Electronics Research Laboratory
University of California
Berkeley, California

LIMITATIONS ON THE TRANSIENT RESPONSE OF POSITIVE REAL FUNCTIONS

by

R. A. Rohrer

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Because the impulse response of a physical network,

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)\, e^{j\omega t}\, d\omega \;,$$

is uniquely determined by the real part Fourier integral[1]

$$h(t) = \frac{2}{\pi} \int_{0}^{\infty} \left[\mathrm{Re}\left\{ H(j\omega) \right\} \right] \cos\omega t \;\; d\omega \;, \qquad (1)$$

it and its integrals can be bounded in many ways. The integrals of the impulse response (step, ramp, ...) can be denoted by

$$h_k(t) = \underbrace{\int_0^t \cdots \int_0^t}_{k \text{ times}} h(t)\, \underbrace{dt \ldots \;\; dt}_{k \text{ times}};$$

hence, the impulse response is $h_o(t)$, the ramp response is $h_2(t)$, etc. The even order responses are given by

$$h_{2n}(t) = \frac{2}{\pi} \int_0^{\infty} (-1)^n \frac{1}{\omega^{2n}} \left[\mathrm{Re}\left\{ H(j\omega) \right\} \right] \cos \omega t \;\; d\omega . \; (2)$$

The cosine inequalities

$$1 - \cos\omega t \geq \frac{1}{4^{a}} \left[1 - \cos\left(2^{a} \omega t\right) \right] \qquad (3)$$

used by Papoulis[2] to bound the frequency attenuation for filters with monotonic step response may be applied here to bound the transient response of positive real network functions. Since for a p.r. function

$$\mathrm{Re}\left\{H(j\omega)\right\} \geq 0 ,$$

we have for n even

$$h_{2n}(0) - h_{2n}(t) = \frac{2}{\pi}\int_0^\infty \frac{1}{\omega^{2n}}\left[\mathrm{Re}\left\{H(j\omega)\right\}\right](1 - \cos\omega t)d\omega$$

$$\geq \frac{2}{\pi}\int_0^\infty \frac{1}{\omega^{2n}}\left[\mathrm{Re}\left\{H(j\omega)\right\}\right]\frac{1}{4^{\alpha}}\left[1 - \cos(2^{\alpha}\omega t)\right] d\omega$$

$$= \frac{1}{4^{\alpha}}\cdot\left[h_{2n}(0) - h_{2n}(2^{\alpha}t)\right]$$

or

$$4^{\alpha}\left[h_{2n}(0) - h_{2n}(t)\right] \geq h_{2n}(0) - h_{2n}(2^{\alpha}t), \quad n \text{ even.} \qquad (4)$$

Similarly,

$$4^{\alpha}\left[h_{2n}(0) - h_{2n}(t)\right] \leq h_{2n}(0) - h_{2n}(2^{\alpha}t), \quad n \text{ odd.} \qquad (5)$$

Taking $n = 0$ and $\alpha = 1$, we have the impulse response bound from (4),

$$4\left[h(0)-h(t)\right] \geq h(0)-h(2t)$$

or

$$h(2t) \geq h(t)-3h(0)\,. \tag{6}$$

An even more interesting bound, however, is that obtained on the ramp response by taking $n=1$ and $\alpha=1$ and employing (5):

$$4\left[h_2(0)-h_2(t)\right] \leq h_2(0)-h_2(2t)\,.$$

Under the obvious condition $h_2(0)=0$, we have

$$h_2(2t) \leq 4h_2(t)\,. \tag{7}$$

But the ramp response is the integral of step response, consequently

$$\int_0^{2t_o} h_1(t)dt \leq 4\int_0^{t_o} h_1(t)\,dt$$

$$\int_0^{t_o} h_1(t)\,dt + \int_{t_o}^{2t_o} h_1(t)\,dt \leq 4\int_0^{t_o} h_1(t)\,dt$$

$$\int_{t_o}^{2t_o} h_1(t)\,dt \leq 3\int_0^{t_o} h_1(t)\,dt\,. \tag{8}$$

From the graphical presentation of a typical step response involved in the realization of a delay function, we may interpret the ripple as the shaded area in Fig. 1, represented by the integral on the right-hand side of (8). Moreover, by one of the usual definitions of rise-time,[3] the area represented by the integral on the left-hand side of (8) may be taken as $t_d - t_r$, the delay minus the rise-time. Hence, we see clearly the interchange which may be obtained between the rise-time, delay, and ripple of a network function with a positive real part,

$$t_d - t_r \leq 3 \text{ (ripple area)}. \tag{9}$$

This is but one of many such bounds on the transient response which may be obtained from (4) and (5).

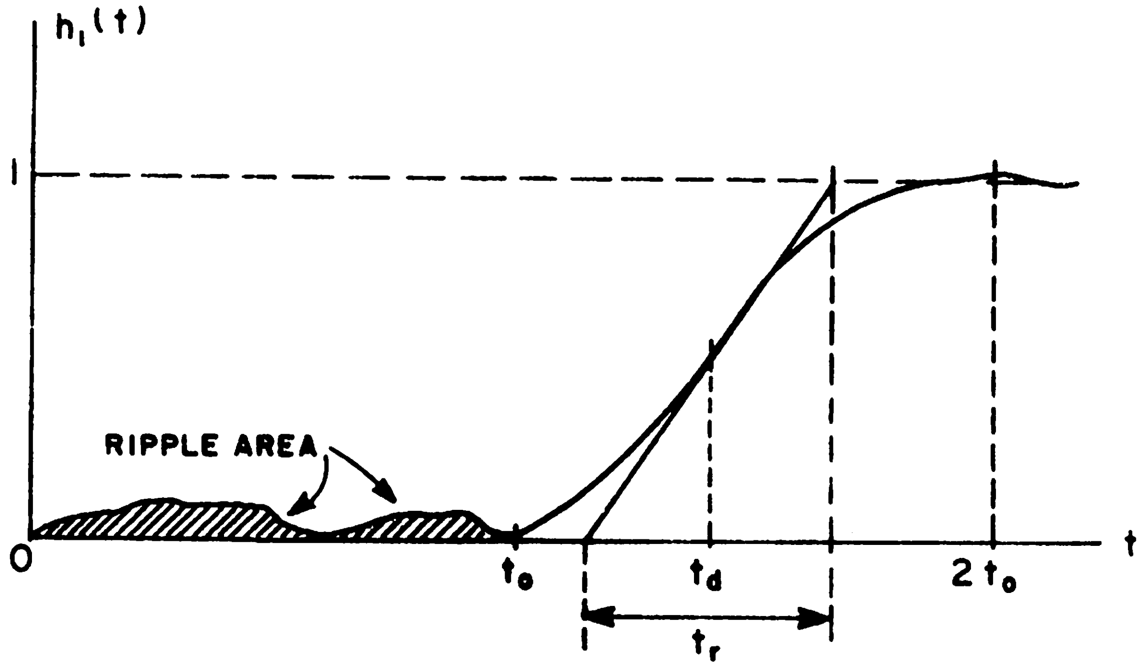

Fig. 1 Delay Function Step Response

REFERENCES

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3. Kuh, E. S. and Pederson, D. O., "Principles of Circuit Synthesis," New York: McGraw-Hill, 1959, pp. 28-29.