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TIME DEPENDENT ENERGY FUNCTIONS AND DISSIPATIVE NETWORKS

by

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ABSTRACT

The energy function approach usually associated with classical mechanics is applied to the single loop, linear, time-varying $R(t) - L(t) - C(t)$ network. Time dependent Lagrangians and Hamiltonians are formulated which lead to a phase plane description of the network behavior. A set of the stability criteria in terms of the time dependence of the network elements is derived.

I. INTRODUCTION

The techniques of classical mechanics have offered many simplifying analytical tools and increased insight in areas where they are applicable. It is unfortunate that many of the most powerful aspects of classical mechanics (e. g., the Hamilton-Jacobi partial differential equation for Hamilton's principle function) have been hitherto mostly confined to conservative (lossless) systems.

Two problems arise in the application of these energy principles to variable networks:

1) inclusion of dissipative effects in the energy formulation, and

2) accounting for non-holonomic constraints.

It is true that Rayleigh's dissipation function allows the inclusion of dissipative effects in the energy formulation, but there the dissipances give rise to a function separate from the Lagrangian. Consequently, the dissipative and the conservative parts of the system are divorced and the formulation seems only useful for determining the governing set of differential equations.

In order to describe the network in terms of strictly differential quantities, it is important that the charge q be chosen as the fundamental coordinate when the series RLC branch is taken as basic (the constraints are then on branch currents at nodes). Similarly, the choice of the parallel RLC structure as basic dictates that the flux ϕ be the fundamental coordinate. The familiar Kirchoff current and voltage laws then take the form of non-holonomic [1] differential constraints:

$$\sum_k a_{jk}\dot{q}_k = 0 \quad (a_{jk} = \pm 1), \quad \text{at node } j \tag{1}$$

$$\sum_k b_{jk}\dot{\phi}_k = 0 \quad (b_{jk} = \pm 1), \quad \text{around loop } j \quad . \tag{2}$$

In systems which contain time-dependent inductances and capacitances (L(t) and C(t)) one may go blithely ahead and write the energy relations as though the constraints were holonomic. These systems might be dubbed "quasi-conservative" in that they can be handled by methods identical to those for time-invariant LC networks [2]. Lossy networks, employing time-dependent resistances (R(t)), on the other hand, definitely require the non-holonomic constraint relations.

Classically the method of Lagrange undetermined multipliers [1] has been used for handling constraints of the form (1) and (2); it too can be used here, but the resulting equations tend to obscure the desired information. A second method for overcoming the problem of non-holonomic constraints is to extend the number of coordinates beyond that dictated by the topology of the system. Properly done, this extension places the resistances in the position of providing holonomic constraints on the coordinates (inductance currents and capacitance voltages) [3, 4]. Again, the differential equations of the system may be found, but few further insights are gained.

The results to be presented represent a thorough exploitation of the energy function concept for a single degree of freedom dissipative network, and are presently being investigated for possible means of extension to higher order systems.

II. FORMULATION OF ENERGY FUNCTIONS

Consider the single-loop lumped linear time-varying network of Fig. 1. The linear second-order differential equation governing its behavior is[+]

$$\frac{d}{dt}(L\dot{q}) + R\dot{q} + \frac{1}{C}q = v \qquad (3)$$

Following Friedman [5] this equation can be recast as[++]

[+] The arguments of the functions will be omitted for notational convenience -- strictly speaking, we should write $R(t)$, $q(t)$, etc.

[++] The end points in these integrals are always to be 0 and t, hence

$$\int \frac{R}{L}\,dt = \int_0^t \frac{R}{L}\,dt.$$

$$\exp\left(-\int \frac{R}{L}dt\right)\left\{\frac{d}{dt}\left[L\dot{q}\exp\left(\int \frac{R}{L}dt\right)\right]+\frac{1}{C}q\exp\left(\int \frac{R}{L}dt\right)\right.$$
$$\left. - v\exp\left(\int \frac{R}{L}dt\right)\right\} = 0. \tag{4}$$

Since only the single loop is being considered the multiplier, $\exp\left(-\int \frac{R}{L}dt\right)$ may be dropped (it is this contrivance which gives rise to the difficulties encountered later with non-holonomic constraints in multi-degree-of-freedom systems). Hence, the governing differential equation in its final form is

$$\frac{d}{dt}\left[L\dot{q}\exp\left(\int \frac{R}{L}dt\right)\right]+\frac{1}{C}q\exp\left(\int \frac{R}{L}dt\right) - v\exp\left(\int \frac{R}{L}dt\right) = 0 . \tag{5}$$

To obtain an "energy" formulation (5) must be shown to be the consequence of some variational principle. The usual method [6] is to employ the definite integral

$$\int_{t_1}^{t_2}\left\{\frac{d}{dt}\left[L\dot{q}\exp\left(\int \frac{R}{L}dt\right)\right]+\frac{1}{C}q\exp\left(\int \frac{R}{L}dt\right)-v\exp\left(\int \frac{R}{L}dt\right)\right\}\delta q\, dt = 0 \tag{6}$$

Under the usual condition that the variation δq vanishes at the end points t_1 and t_2, the first term in the integrand may be integrated by parts yielding+

+ $\frac{d}{dt}(\delta q) = \delta\dot{q}$, see Hildebrand [6].

$$\int_{t_1}^{t_2} \exp\left(\int \frac{R}{L}\, dt\right) \left[- L \dot{q}\, \delta\dot{q} + \frac{1}{C} q\, \delta q - v\, \delta q\right] dt = 0 . \tag{7}$$

Upon multiplication by -1 and removal of the variation from under the integral, (7) becomes

$$\delta \int_{t_1}^{t_2} \exp\left(\int \frac{R}{L} dt\right) \left[\frac{1}{2} L \dot{q}^2 - \frac{1}{2C} q^2 + vq\right] dt . \tag{8}$$

For a demonstration that the extrema in (8) actually lead to the desired differential equation (5) see Appendix I.

The quantity in brackets in (8) is recognized to be the usual Lagrangian [7], hence we may define a modified Lagrangian

$$\hat{L}(q, \dot{q}, t) = \exp\left(\int \frac{R}{L} dt\right) \left[\frac{1}{2} L \dot{q}^2 - \frac{1}{2C} q^2 + vq\right] . \tag{9}$$

The Euler-Lagrange differential equation,

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{q}}\right) - \frac{\partial \hat{L}}{\partial q} = 0 \tag{10}$$

is the expected equation of motion (5).

III. EXTENSIONS

The modified Lagragian (9) leads to a modified Hamiltonian through a Legendre transformation. First we must define the conjugate momentum [8] associated with the canonic coordinate q:

$$p = \frac{\partial \hat{L}(q, \dot{q}, t)}{\partial \dot{q}} \tag{11}$$

$$p = L\dot{q} \exp\left(\int \frac{R}{L} dt\right). \tag{12}$$

From the Legendre transformation,

$$\hat{H}(q, p, t) = p\dot{q} - \hat{L}(q, \dot{q}, t), \tag{13}$$

we obtaine the modified Hamiltonian

$$\hat{H}(q, p, t) = \frac{1}{2L} p^2 \exp\left(-\int \frac{R}{L} dt\right) + \frac{1}{2C} q^2 \exp\left(\int \frac{R}{L} dt\right) - v\,q \exp\left(\int \frac{R}{L} dt\right). \tag{14}$$

Hamilton's canonical equations for this system take the form

$$\dot{q} = \frac{\partial \hat{H}}{\partial p} = \frac{1}{L} p \exp\left(-\int \frac{R}{L} dt\right) \tag{15a}$$

and

$$\dot{p} = -\frac{\partial \hat{H}}{\partial q} = -\frac{1}{C} q \exp\left(\int \frac{R}{L} dt\right) + v \exp\left(\int \frac{R}{L} dt\right). \tag{15b}$$

Hence, the problem can be reformulated in terms of a first order matrix differential equation

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \exp\left(-\int \frac{R}{L} dt\right) \\ -\frac{1}{C} \exp\left(\int \frac{R}{L} dt\right) & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ v \end{pmatrix} \exp\left(\int \frac{R}{L} dt\right). \tag{16}$$

A solution of the Hamilton-Jacobi partial differential equation will yield q(t) [9]; this equation is

$$\hat{H}(q, \frac{\partial S}{\partial q}, t) - \frac{\partial S}{\partial t} = 0 \tag{17}$$

where

$$\frac{\partial S(q, t)}{\partial q} = p. \tag{18}$$

If we consider the homogeneous case, where the drive $v=0$, the Hamilton-Jacobi equation becomes

$$\frac{1}{2L} \exp\left(-\int \frac{R}{L} dt\right)\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2C} \exp\left(\int \frac{R}{L} dt\right) q^2 - \frac{\partial S}{\partial t} = 0. \tag{19}$$

The solution may be taken to be of the form

$$S = \frac{1}{2} \frac{q^2}{f(t)} . \tag{20}$$

Inserting (20) in (19) yields

$$\frac{1}{2} q^2 \left[\frac{1}{L} \exp\left(-\int \frac{R}{L} dt\right) (f)^{-2} + \frac{1}{C} \exp\left(\int \frac{R}{L} dt\right) - (f)^{-2} \frac{df}{dt}\right] = 0 \tag{21}$$

which leads to the homogeneous first order nonlinear differential equation for $f(t)$

$$\frac{df}{dt} - \frac{1}{C} \exp\left(\int \frac{R}{L} dt\right) f^2 - \frac{1}{L} \exp\left(-\int \frac{R}{L} dt\right) = 0 . \tag{22}$$

This equation is recognized to be the Riccati equation [10], which may be transformed into a linear second-order differential equation by the change of variables

$$f(t) = -C\exp\left(-\int \frac{R}{L}dt\right)\frac{\dot{z}}{z} \quad . \tag{23}$$

This substitution leads ultimately to the equation

$$\frac{d}{dt}\left[C\dot{z}\exp\left(-\int\frac{R}{L}dt\right)\right] + \frac{1}{L}z\exp\left(-\int\frac{R}{L}dt\right). \tag{24}$$

The system represented by this equation, when multiplied by the proper impedance normalization (LC^{-1}), has the remarkable property that it creates exactly as much energy as the original homogeneous system (5) dissipates and vice-versa. Hence the two systems used in conjunction (connected in series or similarly fed by the same source) form a conservative oscillatory system.

Suppose now that the term associated with the capacitance does not exist (i. e., $1/C=0$). The Riccati equation (22) then has a closed form solution

$$f(t) = \int\frac{1}{L}\exp\left(-\int\frac{R}{L}dt\right)dt + \alpha \tag{25}$$

where α is an arbitrary constant of integration. Hence,

$$S = \frac{1}{2}q^2\left[\int\frac{1}{L}\exp\left(-\int\frac{R}{L}dt\right)dt + \alpha\right]^{-1} \tag{26}$$

From Hamilton-Jacobi theory, q may now be solved for in terms of time and constants (perhaps related) α and β with the relation

$$\beta = \frac{\partial S(q, t, \alpha)}{\partial \alpha} \quad . \tag{27}$$

Therefore,

$$q(t) = \sqrt{2\beta}\left[\int \frac{1}{L}\exp\left(-\int \frac{R}{L}dt\right) dt + a\right] . \qquad (28)$$

Since in this system only the current is of importance, a is of no consequence, and

$$i(t) = \dot{q}(t) = \sqrt{2\beta}\ \frac{1}{L}\exp\left(-\int \frac{R}{L}dt\right) . \qquad (29)$$

This result could have been, of course, easily obtained by direct integration of the first order linear differential equation, but derived in the above manner it does demonstrate the inherent simplicity of the energy approach. Again analytical mechanics has shown itself to be readily applicable where the answer is also obtainable by straightforward methods, but the hope for the method is not in the providing of exact solutions so much as to provide insights into those problems where perhaps exact solutions cannot be obtained.

IV. STABILITY AND PHASE PLANE PHENOMENA

Returning to Hamilton's canonic equations (15), we can formulate in the usual manner an equation for the orbit in the phase plane:

$$\frac{dq}{dp} = \left(\frac{dq}{dt}\right)\left(\frac{dp}{dt}\right)^{-1} = -\frac{C}{L}\exp\left(-2\int \frac{R}{L}dt\right)\frac{p}{q} . \qquad (30)$$

Unlike the usual orbit equations, (3) still exhibits a time-dependence. It may, however, be partially integrated as follows:

$$\int q\,dq = -\int \frac{C}{L} \exp\left(-2\int \frac{R}{L}\,dt\right) p\,dp \tag{31}$$

$$\frac{1}{2} q^2 = K - \int \frac{C}{L} \exp\left(-2\int \frac{R}{L}\,dt\right) p\,dp \tag{32}$$

The second term on the right can be integrated by parts.

$$\int \frac{C}{L}\left(\exp -2\int \frac{R}{L}\,dt\right) p\,dp = \frac{1}{2}p^2 \frac{C}{L} \exp\left(-2\int \frac{R}{L}\,dt\right) - \frac{1}{2}\int p^2 \frac{d}{dp}\left[\frac{C}{L}\exp\left(-2\int \frac{R}{L}\,dt\right)\right] dp\,. \tag{33}$$

But

$$\frac{d}{dp}\left[\frac{C}{L} \exp\left(-2\int \frac{R}{L}\,dt\right)\right] = \frac{d}{dt}\left[\frac{C}{L} \exp\left(-2\int \frac{R}{L}\,dt\right)\right] \frac{dt}{dp}\,. \tag{34}$$

Therefore,

$$-\frac{1}{2}\int p^2 \frac{d}{dp}\left[\frac{C}{L} \exp\left(-2\int \frac{R}{L}\,dt\right)\right] dp = -\frac{1}{2}\int p^2 \frac{d}{dt}\left[\frac{C}{L} \exp\left(-2\int \frac{R}{L}\,dt\right)\right] dt\,. \tag{35}$$

Finally, the integrated orbit equation (32) is

$$\frac{1}{2}q^2 + \frac{1}{2}p^2 \frac{C}{L} \exp\left(-2\int \frac{R}{L}\,dt\right) = K + \frac{1}{2}\int p^2 \frac{d}{dt}\left[\frac{C}{L}\exp\left(-2\int \frac{R}{L}\,dt\right)\right] dt\,. \tag{36}$$

Substituting the expression for p in terms of $\dot{q}$ (12) gives

$$\frac{1}{2}q^2 + \frac{1}{2}LC\dot{q}^2 = K + \frac{1}{2}\int_{t_o}^{t} \dot{q}^2 L^2 \left[\frac{d}{dt}\left(\frac{C}{L}\right) - \frac{2RC}{L^2}\right] dt \tag{37}$$

K is merely a constant depending upon the observed values of q and $\dot{q}$ at the chosen time origin.

Equation (37) may be used to provide a stability criterion: since q^2 and $\dot{q}^2$ are positive, the left hand side of (37) is positive, K is positive; hence, if the integrand in the last term on the right is always negative, the left hand side must decrease to zero. Moreover, if the integrand is always positive q or $\dot{q}$ increase without bound, and instability ensues. The simple stability criteria are

$$\frac{d}{dt}\left(\frac{C}{L}\right) - \frac{2RC}{L^2} < 0, \ \forall t > t_o \Longrightarrow \text{stability} \tag{38a}$$

$$\frac{d}{dt}\left(\frac{C}{L}\right) - \frac{2RC}{L^2} > 0, \ \forall t > t_o \Longrightarrow \text{instability} \tag{38b}$$

This stability criterion has been previously obtained by Gadsden [11] from an entirely different approach. He also obtained the stability criterion

$$\frac{d}{dt}\left(\frac{C}{L}\right) > 0 \quad \forall t > t_o \Longrightarrow \text{stability} \tag{39}$$

It is easily seen that Gadsden's stability criteria are but the extreme two of the set of bounds on q and $\dot{q}$. Upon re-examination of (30), it is seen that it can be broken up as

$$a\,q\,dq = -\,b\exp\left(-\int \frac{R}{L}dt\right) p\,dp, \tag{40}$$

where

$$\frac{b}{a} = \frac{C}{L}. \tag{41}$$

Integration of (40) as above yields

$$\frac{1}{2}aq^2+\frac{1}{2}b\exp\left(-2\int\frac{R}{L}dt\right)p^2 = K+\frac{1}{2}\int_{t_o}^{t} q^2\left[\frac{da}{dt}\right]dt+\frac{1}{2}\int_{t_o}^{t} p^2\frac{d}{dt}\left[b\exp\left(-2\int\frac{R}{L}dt\right)\right]dt \tag{42}$$

Again substituting in (16a) gives

$$\frac{1}{2}aq^2+\frac{1}{2}bL^2\dot{q}^2=K+\frac{1}{2}\int_{t_o}^{t} q^2\left[\frac{da}{dt}\right]dt+\frac{1}{2}\int \dot{q}^2\left[L^2\frac{db}{dt}-2RLb\right]dt. \tag{43}$$

A simple application of the bounding conditions follows upon the choice of

$$a = \exp(-\alpha t) \tag{44a}$$

$$b = \frac{C}{L}\exp(-\alpha t) \tag{44b}$$

Then from (43) it is easily seen that the growth of $\frac{1}{2}(q^2+LC\dot{q}^2)$ is bounded by $\exp(\alpha t)$ when

$$-\alpha \leq 0, \tag{45a}$$

$$-L^2\alpha - L^2\frac{d}{dt}\left(\frac{C}{L}\right) - 2RC \leq 0, \tag{45b}$$

because then the left hand side of (43), which is now $\frac{1}{2}e^{-\alpha t}(q^2+LC\dot{q}^2)$ is decreasing. Hence, $\frac{1}{2}(q^2+LC\dot{q}^2)$ can always be bounded from above by a series of exponential arcs.

Note that the selection $a=L/C$, $b=1$ yields the stability criterion

$$\frac{d}{dt}\left(\frac{L}{C}\right) < 0, \quad \forall t > t_o . \tag{46}$$

Multiplication of the left hand side of the inequality by the negative number $-C^2L^{-2}$ gives

$$-\frac{C^2}{L^2}\frac{d}{dt}\left(\frac{L}{C}\right) = \frac{d}{dt}\left(\frac{C}{L}\right) > 0 \tag{47}$$

Hence, this particular choice of a and b gives Gadsden's second stability criterion.

It is interesting to note that so long as the elements L and C vary between finite positive values, the system point spirals about the origin in the phase plane in the clockwise direction, similar to what would be encountered in the time-invariant system (see fig. 2). Consider the new set of variables in the phase plane

$$r = \sqrt{q^2 + p^2} \tag{48a}$$

$$\theta = \tan^{-1}\frac{p}{q} \quad . \tag{48b}$$

Then,

$$\dot{\theta} = \frac{q^2}{q^2+p^2}\left(\frac{\dot{p}q - \dot{q}p}{q^2}\right) \quad . \tag{49}$$

Substitution of relations (16) gives

$$\dot{\theta} = \frac{1}{q^2+p^2}\left[-\frac{1}{C}\exp\left(\int\frac{R}{L}dt\right)q^2 - \frac{1}{L}\exp\left(-\int\frac{R}{L}dt\right)p^2\right] \tag{50}$$

Substitution of the inverse transformations of (48),

$$q = r \cos \theta \tag{51a}$$

$$p = r \sin \theta \tag{51b}$$

into (50) yields

$$\dot{\theta} = - \left[\frac{1}{C} \exp\left(\int \frac{R}{L} dt \right) \cos^2 \theta + \frac{1}{L} \exp\left(- \int \frac{R}{L} dt \right) \sin^2 \theta \right] \tag{52}$$

Since $\dot{\theta}$ is always negative, the system point must encircle the origin in the clockwise direction.

An approximate solution to the orbit equation can be obtained if the function

$$\left[\frac{C}{L} \exp\left(-2 \int \frac{R}{L} dt \right) \right]$$

is approximated by a series of steps. Its time derivative then gives rise to a train of impulses and the integrand on the right hand side of (36) merely provides new initial conditions for short arc approximations of the phase plane trajectory. This is not the usual piecewise constant approximation and it should be even more accurate.

V. CONCLUSIONS

If the energy function analysis introduced here can be extended to higher order systems, it should enjoy an extensive domain of applicability. Its utility in answering questions of stability and qualitative behavior of networks could make it a powerful tool of network analysis. Work is currently under way to effect the extension by modifying matrix differential equations of the type (3).

APPENDIX I

To demonstrate rigorously that the variational principle (8) leads to the differential equation (5) consider the integral

$$J_{\bar{c}} = \int_{t_1}^{t_2} \exp\left(\int \frac{R}{L}dt\right)\left[\frac{1}{2}L(\dot{q} + \epsilon\dot{\eta})^2 - \frac{1}{2C}(q+\epsilon\eta)^2 + v(q+\epsilon\eta)\right] dt$$

where $\eta(t)$ is an arbitrary function vanishing at t_1 and t_2. We must show that

$$\delta J_{\bar{c}} = \epsilon\left(\frac{\partial J_{\bar{c}}}{\partial\epsilon}\right)_{\epsilon=0} = 0$$

leads to equation (5):

$$\frac{\partial J_{\bar{c}}}{\partial\epsilon} = \int_{t_1}^{t_2} \exp\left(\int \frac{R}{L}dt\right)\left[L(\dot{q} + \epsilon\dot{\eta})\dot{\eta} - \frac{1}{C}(q+\epsilon\eta)\eta + v\eta\right] dt$$

$$\left(\frac{\partial J_{\bar{c}}}{\partial\epsilon}\right)_{\epsilon=0} = \int_{t_1}^{t_2} \exp\left(\int \frac{R}{L}dt\right)\left[L\dot{q}\dot{\eta} - \frac{1}{C}q\eta + v\eta\right]dt$$

Integrating the first term by parts and setting the right hand side equal to zero yields:

$$\left[\exp\left(\int \frac{R}{L}dt\right) L\dot{q}\eta\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\left\{-\frac{d}{dt}\left[L\dot{q}\exp\left(\int \frac{R}{L}dt\right)\right] - \frac{1}{C}q\exp\left(\int \frac{R}{L}dt\right)\right.$$

$$\left. + v\exp\left(\int \frac{R}{L}dt\right)\right\}\eta\, dt = 0 .$$

The bracketed term vanishes because $\eta(t)$ is zero at the end points, and by virtue of the arbitrariness of η (the fundamental theory of the calculus of variations) the term in braces in the integrand must be zero, hence

$$-\frac{d}{dt}\left[L\dot{q}\exp\left(\int\frac{R}{L}dt\right)\right]-\frac{1}{C}q\exp\left(\int\frac{R}{L}dt\right)+v\exp\left(\int\frac{R}{L}dt\right)=0$$

This equation, except for the minus sign introduced in deriving (8), is the equation of motion (5).

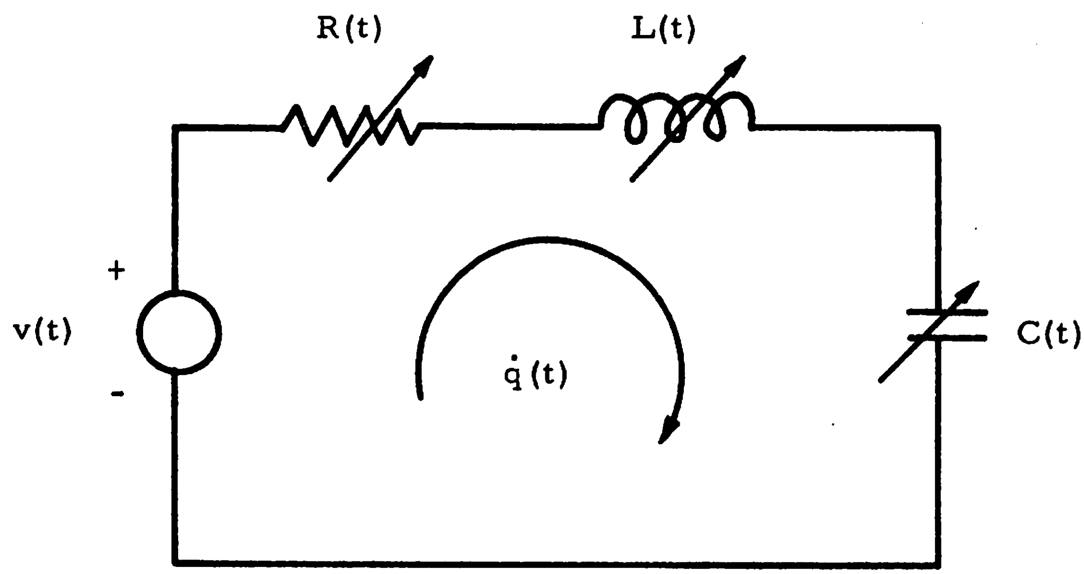

Fig. 1 Time-Varying RLC Loop.

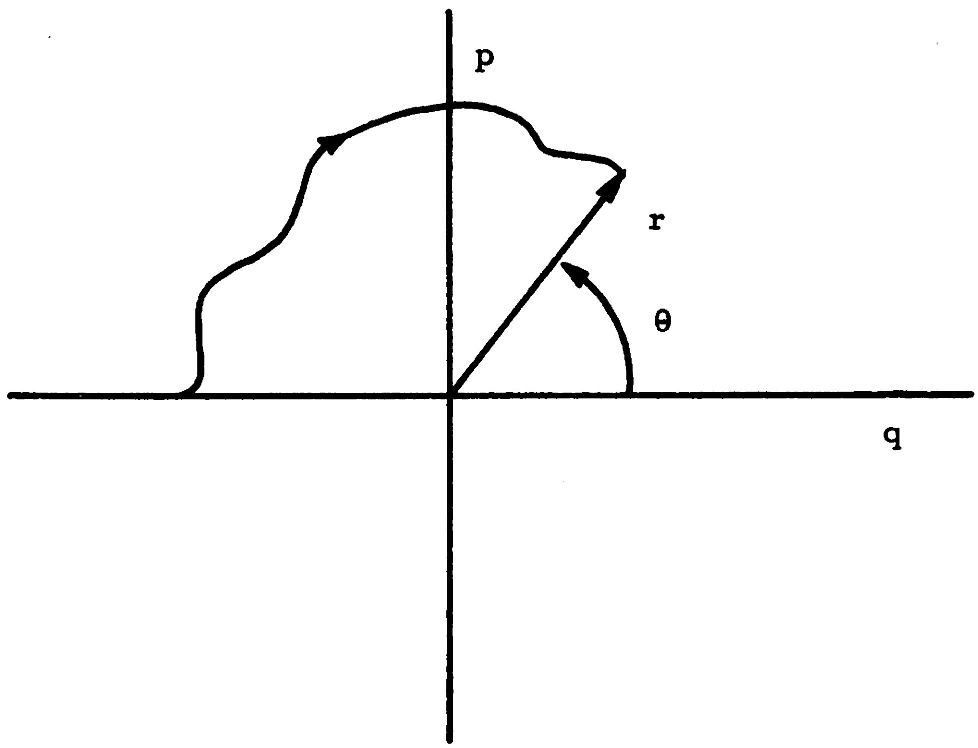

Fig. 2 Phase-Plane and the Associated System Trajectory.

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LIST OF FIGURES

Figure