

then $\underline{u}_i \otimes \underline{u}_j$ is the left eigenvector of K corresponding to the eigenvalue $\lambda_i + \lambda_j$.

Proof

$$\begin{aligned}(\underline{u}_i \otimes \underline{u}_j)'\,\underline{K} &= (\underline{u}_i' \otimes \underline{u}_j')\,(\underline{I} \otimes \underline{C}' + \underline{C}' \otimes \underline{I}) \\ &= \underline{u}_i' \otimes \underline{u}_j'\,\underline{C}' + \underline{u}_i'\,\underline{C}' \otimes \underline{u}_j' \\ &= \lambda_j\,\underline{u}_i' \otimes \underline{u}_j' + \lambda_i\,\underline{u}_i' \otimes \underline{u}_j' \\ &= (\lambda_i + \lambda_j)\,\underline{u}_i' \otimes \underline{u}_j'\end{aligned}$$

which proves the lemma.

LEMMA 2

$$(\underline{u}_i \otimes \underline{u}_j)'\,\underline{d} = \underline{u}_i'\,\underline{D}\,\underline{u}_j$$

Proof

Let $\underline{u}_i = \begin{bmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{in} \end{bmatrix}$

then $\quad (\underline{u}_i \otimes \underline{u}_j)' = (u_{i1}\,\underline{u}_j',\ u_{i2}\,\underline{u}_j', \ldots,\ u_{in}\,\underline{u}_j')$

and $(\underline{u}_i \otimes \underline{u}_j)' = u_{i1}\,\underline{u}_j' \begin{bmatrix} D_{11} \\ \vdots \\ D_{1n} \end{bmatrix} + u_{i2}\,\underline{u}_j' \begin{bmatrix} D_{21} \\ \vdots \\ D_{2n} \end{bmatrix} + \ldots + u_{in}\,\underline{u}_j' \begin{bmatrix} D_{n1} \\ \vdots \\ D_{nn} \end{bmatrix}$

$$= \underline{u}_i\,\underline{D}\,\underline{u}_j$$

The following theorem will now be proven.

Theorem 5

A necessary condition for equation (5) to have a solution is that

$$\underline{D}\,\underline{u} = \underline{0}$$

for every vector $\underline{u}$ such that

$$\underline{C}\,\underline{u} = \underline{0} \text{ or } \underline{C}\,\underline{u} = j\omega\underline{u} .$$

Proof

If there is a $\underline{u}_i$ such that

$$\underline{C}\,\underline{u}_i = 0,$$

then by Lemma 1,

$$(\underline{u}_i \otimes \underline{u}_i)'\,\underline{K} = 0.$$

Thus in order for a solution to exist

$$(\underline{u}_i \otimes \underline{u}_i)'\,\underline{d} = 0,$$

or using Lemma 2

$$\underline{u}_i'\,\underline{D}\,\underline{u}_i = 0.$$

Since $\underline{D}$ is assumed to be positive semi-definite, the minimum value of $\underline{u}_i'\,\underline{D}\,\underline{u}_i$ is zero and can only be assumed along an eigenvector of $\underline{D}$. Thus $\underline{D}\,\underline{u}_i = 0$.

Now, if there is a $\underline{u}_i$ such that

$$\underline{C}\,\underline{u}_i = j\omega\underline{u}_i ,$$

then, since $\underline{C}$ is real, there is also a $\underline{u}_j$ (equal to the complex conjugate of $\underline{u}_i$) such that

$$\underline{C}\,\underline{u}_j = -j\omega\underline{u}_j.$$

Then $(\underline{u}_i \otimes \underline{u}_j)'\,\underline{K} = 0$

and necessarily $(\underline{u}_i \otimes \underline{u}_j)'\,\underline{d} = 0$

or $\underline{u}_i'\,\underline{D}\,\underline{u}_j = 0$.

Let $\underline{u}_i$ be split into its real and imaginary components,

$$\underline{u}_i = \underline{w}_1 + j\underline{w}_2$$

then $\quad \underline{u}_j = \underline{w}_1 - j\underline{w}_2$

and $\quad \underline{u}_i'\,\underline{D}\,\underline{u}_j = \underline{w}_1'\,\underline{D}\,\underline{w}_1 + \underline{w}_2'\,\underline{D}\,\underline{w}_2 = 0 .$

Again since $\underline{D}$ is assumed to be positive semi-definite,

$$\underline{D}\,\underline{w}_1 = \underline{0}$$

and $\quad \underline{D}\,\underline{w}_2 = \underline{0}$

or equivalently $\underline{D}\,\underline{u}_i = \underline{0}$

which completes the proof of the theorem.

Therefore, for k equal to k_1 or k equal to k_2 (if k_2 is finite), $\underline{Q}(k_1)$ or $\underline{Q}(k_2)$ must satisfy the necessary conditions of Theorem 5 in order to guarantee that equation (4) will have a solution. As will be shown later, this fact can help to determine $\underline{Q}(k)$ and subsequently $\underline{P}(k)$.

IV. 3. Treatment of Double-Zero Cases

Systems may be encountered for which $\underline{G}(k_1)$ has two eigenvalues equal to zero, but only one eigenvector. It will be shown in this section that the constraints of Theorem 5 must also be applied to $\underline{Q}(k_1)$ with regard to the generalized eigenvector for the zero eigenvalue.

Assume that the matrix $\underline{C}$ of equation (5) has two eigenvalues equal to zero and that $\underline{C}$ is in the Jordan canonical form (which can always be accomplished by a suitable transformation if necessary). Then

$$\underline{C} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & & \\ & & & \underline{J} & \\ 0 & 0 & & & \end{bmatrix}$$

where $\underline{J}$ is an n-2 x n-2 Jordan matrix. Computing the general form of $\underline{D}$ it is found that

$$-\underline{D} = \underline{Y}\,\underline{C} + \underline{C}'\,\underline{Y} = \begin{bmatrix} 0 & Y_{11} & \dots \\ Y_{11} & 2Y_{12} & \dots \\ \vdots & \vdots & \end{bmatrix}$$

Thus, in order for $\underline{D}$ to be positive semi-definite, it is necessary that Y_{11} be zero. This means that $\underline{Y}$ cannot be positive definite and in order for it to be positive semi-definite, it is necessary that the first row and column of $\underline{Y}$ be zero.* Then Y_{12} equals zero and the second row and column of $\underline{D}$ must be zero which means that the generalized eigenvector of $\underline{C}$ for the zero eigenvalue,

$$\underline{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

---

* It has been assumed that $\underline{Y}$ is symmetric.

must be an eigenvector for $\underline{D}$ for the zero eigenvalue. That is,

$$\underline{D}\,\underline{u}_2 = \underline{0}.$$

The following theorem has, therefore, been proven.

Theorem 6

If $\underline{C}$ has two eigenvalues equal to zero with one eigenvector $\underline{u}_1$ and one generalized eigenvector $\underline{u}_2$, then in addition to the condition of Theorem 5, it is also necessary that

$$\underline{D}\,\underline{u}_2 = 0$$

in order for a solution to exist to equation (5) such that $\underline{Y}$ and $\underline{D}$ are both positive semi-definite.

Note that because of the fact that in this case, $\underline{Y}$ can only be positive semi-definite, it will be impossible to satisfy the conditions of Theorem 3 for systems of this type. It will be possible, however, in many cases to satisfy the conditions of Theorem 4 and verify the modified conjecture.

V. THE CONSTRUCTION OF $\underline{Q}$ (k)

V.1. Construction of $\underline{Q}(k_1)$ and $\underline{Q}(k_2)$

It has been shown that in order for a Lyapunov function to exist which will verify Aizerman's conjecture or the modified conjecture for a particular system, it is necessary that $\underline{Q}(k_1)$ and $\underline{Q}(k_2)$ satisfy the conditions of Theorem 5 and in some cases, Theorem 6. In order to use these conditions to help specify $\underline{Q}(k)$, a technique for constructing a matrix to have certain specified eigenvectors will be described.

Suppose it is desired to construct a matrix $\underline{D}$ to have a set of k eigenvectors $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k$ for the eigenvalue zero. Let

$$\underline{T} = \left[\begin{array}{c:c:c:c:c:c:c} \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_k & \underline{h}_j & \cdots & \underline{h}_{n-k} \end{array}\right]$$

where the $\underline{h}_i$ are a set of vectors linearly independent of the $\underline{u}_j$. In most cases that will be encountered, the $\underline{h}_i$ can be chosen as simple one-element vectors of the form

$$\underline{h}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then let

$$\underline{D} = (\underline{T}^{-1})' \begin{bmatrix} 0 & \cdots & & & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & & 0 & \cdots & 0 \\ \vdots & & \vdots & \underline{B} & \\ 0 & & 0 & & \end{bmatrix} \underline{T}^{-1}$$

where $\underline{B}$ is an arbitrary positive semi-definite n-k x n-k matrix.
Now, since

$$\underline{T}^{-1} \underline{u}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad i^{th} \text{ position}$$

$\underline{D} \, \underline{u}_i = 0$.

Using this procedure, $\underline{Q}(k_1)$ and $\underline{Q}(k_2)$ can be constructed to be as general as possible and still satisfy the necessary condition of Theorem 5. $\underline{Q}(k)$ has not been completely specified, however, because of the arbitrary matrix $\underline{B}$. In the next section, further restrictions will be developed which will help to determine the elements of $\underline{B}$.

V. 2. Further Restrictions

Consider the general form of $\underline{Q}(k)$ given by the relation

$$- \underline{Q}(k) = \underline{P}\,\underline{A} + \underline{A}'\underline{P} + k \left[\underline{P}\,\underline{b}\,\underline{a}' + \underline{a}\,\underline{b}'\,\underline{P} + \frac{\gamma}{2} (\underline{aa}'\,\underline{A} + \underline{A}'\,\underline{aa}')\right] + \frac{\gamma}{2} k^2 \left[\underline{aa}'\,\underline{ba}' + \underline{ab}'\,\underline{aa}'\right]$$

derived in section III. 1. Notice that the terms involving k have elements only in the first row and column due to the presence of the vector $\underline{a}$. This means that $\underline{Q}(k)$ can only vary in the first row and column as the parameter k is varied.

Now suppose that both k_1 and k_2 are finite and $\underline{Q}(k_1)$ and $\underline{Q}(k_2)$ are constructed as described in the previous section. Then all the elements not in the first row and column of $\underline{Q}(k_1)$ and $\underline{Q}(k_2)$ must be equated. This will reduce the arbitrariness of $\underline{Q}(k)$ considerably.

Now, if there is no upper limit k_2, then $\underline{Q}(k)$ must remain positive semi-definite for arbitrarily large values of k. This condition can also be used to help specify $\underline{Q}(k)$ by placing restrictions upon the terms of $\underline{Q}(k)$ involving k. However, this restriction will not be used until the actual determination of the Lyapunov function is attempted which will be described in the next section.

VI. DETERMINATION OF THE LYAPUNOV FUNCTION

VI. 1. Solving For The P_{ij}

The general form of $\underline{Q}(k)$ can be found by letting $\underline{P}$ have undetermined elements P_{ij} and computing $\underline{Q}(k)$ using the differential equations for the particular system under consideration. Equating this $\underline{Q}(k)$ for k equal to k_1 to the $\underline{Q}k_1$ constructed by the method of the previous chapter, a set of equations for the P_{ij} and γ are obtained. Also, another set of equations can be obtained for $\underline{Q}(k_2)$ if there is a finite k_2. Otherwise, additional constraints can be placed upon the general form of $\underline{Q}(k)$ to insure that $\underline{Q}(k)$ remains positive semi-definite for arbitrarily large values of k.

The problem now is to solve these linear equations to determine the P_{ij} and γ. Because of the structure of the equations, they are most easily solved by the method of substitution and elimination of variables. In many cases, $\underline{Q}(k_1)$ and $\underline{Q}(k_2)$ will not be completely specified by the previous constraints placed upon it and there will be undetermined coefficients in the equations for the P_{ij} resulting from this. In some cases, these undetermined coefficients will be specified in the process of solving for the P_{ij} and γ. In other cases, there may be coefficients of this type which will remain completely arbitrary as long as they are selected to maintain the positive semi-definiteness of $\underline{Q}(k_1)$ or $\underline{Q}(k_2)$.

These equations for the P_{ij} do not always have a solution, either because of the fact that Aizerman's conjecture is not true for the particular system under consideration or because of the fact that a Lyapunov function of this type does not exist which will verify the conjecture. In these cases, it will be found that when the equations are solved by substitution and elimination it will be impossible to satisfy all the equations and maintain $\underline{Q}(k)$ positive semi-definite.

VI. 2. Application of Theorem 3 or 4

If a solution can be found to the equations for the P_{ij}, it is necessary to check the Lyapunov function thus determined to see if it will satisfy the conditions of Theorem 3 or Theorem 4 in order to verify Aizerman's conjecture or the modified conjecture. Due to the manner in which $\underline{Q}(k)$ was constructed, it will be at least positive semi-definite for the values of k in the interval (k_1, k_2). $\underline{P}(k)$ can be checked for positive definiteness using the tests of Appendix I. In some special cases, $\underline{P}(k_1)$ will only be positive semi-definite which will require the use of Theorem 4 to prove the a. s. i. l. of the nonlinear system.

In most cases, $\underline{Q}(k)$ will be positive semi-definite and there will be a vector, say $\underline{v}$, in the state space for which

$$\dot{V}(\underline{v}) = -\underline{v}'\,\underline{Q}(k)\underline{v} = 0.$$

It is then necessary to verify that the system point of the nonlinear system cannot remain in this vector. This is accomplished by computing $\dot{\underline{v}}$ using the differential equations of the system. Then, if $\dot{\underline{v}}$ is not co-linear with the vector $\underline{v}$ for $k_1 < \frac{f(x_1)}{x_1} < k_2$, the condition that $\underline{x} = \underline{0}$ is the only invariant set for which $\dot{V}(\underline{x})$ is zero is satisfied.

In some cases, $\underline{Q}(k)$ may have two independent vectors for which $\dot{V}_L(\underline{x})$ will be zero (as in examples 3 and 6). It is then necessary to verify that the system point cannot remain in the plane defined by these vectors.

Once it has been shown that all the conditions of Theorem 3 or 4 are satisfied by the Lyapunov function which has been determined, Aizerman's conjecture or the modified conjecture has been verified for the system under consideration. This procedure will be demonstrated by several third and fourth order examples in the following section.

VII. EXAMPLES

VII. 1. Example 1

Consider the third order system shown in Figure 2. The differential equations for a set of state variables describing this system are the following:

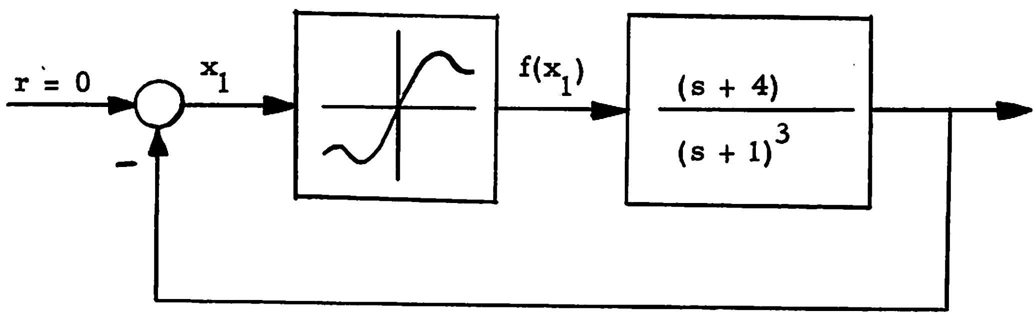

Figure 2

$$\dot{\underline{x}} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 3 \\ -\frac{f(x_1)}{x_1} & 0 & -1 \end{bmatrix} \underline{x}$$

The matrix $\underline{G}(k)$ for the linearized system is then determined by replacing $f(x_1)$ by kx_1. The linearized system is asymptotically stable for

$$-\frac{1}{4} < k < 8.$$

For $k = \frac{1}{4}$, there is one $\lambda = 0$; and for $k = 8$, there are imaginary eigenvalues at $\pm j\sqrt{11}$.

It is easily determined that

$$\underline{G}(k_2)\ \underline{u} = -j\sqrt{11}\ \underline{u}$$

for $\underline{u} = \begin{bmatrix} \frac{5}{4} - \frac{1}{4}\ j\sqrt{11} \\ 3 \\ 1 + j\sqrt{11} \end{bmatrix}$

As demonstrated in the proof of Theorem 5, $\underline{u}$ can be separated into its real and imaginary components. Furthermore, any eigenvector can be multiplied by a constant. Therefore, take

$$\underline{u}_1 = \begin{bmatrix} \frac{5}{4} \\ 3 \\ 1 \end{bmatrix}, \quad \underline{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

$$\underline{T}_2 = \begin{bmatrix} \frac{5}{4} & -1 & 0 \\ 3 & 0 & 0 \\ 1 & 4 & 1 \end{bmatrix}, \underline{T}_2^{-1} = \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ -1 & \frac{5}{12} & 0 \\ 4 & -2 & 1 \end{bmatrix}$$

Then

$$\underline{Q}(k_2) = (\underline{T}_2^{-1})' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{bmatrix} \underline{T}_2^{-1}$$

$$\underline{Q}(k_2) = \begin{bmatrix} 16a & -8a & 4a \\ -8a & 4a & -2a \\ 4a & -2a & a \end{bmatrix}$$

Now for $k = -\frac{1}{4}$, $\underline{u}_1 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$

Take $\underline{T}_1 = \begin{bmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $\underline{T}_1^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{3}{4} & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{bmatrix}$

Notice, that since $\underline{T}_1^{-1}$ has an identity 2×2 sub-matrix, and since these elements of $\underline{Q}(k_1)$ and $\underline{Q}(k_2)$ are going to be equated, $\underline{Q}(k_1)$ can be taken directly as

$$\underline{Q}(k_1) = (\underline{T}_1^{-1})' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4a & -2a \\ 0 & -2a & a \end{bmatrix} \underline{T}_1^{-1}$$

$$\underline{Q}(k_1) = \begin{bmatrix} \frac{25}{16}a & \frac{-5}{2}a & \frac{5}{4}a \\ \frac{-5}{2}a & 4a & -2a \\ \frac{5}{4}a & -2a & a \end{bmatrix}$$

Since a is only a multiplicative constant in both $\underline{Q}(k_1)$ and $\underline{Q}(k_2)$, it may be taken equal to one. Then, computing the general form of $\underline{Q}(k)$ using

$$\underline{P}(k) = \begin{bmatrix} P_{11} + \frac{\gamma}{2}k & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix}$$

it is found that

$\underline{Q}(k) =$

$$\begin{bmatrix} 2P_{11}+k\gamma+2P_{13} & 2P_{12}-P_{11}+k(P_{23}-\frac{\gamma}{2}) & 2P_{13}-P_{11}-3P_{12}+k(P_{33}-\frac{\gamma}{2}) \\ 2P_{12}-P_{11}+k(P_{23}-\frac{\gamma}{2}) & 2(P_{22}-P_{12}) & 2P_{23}-P_{12}-P_{13} \\ 2P_{13}-P_{11}-3P_{12}+k(P_{33}-\frac{\gamma}{2}) & 2P_{23}-P_{12}-P_{13} & 2(P_{33}-3P_{23}-P_{13}) \end{bmatrix}$$

Equating this $\underline{Q}(k)$ for k equal to k_1 to the $\underline{Q}(k_1)$ and similarly for $\underline{Q}(k)$ for k equal to k_2 and solving the linear equations thus obtained, it is found that

$$\underline{P}(k) = \begin{bmatrix} 1+\frac{23}{24}k & -\frac{5}{6} & -\frac{1}{12} \\ -\frac{5}{6} & \frac{7}{6} & \frac{7}{24} \\ -\frac{1}{12} & \frac{7}{24} & \frac{31}{24} \end{bmatrix}$$

It is easily verified that $\underline{P}(k)$ is positive definite for $k_1 \leq k \leq k_2$. The only vector $\underline{x}$ for which

$$\underline{Q}(k)\underline{x} = \begin{bmatrix} 2+\frac{7}{4}k & -\frac{8}{3}-\frac{2}{3}k & \frac{4}{3}+\frac{1}{3}k \\ -\frac{8}{3}-\frac{2}{3}k & 4 & -2 \\ \frac{4}{3}+\frac{1}{3}k & -2 & 1 \end{bmatrix} \underline{x} = 0$$

is the vector $\underline{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Calculating the derivative of this vector using the differential equations of the system, it is found that

$$\dot{\underline{x}} = \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} .$$

Therefore, $\underline{x} = \underline{0}$ is the only invariant set for which $\dot{V}(\underline{x})$ equals zero. The conditions of Theorem 3 are satisfied and the nonlinear system is a. s. i. l. for all $f(x_1)$ satisfying

$$k_1 < \frac{f(x_1)}{x_1} < k_2 .$$

This then verifies Aizerman's conjecture for this system.

VII. 2. Example 2

The linearized version of the nonlinear control system of Figure 3 is stable for $k > -\frac{1}{2}$.

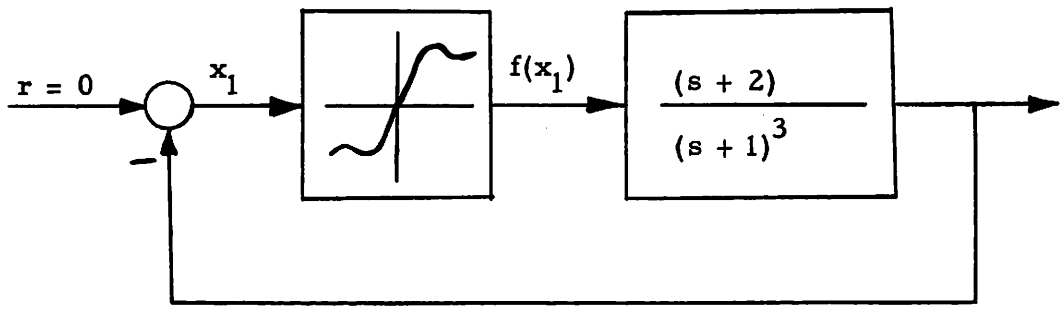

Figure 3

A possible linearized $\underline{G}(k)$ matrix for this system is

$$\underline{G}(k) = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ -k & 0 & -1 \end{bmatrix}$$

For $k = -\frac{1}{2}$, there is an eigenvalue $\lambda = 0$ and an eigenvector

$$\underline{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Take

$$\underline{T_1} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \underline{T_1}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$\underline{Q}(k_1) = (\underline{T_1}^{-1})' \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & 1 \end{bmatrix} \underline{T_1}^{-1} = \begin{bmatrix} \frac{1}{4}(\alpha+2\beta+1) - \frac{1}{2}(\alpha+\beta) & -\frac{1}{2}(\beta+1) \\ -\frac{1}{2}(\alpha+\beta) & \alpha & \beta \\ -\frac{1}{2}(\beta+1) & \beta & 1 \end{bmatrix}$$

Using the general form for $\underline{P}(k)$, it is found that

$\underline{Q}(k) =$

$$\begin{bmatrix} 2P_{11}+k(2P_{13}+\gamma) & 2P_{12}-P_{11}+k(P_{23}-\frac{\gamma}{2}) & 2P_{13}-P_{12}-P_{11}+k(P_{33}-\frac{\gamma}{2}) \\ 2P_{12}-P_{11}+k(P_{23}-\frac{\gamma}{2}) & 2(P_{22}-P_{12}) & 2P_{23}-P_{12}-P_{13}-P_{22} \\ 2P_{13}-P_{12}-P_{11}+k(P_{33}-\frac{\gamma}{2}) & 2P_{23}-P_{12}-P_{13}-P_{22} & 2(P_{33}-P_{13}-P_{23}) \end{bmatrix}$$

In order for $\underline{Q}(k)$ to remain positive semi-definite as k approaches infinity, it is necessary that

$$P_{23} = \frac{\gamma}{2} = P_{33} \quad \text{and} \quad 2P_{13} + \gamma > 0.$$

Equating this $\underline{Q}(k)$ for $k = k_1$ to $\underline{Q}(k_1)$ above and solving for the P_{ij} and γ by substitution and elimination, it is found that there is a solution for arbitrary α and β. In fact,

$$P_{13} = -\frac{1}{2} \quad \text{and} \quad \gamma = \frac{\alpha}{6} + \beta - \frac{5}{6}.$$

So that in order for $2P_{13} + \gamma > 0$,

it is necessary that $\frac{\alpha}{6} + \beta \;\; \frac{11}{6} > 0.$

α and β must also be restricted so that $\underline{Q}(k)$ is positive semi-definite, that is,

$$\alpha - \beta^2 > 0 .$$

If β is taken as one and α as six, both the above conditions are satisified and

$$\underline{P}(k) = \begin{bmatrix} \frac{7}{6} + \frac{7}{12}k & -\frac{7}{6} & -\frac{1}{2} \\ -\frac{7}{6} & \frac{11}{6} & \frac{13}{12} \\ -\frac{1}{2} & \frac{13}{12} & \frac{13}{12} \end{bmatrix}$$

$$\underline{Q}(k) = \begin{bmatrix} \frac{7}{3} + \frac{k}{6} & -\frac{7}{2} & -1 \\ -\frac{7}{2} & \frac{2}{3} & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

In this case, $\underline{P}(k)$ is positive definite for $k \geq -\frac{1}{2}$, and $\underline{Q}(k)$ is positive definite for $k > -\frac{1}{2}$. Thus, all the conditions of Theorem 3 are satisfied and the nonlinear system is a. s. i. l. for

$$\frac{f(x_1)}{x_1} > -\frac{1}{2}$$

which verifies Aizerman's conjecture for this system.

VII. 3. Example 3

The linearized version of the nonlinear control system of Figure 4 has two eigenvalues equal to zero for $k = 0$ and is asymptotically stable for $k > 0$.

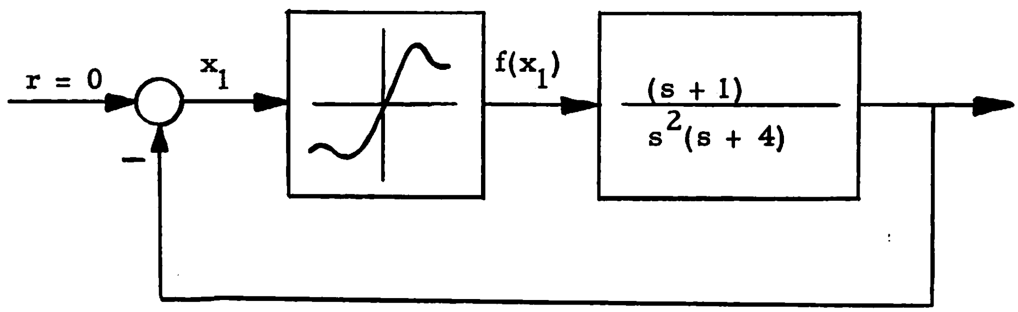

Figure 4

A possible $\underline{G}(k)$ matrix for this system is

$$\underline{G}(k) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ -k & 0 & -4 \end{bmatrix}$$

For $k = 0$, the eigenvector $\underline{u_1}$ and the generalized eigenvector $\underline{u_2}$ for $\lambda = 0$ are

$$\underline{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \underline{u_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, $\underline{Q}(k_1)$ must have the form

$$\underline{Q}(k_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

Using the general form of $\underline{P}(k)$ and including the restriction that the first row and column of $\underline{P}(0)$ equal zero (developed in Section IV. 3), it is found that

$$\underline{Q}(k) = \begin{bmatrix} 0 & k(P_{23} - \frac{\gamma}{2}) & k(P_{33} - \frac{\gamma}{2}) \\ k(P_{23} - \frac{\gamma}{2}) & 0 & 4P_{23} - P_{22} \\ k(P_{33} - \frac{\gamma}{2}) & 4P_{23} - P_{22} & 2(4P_{33} - P_{23}) \end{bmatrix}$$

In order for $\underline{Q}(k)$ to be positive semi-definite as k approaches infinity, it is necessary that

$$P_{23} = \frac{\gamma}{2} = P_{33}.$$

Then equating this $\underline{Q}(k)$ to $\underline{Q}(k_1)$ with $\alpha = 6$, it is found that

$$\underline{P}(k) = \begin{bmatrix} k & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \underline{Q}(k) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Now $\dot{V}(\underline{x}) = 0$ when $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$

Then $\dot{\underline{x}} = \begin{bmatrix} x_2 \\ 0 \\ -f(x_1) \end{bmatrix}$

It is, therefore, impossible for $\underline{x}$ to remain in the plane $x_3 = 0$ where $\dot{V}(\underline{x}) = 0$ for $k > 0$ unless $\underline{x} = \underline{0}$. Consequently, $\underline{x} = \underline{0}$ is the only invariant set for which $\dot{V}(\underline{x}) = 0$. The conditions of Theorem 4 are satisfied and the modified conjecture is true for this system. That is, the nonlinear system is a. s. i. l. for all $f(x_1)$ such that

$$\frac{f(x_1)}{x_1} > \epsilon$$

where ϵ is an arbitrarily small positive number.

VII. 4. Example 4

Consider the nonlinear control system of Figure 5.

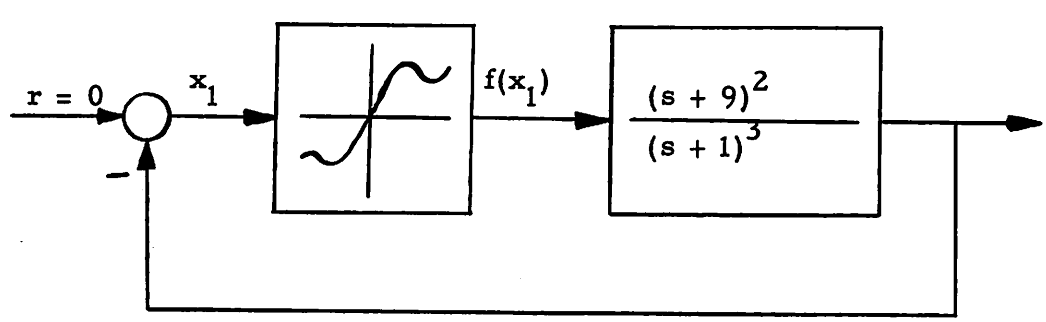

Figure 5

For this system, the following matrix is obtained as the linearized $\underline{G}(k)$.

$$\underline{G}(k) = \begin{bmatrix} -1-k & 1 & 1 \\ -8k & -1 & 8 \\ -8k & 0 & -1 \end{bmatrix}$$

This matrix has eigenvalues with negative real parts for

$$-\frac{1}{81} < k < \frac{2}{3}$$

or for $\frac{2}{3} < k$.

Thus, there are two separate ranges of k for which the linearized system is asymptotically stable. Only the first interval will be considered in this example.

For $k = \frac{2}{3}$ there are imaginary eigenvalues at $\lambda = \pm j\sqrt{15}$ with the eigenvector

$$\underline{u} = \begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix} + j\, 2\sqrt{15} \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

Take $\underline{T}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -\frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix}$ $\qquad \underline{T}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \frac{4}{3} & -\frac{1}{2} & 1 \end{bmatrix}$

$$\underline{Q}(k_2) = (\underline{T}_2^{-1})' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{T}_2^{-1} = \begin{bmatrix} \frac{16}{9} & -\frac{2}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{4} & \frac{1}{2} \\ \frac{4}{3} & -\frac{1}{2} & 1 \end{bmatrix}$$

For $k = -\frac{1}{8}$, there is an eigenvalue $\lambda = 0$ with

$$\underline{u} = \begin{bmatrix} 9 \\ 8 \\ \frac{8}{9} \end{bmatrix}$$

Take $\underline{T}_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{8}{9} & 1 & 0 \\ \frac{8}{81} & 0 & 1 \end{bmatrix}$ $\qquad \underline{T}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{8}{9} & 1 & 0 \\ -\frac{8}{81} & 0 & 1 \end{bmatrix}$

$$\underline{Q}(k_1) = (\underline{T}_1^{-1})' \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \qquad \underline{T_1}^{-1} = \begin{bmatrix} \frac{784}{(81)^2} & -\frac{14}{81} & \frac{28}{81} \\ -\frac{14}{81} & \frac{1}{4} & -\frac{1}{2} \\ \frac{28}{81} & -\frac{1}{2} & 1 \end{bmatrix}$$

Using a general form for $\underline{P}(k)$ it is found that

$$\underline{Q}(k) = \begin{bmatrix} 2P_{11} & 2P_{12}-P_{11} & 2P_{13}-P_{11}-8P_{12} \\ 2P_{12}-P_{11} & 2(P_{22}-P_{12}) & 2P_{23}-P_{12}-P_{13}-8P_{22} \\ 2P_{13}-P_{11}-8P_{12} & 2P_{23}-P_{12}-P_{13}-8P_{22} & 2(P_{33}-P_{13}-8P_{23}) \end{bmatrix}$$

$$+k \begin{bmatrix} \gamma+2P_{11}+16P_{12}+16P_{13} & P_{12}+8P_{22}+8P_{23}-\frac{\gamma}{2} & P_{13}+8P_{23}+8P_{33}-\frac{\gamma}{2} \\ P_{12}+8P_{22}+BP_{23}-\frac{\gamma}{2} & 0 & 0 \\ P_{13}+8P_{23}+8P_{33}-\frac{\gamma}{2} & 0 & 0 \end{bmatrix}$$

$$+ k^2 \begin{bmatrix} \gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By letting $k_1 = -\frac{1}{81}$ in this $\underline{Q}(k)$ and equating it to the $\underline{Q}(k_1)$ above and letting $k_2 = \frac{2}{3}$ and equating it to the $\underline{Q}(k_2)$ above, nine linear equations in the P_{ij} and γ are obtained which can be solved by substitution and elimination to find that

$$\underline{P}(k) = \begin{bmatrix} \frac{8}{121} + \frac{128k}{121} & \frac{-7}{121} & \frac{-2}{121} \\ \frac{-7}{121} & \frac{65}{968} & \frac{-9}{484} \\ \frac{-2}{121} & \frac{-9}{984} & \frac{81}{242} \end{bmatrix}$$

and

$$\underline{Q}(k) = \begin{bmatrix} \frac{16}{121} + \frac{128}{121}k + \frac{256}{121}k^2 & -\frac{2}{11} - \frac{8}{11}k & \frac{4}{11} + \frac{16}{11}k \\ \frac{-2}{11} - \frac{8}{11}k & \frac{1}{4} & \frac{-1}{2} \\ \frac{4}{11} + \frac{16}{11}k & -\frac{1}{2} & 1 \end{bmatrix}$$

It is easily verified that $\underline{P}(k)$ is positive definite for all k such that $-\frac{1}{81} < k < \frac{2}{3}$. $\underline{Q}(k)$ is positive semi-definite and $\underline{Q}(k)\,\underline{x} = \underline{0}$ only if

$$\underline{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Since $\dot{\underline{x}} = \begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix}$, the point $\underline{x} = \underline{0}$ is the only invariant set for which $\dot{V}(\underline{x}) = 0$. Because of the fact that $\underline{P}(k_1)$ is not positive definite, Theorem 4 will have to be used. Thus the nonlinear system is a. s. i. l. for

$$-\frac{1}{81} + \epsilon < \frac{f(x_1)}{x_1} < \frac{2}{3}$$

and the modified conjecture is true for this system.

VII. 5. Example 5

The third order nonlinear control system of Figure 6 has two poles at $s = 0$ in the transfer function of its linear part. The linearized version is asymptotically stable for all $k > 0$.

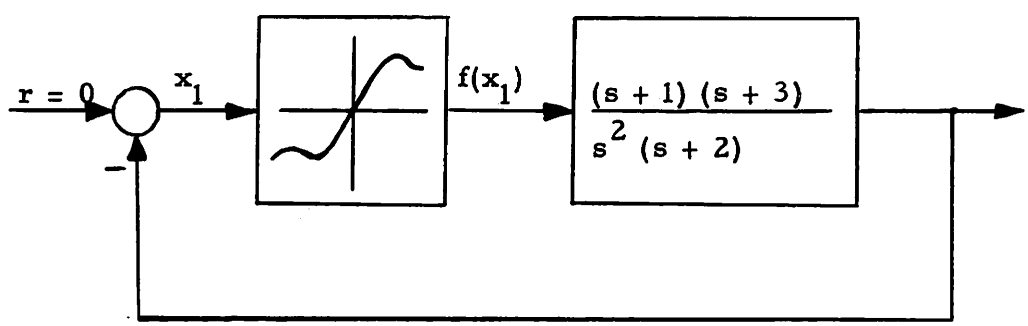

Figure 6

For this system,

$$\underline{G}(k) = \begin{bmatrix} -k & 1 & 1 \\ -k & 0 & 1 \\ -k & 0 & -2 \end{bmatrix}$$

For $k = 0$, the eigenvector and generalized eigenvector for $\gamma = 0$ are

$$\underline{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \underline{u_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Take $\underline{Q}(k_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{bmatrix}$

Using the general form of $\underline{P}(k)$ again with P_{12} and P_{13} equal to zero

$\underline{Q}(k) =$

$$\begin{bmatrix} 2kP_{11} + \gamma k^2 & -P_{11} + k(P_{22} + P_{23} - \frac{\gamma}{2}) & -P_{11} + k(P_{23} + P_{33} - \frac{\gamma}{2}) \\ -P_{11} + k(P_{22} + P_{23} - \frac{\gamma}{2}) & 0 & 2P_{23} - P_{22} \\ -P_{11} + k(P_{23} + P_{33} - \frac{\gamma}{2}) & 2P_{23} - P_{22} & 4P_{33} - 2P_{23} \end{bmatrix}$$

In order for $\underline{Q}(k)$ to be positive semi-definite as k approaches infinity, it is necessary that

$$P_{22} + P_{23} = \frac{\gamma}{2} = P_{23} + P_{33}$$

Equating this $\underline{Q}(k)$ for $k = k_1$ to $\underline{Q}(k_1)$ above and setting α equal to three, it is found that

$$\underline{P}(k) = \begin{bmatrix} \frac{3}{2}k & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \quad \text{and} \quad \underline{Q}(k) = \begin{bmatrix} 3k^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

$\dot{V}(\underline{x}) = 0$ only for $\underline{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$

Then $\underline{\dot{x}} = \begin{bmatrix} x_2 \\ 0 \\ 0 \end{bmatrix}$

Thus $\underline{x} = \underline{0}$ is the only invariant set for which $\dot{V}(x) = 0$. All the conditions of Theorem 4 are satisfied and the modified conjecture is true for this system.

VII. 6. Example 6

The linearized version of the nonlinear control system of Figure 7 has one zero eigenvalue and two purely imaginary eigenvalues for $k = -\frac{21}{2}$. However, the linearized system is stable for all $k > -\frac{21}{2}$.

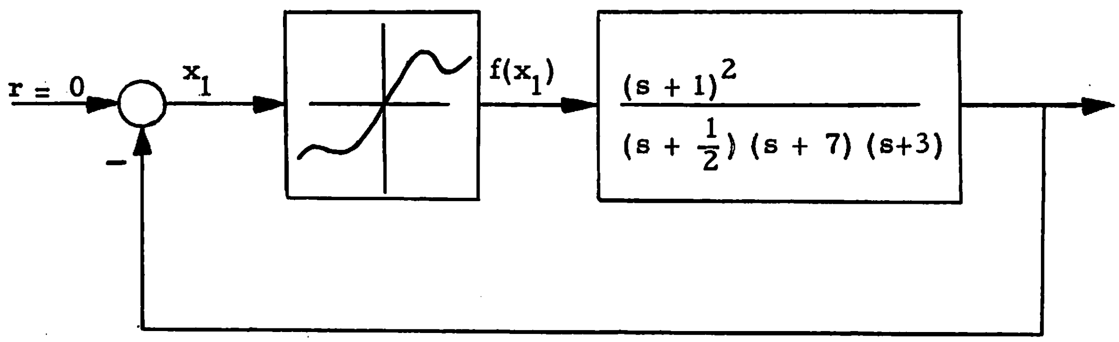

Figure 7

A possible $\underline{G}(k)$ matrix for this system is

$$\underline{G}(k) = \begin{bmatrix} -7-k & 1 & 1 \\ 2k & -3 & -2 \\ -\frac{k}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

In order to impose the constraints of Theorem 5 upon $\underline{Q}(k_1)$, it is necessary to make $\underline{Q}(k_1)$ equal the zero matrix since it must have three independent eigenvectors for the eigenvalue zero. Using the general form of $\underline{P}(k)$, it is found that

$$\underline{Q}(k) = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix}$$

$$
\text{where}\quad \left[\begin{array}{l}
q_{11} = 14P_{11} + k(2P_{11}-4P_{12} + P_{13}) + \frac{\gamma}{2} k^2 \\
q_{12} = 10P_{12}-P_{11} + k(P_{12}-2P_{22} + \frac{1}{2} P_{23} - \frac{\gamma}{2}) \\
q_{13} = \frac{15}{2}P_{13} + 2P_{12} - P_{11} + k(P_{13}-2P_{23} + \frac{1}{2} P_{33} - \frac{\gamma}{2}) \\
q_{22} = 2(3P_{22}-P_{12}) \\
q_{23} = \frac{7}{2} P_{23} + 2P_{22} - P_{12} - P_{13} \\
q_{33} = P_{33} + 4P_{23} - 2P_{13}
\end{array}\right]
$$

The elements in all but the first row and column of $\underline{Q}(k)$ must be set equal to zero. Then, in order for $\underline{Q}(k)$ to remain positive definite as k approaches infinity, the coefficients of k in all but the one-one position must be set equal to zero. Consequently, all the constant elements of $\underline{Q}(k)$ must be zero except for the term in the one-one position. Furthermore, when $k = -\frac{21}{2}$, this term must also be zero.

Arbitrarily set $P_{33} = 1$. The equations obtained above may then be solved to determine that

$$
\underline{P}(k) = \begin{bmatrix} \frac{75}{4} + k & \frac{15}{8} & 2 \\ \frac{15}{8} & \frac{5}{8} & \frac{3}{4} \\ 2 & \frac{3}{4} & 1 \end{bmatrix}
$$

and
$$
\underline{Q}(k) = \begin{bmatrix} (k + \frac{21}{2})(2k + 25) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

Now, $\underline{Q}(k)$ is positive semi-definite for $k > \frac{21}{2}$, and $\underline{Q}(k)\,\underline{x} = \underline{0}$ only if $x_1 = 0$. In this case

$$
\underline{\dot{x}} = \begin{bmatrix} x_2 + x_3 \\ -3x_2 - 2x_3 \\ -\frac{x_3}{2} \end{bmatrix}
$$

In order to remain in the plane $x_1 = 0$, it would be necessary for $x_2 = - x_3$.

$$\text{Then } \dot{\underline{x}} = \begin{bmatrix} 0 \\ x_3 \\ \frac{-x_3}{2} \end{bmatrix}$$

Consequently, x_2 could not remain equal to $-x_3$ and the system could not remain in the plane $x_1 = 0$. Therefore, $\underline{x} = \underline{0}$ is the only invariant set for which $\dot{V}(\underline{x}) = 0$. The conditions of Theorem 3 are, therefore, satisfied and the nonlinear system is a. s. i. l. for all $f(x_1)$ such that

$$\frac{f(x_1)}{x_1} > - \frac{21}{2},$$

and Aizerman's conjecture is true for this system.

VII. 7. Example 7

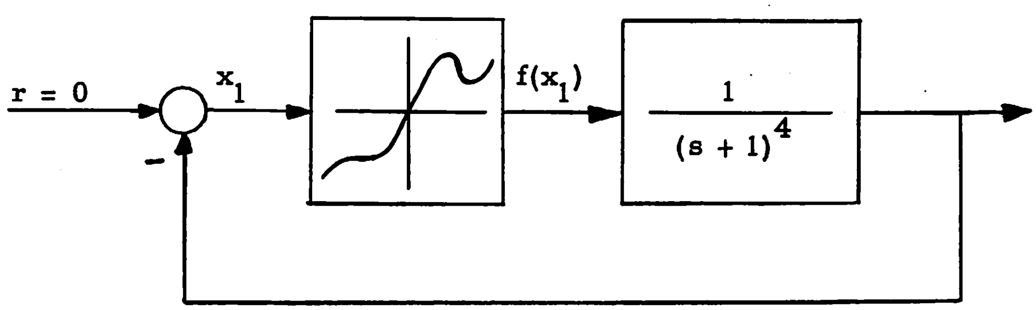

Figure 8

For the control system of Figure 8, the following $\underline{G}(k)$ matrix was obtained.

$$\underline{G}(k) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -k & 0 & 0 & -1 \end{bmatrix}$$

$\underline{G}(k)$ has eigenvalues in the left-half plane for

$$-1 < k < 4.$$

For k equal to 4, there are imaginary poles at $\pm j$. An eigenvector for these poles is

$$\underline{u} = \begin{bmatrix} 1 \\ 1+j \\ 2j \\ -2(1-j) \end{bmatrix}$$

Let $\underline{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$, $\underline{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}$, and $\underline{T}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}$

$$\underline{Q}(4) = (\underline{T_2}^{-1})' \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & \beta & 1 \end{bmatrix} \underline{T_2}^{-1}$$

$$\underline{Q}(4) = \begin{bmatrix} 4\alpha + 16\beta + 16 & -4(\alpha + 3\beta + 2) & 2\alpha + 4\beta & 2\beta + 4 \\ 4[\alpha + 3\beta + 2] & 4(\alpha + 2\beta + 1) & -2(\alpha + \beta) & -2(\beta + 1) \\ 2\alpha + 4\beta & -2(\alpha + \beta) & \alpha & \beta \\ 2\beta + 4 & -2(\beta + 1) & \beta & 1 \end{bmatrix}$$

Now for k equal to minus one, $\underline{G}(k)$ has an eigenvalue equal to zero and the eigenvector,

$$\underline{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Take $\underline{T}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$$\text{and} \quad \underline{Q}(-1) = (\underline{T}_1^{-1})' \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4(\alpha + 2\beta + 1) & -2(\alpha+\beta) & -2(\beta+1) \\ 0 & -2(\alpha+\beta) & \alpha & \beta \\ 0 & -2(\beta+1) & \beta & 1 \end{bmatrix} \underline{T}_1^{-1}$$

$$\underline{Q}(-1) = \begin{bmatrix} \alpha + 2\beta + 1 & -(2\alpha + 4\beta + 2) & \alpha + \beta & \beta + 1 \\ -(2\alpha + 4\beta + \beta) & 4(\alpha + 2\beta + 1) & -2(\alpha+\beta) & -2(\beta+1) \\ \alpha + \beta & -2(\alpha+\beta) & \alpha & \beta \\ \beta + 1 & -2(\beta+1) & \beta & 1 \end{bmatrix}$$

Using the general form for $\underline{P}(k)$, the following $\underline{Q}(k)$ is obtained,

$\underline{Q}(k)$ =

$$\begin{bmatrix} 2P_{11}+k(2P_{14}+\gamma) & 2P_{12}-P_{11}+1(P_{24}-\frac{\gamma}{2}) & 2P_{13}-P_{12}+kP_{34} & 2P_{14}-P_{13}+kP_{44} \\ 2P_{12}-P_{11}+k(P_{24}-\frac{\gamma}{2}) & 2(P_{22}-P_{12}) & 2P_{23}-P_{13}-P_{22} & 2P_{24}-P_{14}-P_{23} \\ 2P_{13}-P_{12}+kP_{34} & 2P_{23}-P_{13}-P_{22} & 2(P_{33}-P_{23}) & 2P_{34}-P_{24}-P_{33} \\ 2P_{14}-P_{13}+kP_{44} & 2P_{24}-P_{14}-P_{23} & 2P_{34}-P_{24}-P_{33} & 2(P_{44}-P_{34}) \end{bmatrix}$$

Equating this $\underline{Q}(k)$ for $k = -1$ and $k = 4$ to the $\underline{Q}(-1)$ and $\underline{Q}(4)$ above, a set of linear equations for the P_{ij} and γ is obtained. Solving these equations by substitution and elimination, it is found that there is a solution if

$$\gamma + 2\beta = \frac{1}{2}.$$

Thus, any α and β may be chosen as long as they satisfy this equation and satisfy the constraint

$$\alpha - \beta^2 > 0$$

to insure that $\underline{Q}(k)$ is positive semi-definite. Take $\beta = 0$ and $\alpha = \frac{1}{2}$.

$$\underline{P}(k) = \begin{bmatrix} \frac{12}{5} + \frac{19}{20}k & -1 & -\frac{1}{5} & \frac{7}{10} \\ -1 & 2 & \frac{2}{5} & -\frac{9}{20} \\ -\frac{1}{5} & \frac{2}{5} & \frac{13}{20} & \frac{1}{10} \\ \frac{7}{10} & -\frac{9}{20} & \frac{1}{10} & \frac{3}{5} \end{bmatrix}$$

$\underline{P}(k)$ is positive definite for $k \geq -1$.

Since $\underline{Q}(k) = \begin{bmatrix} \frac{24}{5} + \frac{33}{10}k & -\frac{22}{5} - \frac{7}{5}k & \frac{3}{5} + \frac{1}{10}k & \frac{8}{5} + \frac{3}{5}k \\ -\frac{22}{5} - \frac{7}{5}k & 6 & -1 & -2 \\ \frac{3}{5} + \frac{1}{10}k & -1 & \frac{1}{2} & 0 \\ \frac{8}{5} + \frac{3}{5}k & -2 & 0 & 1 \end{bmatrix}$

$\underline{Q}(k)\,\underline{x}$ is zero only when $\underline{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}$.

But, then $\dot{\underline{x}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$.

Therefore, $\underline{x} = \underline{0}$ is the only invariant set for which $\dot{V}(\underline{x}) = 0$. All the conditions of Theorem 3 are satisfied and Aizerman's conjecture is true for this system.

VII. 8. Example 8

The fourth order example of Figure 9 is similar to the preceding example.

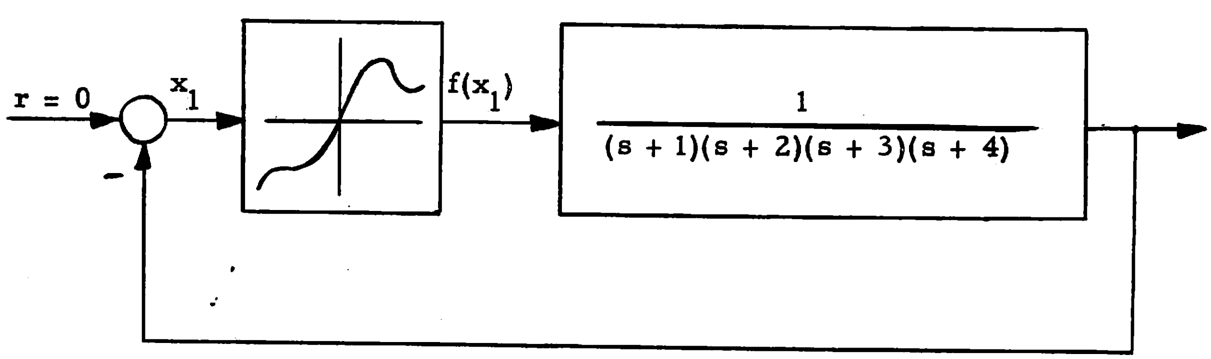

Figure 9

The linearized system is stable for k in the interval $(-24, 126)$. Proceeding exactly as was done in the preceding example, it is found that there is a solution for this case if

$$\alpha + 2\beta = 9.$$

Then choosing $\alpha = 0$ and $\alpha = 9$, it is found that

$$\underline{P}(k) = \begin{bmatrix} \frac{1017}{10} + \frac{5}{8}k & \frac{81}{10} & \frac{243}{40} & \frac{87}{40} \\ \frac{81}{10} & \frac{333}{10} & \frac{99}{40} & \frac{9}{40} \\ \frac{243}{40} & \frac{99}{40} & \frac{93}{40} & \frac{3}{10} \\ \frac{87}{40} & -\frac{9}{40} & \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

$$\underline{Q}(k) = \begin{bmatrix} \frac{1017}{5} + \frac{81}{10}k & -\frac{387}{5} - \frac{21}{10}k & \frac{81}{5} + \frac{3}{10}k & \frac{24}{5} + \frac{1}{5}k \\ -\frac{387}{5} - \frac{21}{10}k & 117 & -27 & -6 \\ \frac{81}{5} + \frac{3}{10}k & -27 & 9 & 0 \\ \frac{24}{5} + \frac{1}{10}k & -6 & 0 & 1 \end{bmatrix}$$

Once again, the conditions of Theorem 3 are satisfied and Aizerman's conjecture is true for this system.

VII. 9. Example 9

The nonlinear control system of Figure 10 has two zeros and four poles in the transfer function of its linear part.

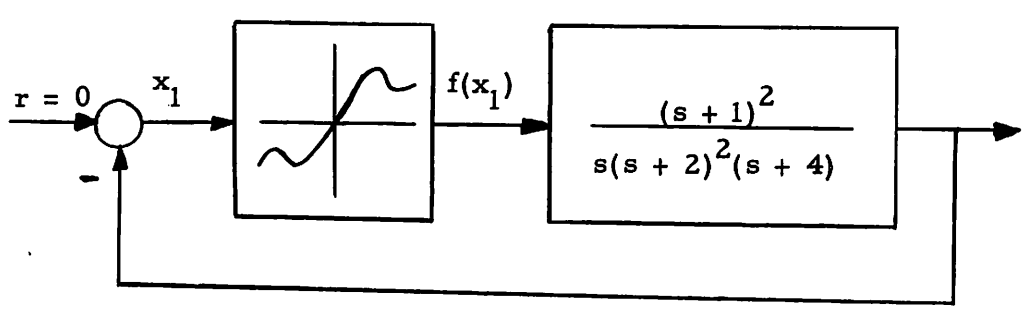

Figure 10

A possible $\underline{G}(k)$ matrix for this system is

$$\underline{G}(k) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & -2 & -1 \\ -k & 0 & 0 & -4 \end{bmatrix}$$

$\underline{G}(k)$ has eigenvalues in the left-half plane for any $k > 0$. For $k = 0$, $\underline{G}(k)$ has an eigenvalue equal to zero with an associated eigenvector,

$$\underline{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\underline{T_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

and $\underline{Q}(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & B_{11} & B_{12} & B_{13} \\ 0 & B_{12} & B_{22} & B_{23} \\ 0 & B_{13} & B_{23} & B_{33} \end{bmatrix}$

Using the general form of $\underline{P}(k)$, it is found that

$$\underline{Q}(k) = \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{12} & q_{22} & q_{23} & q_{24} \\ q_{13} & q_{23} & q_{33} & q_{34} \\ q_{14} & q_{24} & q_{34} & q_{44} \end{bmatrix}$$

where $q_{11} = 2kP_{14}$

$$q_{12} = 2P_{12} - P_{11} + k\,(P_{24} - \frac{\gamma}{2})$$
$$q_{13} = 2P_{13} + P_{12} - P_{11} + k\,(P_{34} - \frac{\gamma}{2})$$
$$q_{14} = 4P_{14} + P_{13} + P_{12} - P_{11} + k\,(P_{44} - \frac{\gamma}{2})$$
$$q_{22} = 2(2P_{22} - P_{12})$$
$$q_{23} = 4P_{23} + P_{22} - P_{13} - P_{12}$$
$$q_{24} = 6P_{24} + P_{23} + P_{22} - P_{12} - P_{14}$$
$$q_{33} = 2(2P_{33} + P_{23} - P_{13})$$
$$q_{34} = 6P_{34} + P_{33} + P_{23} + P_{24} - P_{13} - P_{14}$$
$$q_{44} = 2(4P_{44} + P_{34} + P_{24} - P_{14})$$

In order to insure that $\underline{Q}(k)$ remains positive semi-definite for arbitrarily large k, it is necessary to require that

$$P_{44} = \frac{\gamma}{2} = P_{34} = P_{24}\,.$$

Then by equating $\underline{Q}(k)$ for $k = 0$ to the $\underline{Q}(0)$ above and solving the equations thus obtained for the P_{ij} and γ, it is found that there is a solution if

$$\frac{3}{8}\,B_{11} + \frac{1}{2}\,B_{12} - 2B_{13} + B_{33} = 4$$

and $\frac{1}{4}\,B_{11} + \frac{1}{6}\,B_{12} - \frac{7}{6}\,B_{13} - \frac{1}{4}\,B_{22} + B_{23} = \frac{3}{2}$.

Let $B_{12} = B_{13} = B_{23} = 0$ and $B_{33} = 1$. Then, solving the two equations above, it is found that

$$B_{11} = 8 \text{ and } B_{22} = 2.$$

Then, it is found that

$$\underline{P}(k) = \begin{bmatrix} 16 + \frac{k}{4} & 8 & 4 & 1 \\ 8 & 6 & \frac{3}{2} & \frac{1}{4} \\ 4 & \frac{3}{2} & \frac{7}{4} & \frac{1}{4} \\ 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$\text{and} \quad \underline{Q}(k) = \begin{bmatrix} 2k & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All the conditions of Theorem 3 are satisfied and Aizerman's conjecture is true for this system.

VIII. CONCLUSION

A direct procedure for the determination of Lyapunov functions of the Lur'e type which verify Aizerman's conjecture for particular nonlinear systems has been developed. This procedure will, in fact, determine the function if it exists for the system under consideration. However, it is not known in general that a function of this type will exist to verify Aizerman's conjecture for all systems for which the conjecture is true. This procedure then represents a method for examining a sufficient condition for Aizerman's conjecture to be true.

This procedure is cumbersome to apply to fourth and higher order systems since the number of equations that must be solved for the parameters of the Lyapunov function becomes very large. Nevertheless, this procedure furnishes a straightforward method of determining the function which is seriously needed for higher order systems since it is practically impossible to determine a suitable function by other methods of generating Lyapunov functions which inherently rely on a certain amount of intuition or trial and error.

APPENDIX I: COUNTEREXAMPLES

Krasovskii [21] developed the following counterexample to Aizerman's conjecture. He considered a second order system

$$\dot{x} = x + y + f(x)$$
$$\dot{y} = - x - y$$

with $$f(x) = -\frac{e^{-2x}}{1 + e^{-x}} \quad \text{for} \quad x \geq 1$$

$$f(x) = -\frac{e^{-2}}{1 + e^{-1}}\, x \quad \text{for } x > 1$$

It is easily verified that a solution to this system for $x(0)$ equal to one is

$$x(t) = t - e^{x} + 1 + e$$

$$y(t) = e^{-x(t)} - x(t)$$

Thus $x(t)$ goes to infinity and $y(t)$ goes to minus infinity as t goes to infinity. However, Aizerman's conjecture proposes that this system would be a. s. i. l. for any nonlinearity such that

$$\frac{f(x)}{x} > 0 \quad .$$

Since this condition is satisfied by this unstable system, Aizerman's conjecture is false in this case.

Pliss[4] has shown that many counterexamples exist in third order cases. In particular, for the system

$$\dot{x} = y - f(x)$$
$$\dot{y} = z - x$$
$$\dot{z} = -ax - bf(x)$$

Pliss has shown with a very long and detailed mathematical argument that it is possible for a periodic solution to exist for this system with $f(x)$ satisfying the conditions of Aizerman's conjecture (and also the modified conjecture of this paper).

APPENDIX II: TESTS FOR POSITIVE DEFINITENESS OF A MATRIX

The matrix

$$\underline{P} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & \cdot \\ \vdots & \vdots & & \vdots \\ P_{2n} & \cdot & \cdots & P_{nn} \end{bmatrix}$$

is positive definite if all the principal minors of $\underline{P}$ are positive, that is, if

$$P_{11} > 0, \quad \begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} > 0, \; \ldots\ldots, \quad \left| \underline{P} \right| > 0.$$

The matrix $\underline{P}$ is positive semi-definite if all the principal minors are non-negative.

APPENDIX III: KRONECKER PRODUCTS

The Kronecker product of two matrices $\underline{A}$ and $\underline{B}$ is defined as

$$\underline{A} \otimes \underline{B} = \begin{bmatrix} A_{11}\,\underline{B} & A_{12}\,\underline{B} & \cdots & A_{1n}\,\underline{B} \\ A_{21}\,\underline{B} & A_{22}\,\underline{B} & & \vdots \\ A_{n1}\,\underline{B} & \cdots & & A_{nn}\,\underline{B} \end{bmatrix}$$

Thus if $\underline{A}$ and $\underline{B}$ are both nxn matrices $\underline{A} \otimes \underline{B}$ is a $n^2 \times n^2$ matrix.

Consider a second order example of the equation,

$$\underline{Y}\,\underline{C} + \underline{C}'\,\underline{Y} = -\,\underline{D}\,.$$

The equations for the coefficients are

$$\begin{aligned} Y_{11} C_{11} + Y_{12} C_{21} + C_{11} Y_{11} + C_{21} Y_{21} &= -\,D_{11} \\ Y_{11} C_{12} + Y_{12} C_{22} + C_{11} Y_{12} + C_{21} Y_{22} &= -\,D_{12} \\ Y_{21} C_{11} + Y_{22} C_{12} + C_{12} Y_{11} + C_{22} Y_{21} &= -\,D_{21} \\ Y_{21} C_{12} + Y_{22} C_{22} + C_{12} Y_{12} + C_{22} Y_{22} &= -\,D_{22} \end{aligned}$$

These equations can be written in the vector form

$$\begin{bmatrix} C_{11} + C_{11} & C_{21} & C_{21} & 0 \\ C_{12} & C_{11} + C_{22} & 0 & C_{21} \\ C_{12} & 0 & C_{11} + C_{22} & C_{21} \\ 0 & C_{12} & C_{12} & C_{22} + C_{22} \end{bmatrix} \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{21} \\ Y_{22} \end{bmatrix} = - \begin{bmatrix} D_{11} \\ D_{12} \\ D_{21} \\ D_{22} \end{bmatrix}$$

The matrix on the left can be expressed as

$$\underline{I} \otimes \underline{C}' + \underline{C}' \otimes \underline{I}\,.$$

This procedure can be generalized for matrices of larger dimension.

Some properties of Kronecker products are the following:

(1) If λ_i is an eigenvalue of $\underline{A}$ and μ_j is an eigenvalue of $\underline{B}$, then $\lambda_i \mu_j$ is an eigenvalue of $\underline{A} \otimes \underline{B}$.

(2) If λ_i and λ_j are eigenvalues of $\underline{A}$, then $\lambda_i + \lambda_j$ is an eigenvalue of $\underline{I} \otimes \underline{A} + \underline{A} \otimes \underline{I}$.

(3) $(\underline{A} \otimes \underline{B})\,(\underline{C} \otimes \underline{D}) = \underline{AC} \otimes \underline{BD}$.

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