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EXPERIMENTAL STUDY OF PWM CONTROL SYSTEMS

by

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SUMMARY

This report describes in detail the approximations involved in the statistical study of Pulse Width Modulated Feedback Systems. The exact and the approximate systems are simulated on a digital computer and various statistical quantities are measured and compared. It is shown in this study that the three approximations used in the study of the earlier paper[1] are well justified and yield results within reasonable accuracy. Furthermore theoretical study of the errors involved in the approximation is also discussed with possible modifications for increased accuracy in the analysis.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

I Introduction

In a preceding article,[1] the statistical study of PWM Control Systems has been thoroughly discussed. In this work certain approximations have been made to simplify the analysis. These studies are based on the following three approximations. i) The first approximation involves the replacement of the pulse width modulator by a sampling switch, and a hold circuit followed by a saturating type non-linear element, ii) the second approximation involves the replacement of the non-linear element by equivalent linear element or the use of the so-called separable method and, iii) this approximation involves the statistical independence of the pulses at the output of the saturating element. In this approximation the correlation functions can be evaluated with the use of only the first-order probability density.

These approximations are used in the study of PWM control systems in the preceding paper.[1] In this report we will study these assumptions and the approximations arising from them in a detailed manner and to indicate their validity and to estimate the error involved in using the various models of the original systems. These studies are mainly performed on a digital computer, IBM 704, because of the ease of calculations and the results shown indicate the promising approach initiated in the earlier work.[1] To systemize the experimental study we will briefly discuss the following topics.

II Procedure for Evaluating the Three Approximations by Simulation

In this part, the three approximations mentioned earlier will be discussed in more detail as well as the procedure performed on the digital computer to measure the error involved in this approximation. The system under consideration is shown in Figure 1.

1) The first approximation as noticed from Figures 1 and 2, involves replacing the pulses as indicated earlier at the output of the PWM system which have a varying pulse width by varying amplitude. This approximation facilitates the calculation of the auto-correlation function at the output of the pulse width modulator. This approximation is referred to as the equal area approximation. To check its validity, we perform the following tests:

a) For the exact system we obtain $\phi_{44}^{(E)}(\tau)$, the auto-correlation function of the output. We also obtain the auto-correlation function of the approximate system as shown in Figure 2, denoted as $\phi_{44}^{(A)}(\tau)$. Then we compare these to find the error involved under the assumption that the time constant of the plant following the PWM is at least twice the sampling period.

b) We also obtain the auto-correlation function of the error signal both for the exact and the approximate models, denoted by $\phi_{22}^{(E)}(\tau)$ and $\phi_{22}^{(A)}(\tau)$ respectively and compare these curves to obtain the error of approximation.

The above two measurements would give us an idea of the error involved in the equal area approximation.

2) The second approximation based on the concept of separable processes[6] is needed to obtain the cross-correlation between the input and output for a feedback PWM system. This approximation has been utilized as shown in Figure 3 and Equations 79 and 80 of Reference 1. To check the

validity of this approximation the following is performed:

We want to find the relationship as expressed by Equation 79 of Reference 1

$$\phi_{13}(\tau) = f\left[\phi_{12a}(\tau)\right] \tag{1}$$

The above can be written by linearizing the non-linear element as (Figure 3) Equation 80 of Reference 1

$$\phi_{13}(\tau) = k \cdot \phi_{12a}(\tau) \tag{2}$$

We check the validity of the above approximation by obtaining $\phi_{13}(\tau)$ and $\phi_{12a}(\tau)$ for the system in Figure 2. Then we check whether Equation (2) is a good approximation and what error is involved. Thus we obtain the error of approximation involved in the calculation of the cross-correlation function.

3) To check the validity of the third approximation that the pulses at the output of the nonlinear element are statistically independent we perform the following:

We obtain the auto-correlation function $\phi_{44}(\tau)$ for the system of Figure 2 and compare it with the theoretical one which can be calculated from Equation 75 of Reference 1,

$$\Phi_{44}(s) = \mathcal{L}\left[\phi_{44}(\tau)\right] = \frac{W(e^{sT} + e^{-sT} - 2)}{Ts^2} \cdot G(s) \cdot G(-s) \tag{3}$$

The above will indicate the error introduced in this approximation. In the next section, the techniques of simulation will be discussed.

III Simulation on IBM 704

The exact and approximate systems shown in Figures 1 and 2 are simulated on the digital computer. The input Gaussian process (whose correlation function is assumed to be $\sigma^2 e^{-\beta|\tau|}$) is generated by a random number generator and a digital computer subroutine is used. To get this correlation function, we used a filter of the form $\sigma/s+\beta$. Thus, for simulation purposes the exact and approximate systems are shown in Figures 4 and 5.

The random numbers having a Gaussian distribution are called X(K). Those numbers are fed into a filter as shown in the figures to generate the desired correlation for $X_1(K)$. Using rectangular integration rules, we can write

$$X_1(K) = T_c X(K-1) + (1-T_c)X_1(K-1) \tag{4}$$

where T_c is the sampling interval on the IBM 704.

We also have the figures (assuming $G(s) = \dfrac{1}{\nu s + 1}$)

$$X_2(K) = X_1(K) - X_4(K) \tag{5}$$

and

$$X_4(K) = (T_c/\nu)X_3(K-1) + (1-T_c/\nu)X_4(K-1) \tag{6}$$

The relationships between $X_3(K)$ and $X_2(K)$ are quite different for the exact and the approximate model and the digital computer programs indicating this are readily obtained.

The correlation functions $\phi_{44}^{(E)}(\tau)$, $\phi_{44}^{(A)}(\tau)$, $\phi_{22}^{(E)}(\tau)$ and $\phi_{22}^{(A)}(\tau)$ are also obtained from the computer program. The input correlation $\phi_{11}(\tau)$ is obtained on the computer to be checked with the theoretically assumed one, namely,

$$\phi_{11}(\tau) = \sigma^2 e^{-\beta|\tau|} \tag{7}$$

Furthermore, for the approximate model we also obtained the correlation functions

$$\phi_{13}(\tau),\ \phi_{15}(\tau),\ \phi_{33}(\tau),\ \phi_{55}(\tau) \text{ and } \phi_{53}(\tau)$$

In the above situation study we assumed:

$$\begin{aligned} \beta &= 1 \\ \nu &= 1 \\ T &= 0.2 \text{ sec.} \\ T_c &= 0.04 \text{ sec.} \end{aligned} \tag{8}$$

IV Accuracy of the Results

It is evident that the accuracy of the measurements depends on a) the length of the sampling interval T_c and, b) on the length of the total sample. The latter length is limited by the memory of the computing machine. To check the above accuracy we compared the input correlation function given by the computer with the theoretically assumed. As noticed from Figure 6 this error is very small. The discrepancy gives an indication of the error anticipated in the calculation of the other correlation functions.

In one experimental study, we used 4000 samples at .04 seconds apart. The sampling period of the system in Figure 4 is .2 sec. The error at point (2) which is fed into the PWM and the linear plant is taken only at .2 sec. intervals. This seems quite satisfactory if we make the computer sampling period .04 sec. Four thousand samples of X, X_1, X_2, X_3, X_4, and X_5, are stored in memory to be utilized later to calculate the correlation functions for values of τ between .04 seconds to 2 seconds. The whole procedure is done on the Fortran Computer Program.

The dependence of experimental variance on correlation time τ and sampling interval T_c is known in the literature[2,3] and for those chosen values the error involved is very small. This error is given in terms of standard deviation of one measured correlation function and is less than 10 per cent of the true variance. These errors are quite standard and do not affect the input correlation substantially from the theoretical input. This is an indication of the good accuracy of the results.

V Discussion of Experimental Results

We start the discussion by comparing the auto correlation function of the input as obtained in the simulated system with the theoretically assumed one. This is shown in Figure 6, where it is noticed that for small values of "τ" there is no appreciable difference. This difference increases percentwise as "τ" becomes larger as is expected because of the finite sample length which can be obtained on the computer study. We must also realize that the random numbers generated in the experimental study is a completely deterministic process and any set of random numbers can be generated at will. The experimental curve as noticed from the figure is therefore a reasonably good fit, and now we can study the other experimental curves to obtain an idea of the errors involved in our three approximations.

1) To investigate the equal area approximation we have plotted the auto-correlation function of the output and error for the exact as well as the approximate model simulated on the digital computer. These are shown

in Figures 7 and 8 respectively. The actual simulated input, output and error were also printed out over a ten second period for both the exact and the approximate model to study any major differences. This proved to be very satisfactory. The exact model simulation yielded input, output and error and is shown in Figure 9 over a 10 second period. Figures 10 and 11 respectively show the output and error for both the exact and approximate model over a ten second period. Studying these, it seems that there is essentially very little difference and is quite safe to make equal area approximation. By re-examining Figure 7 we see that the Mean Square Value at the output for the approximate case is about 10 per cent higher than the exact case. For large values of τ (i.e. $\tau > .6$) there is no appreciable difference between the two output correlation functions. This small error can be expected. Similarly, if we study the error auto-correlation function in Figure 8 we also notice about 10 per cent difference in mean square values and there is also a difference in shape of correlation functions between $\tau = .6$ sec. and $\tau = .8$ sec., otherwise there is no essential difference. These changes are again to be expected. Hence we can say that the approximate method gives about 10 per cent higher MSVs than the exact method but is within reason.

2) The second approximation is used to linearize the non-linear element to calculate certain cross-correlation functions and depends upon the relationship (see Figure 5),

$$\phi_{13}(\tau) = f\left[\phi_{15}(\tau)\right] \qquad (9)$$

The cross-correlation functions $\phi_{13}(\tau)$ and $\phi_{15}(\tau)$ are shown in Figure 12.

Also, the ratio $\phi_{13}(\tau)/\phi_{15}(\tau)$ versus t is shown. We notice that this ratio almost remains a constant and at some values it deviates to a maximum of about 6 per cent. We indeed find that

$$\frac{\phi_{13}(\tau)}{\phi_{15}(\tau)} = 4.93 \tag{10}$$

This was expected because Gaussian Process is a Separable Process[7] and we are able to write

$$\phi_{13}(\tau) = K.\phi_{15}(\tau) \tag{11}$$

This proves that we are justified in linearizing the non-linear element for calculation of cross-correlations as given in Part H of reference 1.

3) The third approximation which assumes statistical independence of pulses at the output of Pulse Width Modulator is also checked by simulation. We note that what we are really interested in is the value of $\phi_{44}(0)$ because this is the only place where we use this approximation. We have shown that $\Phi_{44}(s)$ can be written in terms of the MSV of error and G(s). This is given by equation (3) and can be written as

$$\Phi_{44}(s) = \frac{W(e^{sT} + e^{-sT} - 2)}{Ts^2} \cdot G(s) \cdot G(-s) \tag{12}$$

where

$$W = \frac{\alpha^2 \sigma_2^2}{T^2} \left[2\phi\left(\frac{T}{\alpha\sigma_2}\right) - 1 \right] + \left(1 - \phi\left(\frac{T}{\alpha\sigma_2}\right)\right) - \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha\sigma_2}{T} \cdot e^{-\frac{T^2}{2\alpha^2\sigma_2^2}} \tag{13}$$

In our case

$$G(s) = \frac{1}{s+1} \tag{14}$$

Hence we can write

$$\Phi_{44}(s) = \frac{W(e^{sT} + e^{-sT} - 2)}{Ts^2} + \frac{W(e^{sT} + e^{-sT} - 2)}{T(1 - s^2)} \tag{15}$$

$$= \Phi_{44}^{(1)}(s) \qquad + \quad \Phi_{44}^{(2)}(s) \tag{16}$$

We also know that

$$\phi(\tau) = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \Phi(s) e^{s\tau} ds \tag{17}$$

By complex plane integration we can get

$$\phi_{44}^{(1)}(\tau) = \frac{W}{T}\Big[(T-\tau)(1(\tau)-1(\tau-T)) + (T+\tau)(1(-\tau)-1(-\tau-T))\Big] \tag{18}$$

$$\phi_{44}^{(2)}(\tau) = \frac{W}{T}\Big[-2e^{-\tau}1(\tau)-2e^{\tau}1(-\tau) + e^{-\tau+T}1(\tau-T) + e^{\tau-T}1(-\tau+T) + e^{\tau+T}1(-\tau-T) + e^{-\tau-T}1(\tau+T)\Big] \tag{19}$$

where $1(\tau)$ is the unit step function.

Hence we can get

$$\phi_{44}(\tau) = \phi_{44}^{(1)}(\tau) + \phi_{44}^{(2)}(\tau) \tag{20}$$

For a given value of the input MSV, the value of W is a constant because T and α are already chosen to be 0.2 sec. and 1 respectively. Equation (20) can now be plotted as $\phi_{44}(\tau)$ versus τ. The results are tabulated below and are plotted in Figure 13.

τ sec.	0	.2	.4	.6	1.0	1.4	1.6	2.0
$\phi_{44}(\tau)$	.0936W	.082W	.067W	.055W	.037W	.025W	.02W	.0136W

We have already simulated the approximate model from which above correlation $\phi_{44}(\tau)$ has been obtained. In the simulated case we know from the digital computer study that $\phi_{44}(0) = 0.0159$. This is plotted in Figure 13.

In the theoretical case we need W as seen in the above table. To get W we need σ_2 as is obvious by looking at equation (13). For G(s) as given by equation (14) and T = 0.2 sec. and $\alpha = 1$ we can solve equation (103) in example in earlier paper.[1] We also need the MSV of input which is obtained from the simulation study and is given by $\sigma_1^2 = 0.0224$. This is approximate. This yields the value of $\sigma_2 \cong .09$. Using this value in equation (13) we have approximate value of W = 0.2.

We must remember that we are dealing with very small numbers numerically and hence there are bound to be errors involved. With this value of W we can calculate the theoretical output correlation function as given in the table below. The theoretical as well as the simulated output correlation functions are tabulated in the table below and plotted in Figure 13. This is for the approximate model.

τ	0	.2	.4	.6	1.0	1.4	1.6	2.0 sec.
Theoretical $\phi_{44}(\tau)$	.0187	.0164	.0134	.011	.0074	.005	.004	.0027
Simulated $\phi_{44}(\tau)$ on Approx. Model	.016	.014	.011	.008	.0043	.0025	.002	.0013

Figure 13 shows that the results are reasonably close. The mean square values are within 15 per cent of each other. The shape of both the correlation functions is the same. There is some error involved in the determination of W and it is quite possible that the error involved is much lower. The similar shape of both correlation functions is an excellent sign of the reasonableness of this approximation. Hence we conclude that the approximation that assumes statistical independence of pulses at the output of PWM is within reason for our study.

VI Theoretical Consideration of Errors

1) Andeen[4] implies in equal area approximation that one can replace the Pulse Width Modulator by Pulse Amplitude Modulator if the dominant time constant of the linear plant following the Pulse Width Modulator is large (at least twice) compared with the sampling period. This is not generally true as shown by Kadota.[5] He shows that it entirely depends on the form of G(s). This is shown by δ-function expansion. If the linear plant is of the form 1/s+1 as we have assumed for our examples, the equal area approximation is indeed true as will be seen below. If G(s) is of higher order then this approximation is valid for some values of t and error for other values of t has to be evaluated. Let us consider this in some detail.[5] The one efficient way of studying this approximation is to look at the output of the linear plant G(s) when the input to it are pulses coming out of the Pulse Width Modulator and also the case when PWM is replaced by PAM. This will give us some idea as to when the equal area approximation is valid. If the input to the linear plant is f(t) we can write the output x(t) as

$$x(t) = \int_{t_o}^{t} g(t-\gamma)\cdot f(\gamma)d\gamma \tag{21}$$

assuming the initial conditions to be zero at $t = t_o$.

This above integral can be expanded in the form of δ-function expansion. We can write

$$x(t) = \int_{t_o}^{t} g(t-n)f(n)dn \tag{22}$$

where

$$f(n) = \sum_{i=o}^{\infty} a_i \, \delta^{(k)}(n-t_o) \tag{23}$$

$$a_i = \frac{(-1)^i}{i} \int_{t_o}^{t} \delta(\gamma - t_o)f(\gamma)d\gamma \tag{24}$$

Where $\delta^{(k)}$ is the k^{th} derivative of δ-function and these are symbolic functions defined in the sense of distributions. The δ-function expansion can also be obtained by rewriting equation (21)

$$x(t) = \int_{t_o}^{t} d\gamma \int_{t_o}^{t} \delta(n-\gamma)g(t-n)f(n)dn \tag{25}$$

and formally expanding $\delta(n-\gamma)$ in Taylor's expansion with respect to γ at $\gamma = t_o$. The output $x(t)$ can be interpreted for input $f(n)$, $t_o < n < t$, as the sum of response of the system to δ-function and its higher derivatives at $n = t_o$ whose moments are given by weighted integrals of $f(n)$ from t_o to t.

In case of the exact model of PWM system we have the input to the linear plant as

$$f(t) = H(t) - H(t-h) \quad , \quad h > 0 \tag{26}$$

where h is the width of the pulse.

Expanding equation (26) in δ-function expansion we can write equation (22) as ($t_o = 0$)

$$x(t) = \int_{t_o}^{t} g(t-\gamma)\left[t\,\delta(t) - \frac{1}{2}t^2\,\delta^{(1)}(t) + \ldots.\right]d\gamma$$

$$= tg(t) - \frac{1}{2}t^2\,g^{(1)}(t) + \ldots., \quad t<h \tag{27}$$

$$x(t) = \int_{t_o}^{t} g(t-\gamma)\left[h\,\delta(t) - \frac{1}{2}h^2\,\delta^{(1)}(t) + \ldots.\right]d\gamma$$

$$= hg(t) - \frac{1}{2}h^2 g^{(1)}(t) + \ldots., \quad t \geq h \tag{28}$$

This gives us the output of the linear plant.

In the case of the approximate model of the PWM system where we replace the PWM with the PAM we can write the input to the linear plant as

$$f_a(t) = \frac{h}{T}\left[H(t) - H(t-T)\right] \tag{29}$$

Hence the output becomes

$$x_a(t) = \frac{h}{T}\int_{0}^{T} g(t-\gamma)d\gamma \tag{30}$$

Equations (27) and (30) should then be used for any value of G(s) and error evaluated. In the case of the linear plant of the first order we prove that the Andeen approximation is good.

<u>Example:</u>

Let $G(s)=\dfrac{1}{vs+1}$

Then $g(t) = \dfrac{1}{v}\cdot e^{-t/v}$ (31)

Exact model output for this g(t) is then given by equations (27) and (28)

$$x(t) = t/v \cdot e^{-t/v} + t^2/2v^2 \cdot e^{-t/v} \ldots, \quad t \leq h \tag{32}$$

$$x(t) = h/v \cdot e^{-t/v} + h^2/2v^2 \cdot e^{-t/v} \ldots, \quad t \geq h \tag{33}$$

Approximate model output is given by equation (30)

$$x_a(t) = h/v \cdot e^{-t/v} + hT/2v^2 \cdot e^{-t/v} \tag{34}$$

We now have to compare the outputs given by equations (32) and (34).

We notice that for $t \geq h$ the first term of the expansions is the same and therefore error is in higher order terms and now if we make v large compared to T, say ($v \geq 2T$), the error would be small and actually is given by

$$\text{error} = \left[\frac{hT}{2v^2} - \frac{h^2}{2v^2} \right] e^{-t/v} + \ldots, \quad t \leq h \tag{35}$$

Compared to the first term this is small. Hence, we can say that the equal area approximation as defined previously is all right for the first-order linear plant.

For higher-order plants this above study does give an indication that the equal area approximation is always good provided the second- or higher-order terms in the δ-function expansion can be neglected. For $t \leq h$, the error will increase at $t \to 0$. Error in this case is given by

$$\text{error} = e^{-t/v} \left[\frac{(t-h)}{v} + \frac{(t^2-hT)}{v^2} + \ldots \right], \quad t \leq h \tag{36}$$

Kadota[5] has shown that for a second-order linear plant the approximation is only valid for $t \gg h$ but there error is multiplied by the factor of 2 at $t = h$ and the approximation is not valid for $t < h$. In this case the second-order term can not be neglected and the error will be introduced for $t < h$. So, if we consider a second-order linear plant we must keep this fact in mind. In general, we can only make the equal area approximation without introducing any serious error if second- and higher-order terms in the δ-function expansion can be neglected.

2) Nuttal[6] has proved that the separability property of the input Random Process is a necessary and sufficient condition that the cross-correlation function across the non-linear element is proportional to the auto-correlation function of the input to the non-linear element. This has been proved exclusively for the Gaussian Process by Bussgang.[7] In the experimental verification we noted that there is a very slight deviation. This is entirely due to the finite length of the samples taken. Hence, the cross-correlation functions which have been obtained using this fact introduce no error other than any error introduced in the calculation of the value K.

VII Conclusions

The experimental study performed in this paper indicate that the approximations, which are required for the study of PWM systems for statistical inputs, is within reasonable accuracy. It is shown, in most cases, the error involved is about 10 per cent of the exact values. This error could be further reduced if higher-order terms in the equivalent area approximation are considered. The theoretical justification of the approximations discussed in this paper is of importance in future studies of such systems. Most of the relationships obtained in the analysis can be extended to any order system and can be solved with the aid of the computer.

Although the approximation error is shown to be within engineering accuracy for the particular system simulated in this study, it is felt that those approximations could be improved further for higher-order systems. Thus, the analysis procedure is fairly general and could be safely attempted for any system.

In conclusion, one can state that within the framework of these approximations, the analysis of PWM systems as well as other non-linear discrete systems for statistical inputs, can be readily achieved.

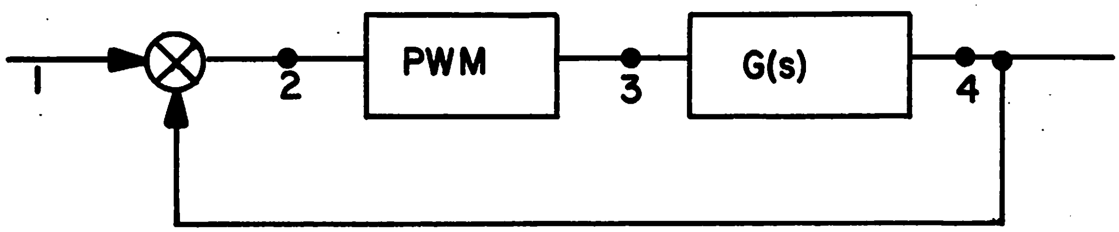

FIGURE 1

GENERAL CLOSED LOOP PWM SYSTEM

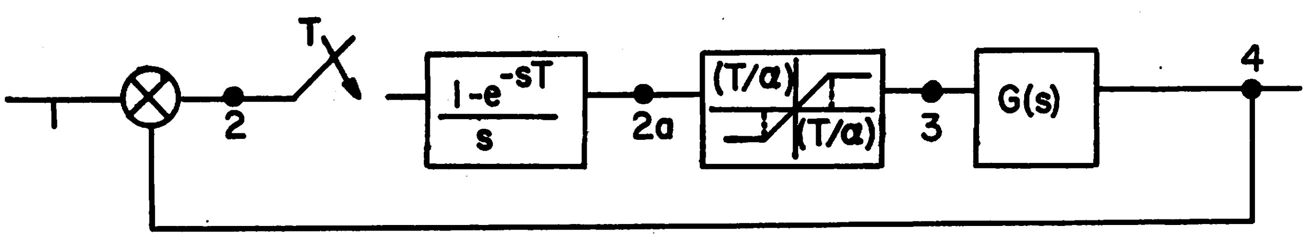

FIGURE 2

APPROXIMATE MODEL OF CLOSED LOOP PWM SYSTEM

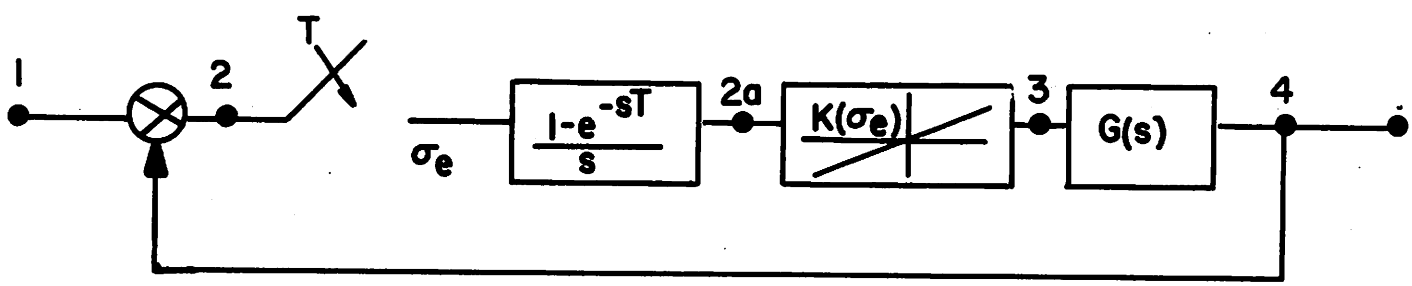

FIGURE 3

APPROXIMATE MODEL FOR CALCULATION OF CROSS CORRELATION FUNCTION

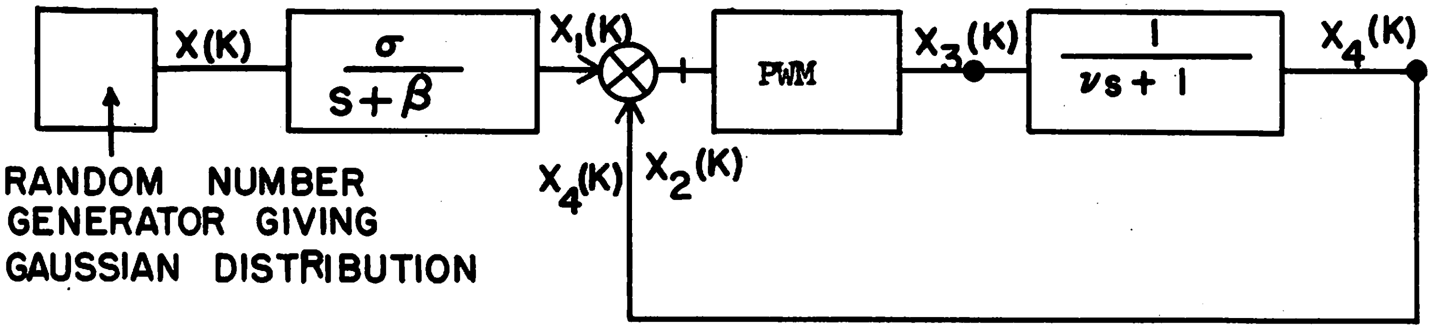

FIGURE 4

EXACT MODEL OF PWM SYSTEM FOR SIMULATION ON DIGITAL COMPUTER

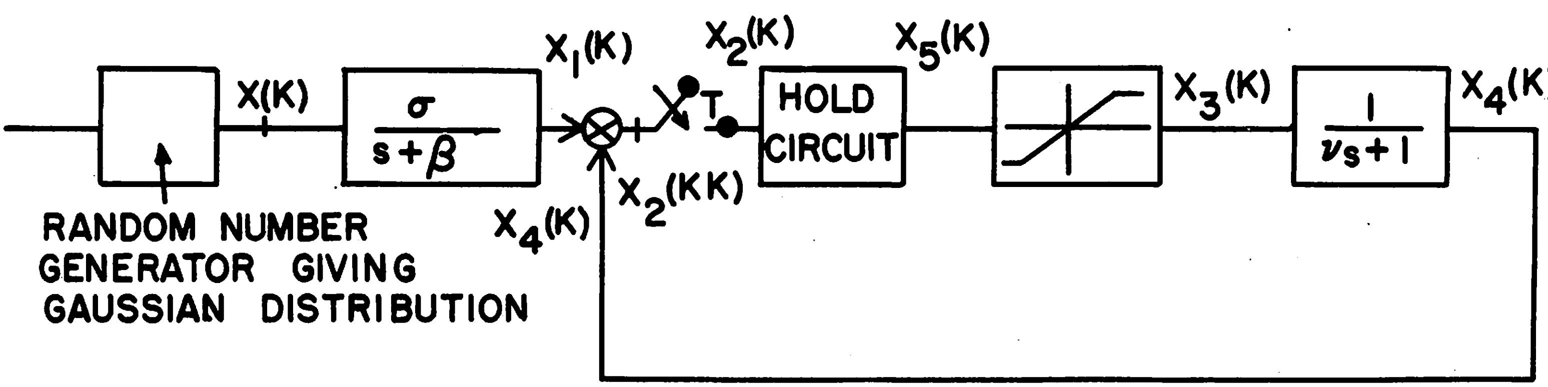

FIGURE 5
APPROXIMATE MODEL OF PWM SYSTEM FOR SIMULATION ON DIGITAL COMPUTER

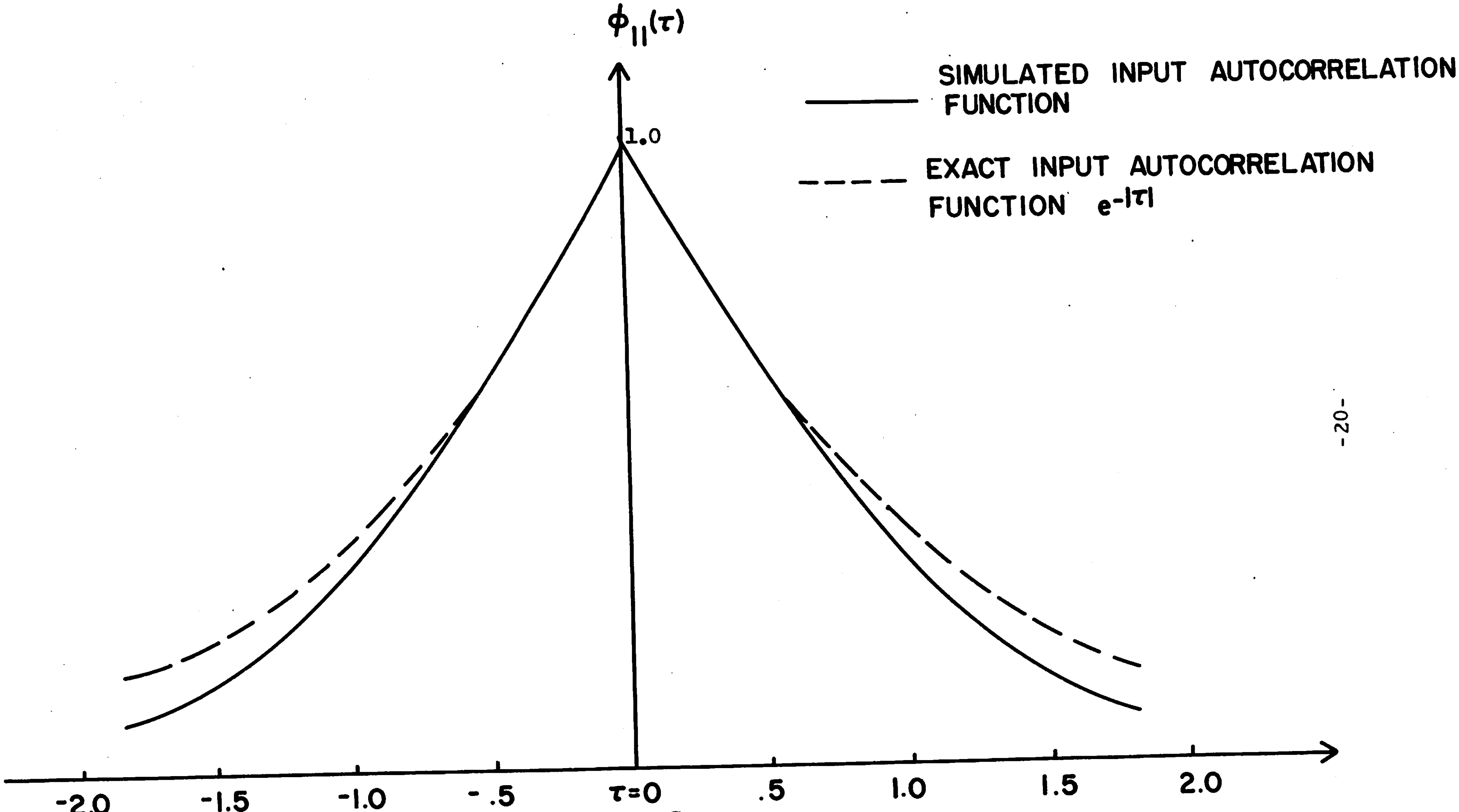

FIGURE 6

INPUT AUTOCORRELATION FUNCTION OBTAINED BY SIMULATION AS WELL AS THEORETICALLY ASSUMED ONE

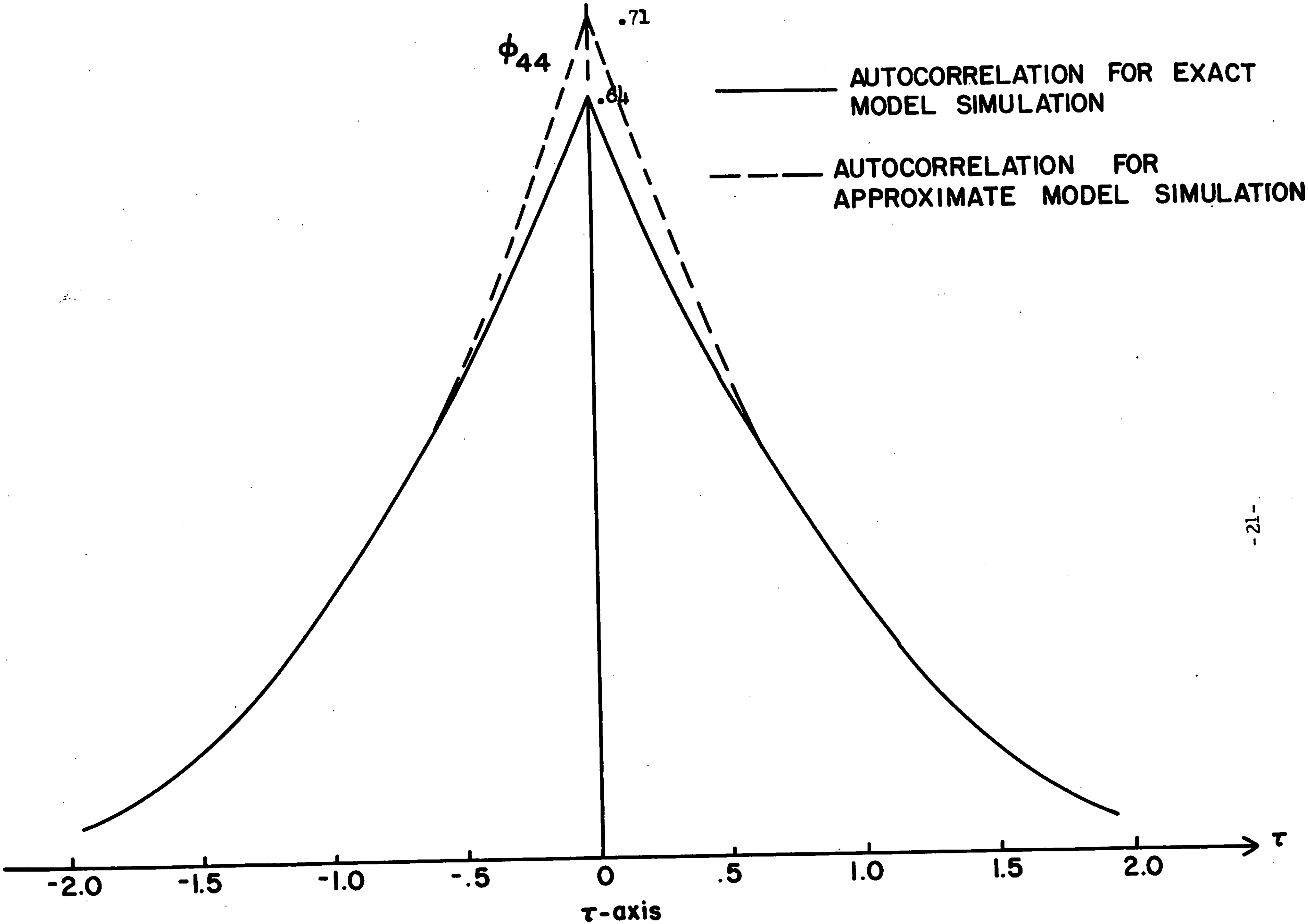

FIGURE 7

OUTPUT AUTOCORRELATION FUNCTION BY SIMULATION OF BOTH THE EXACT AND THE APPROXIMATE MODEL

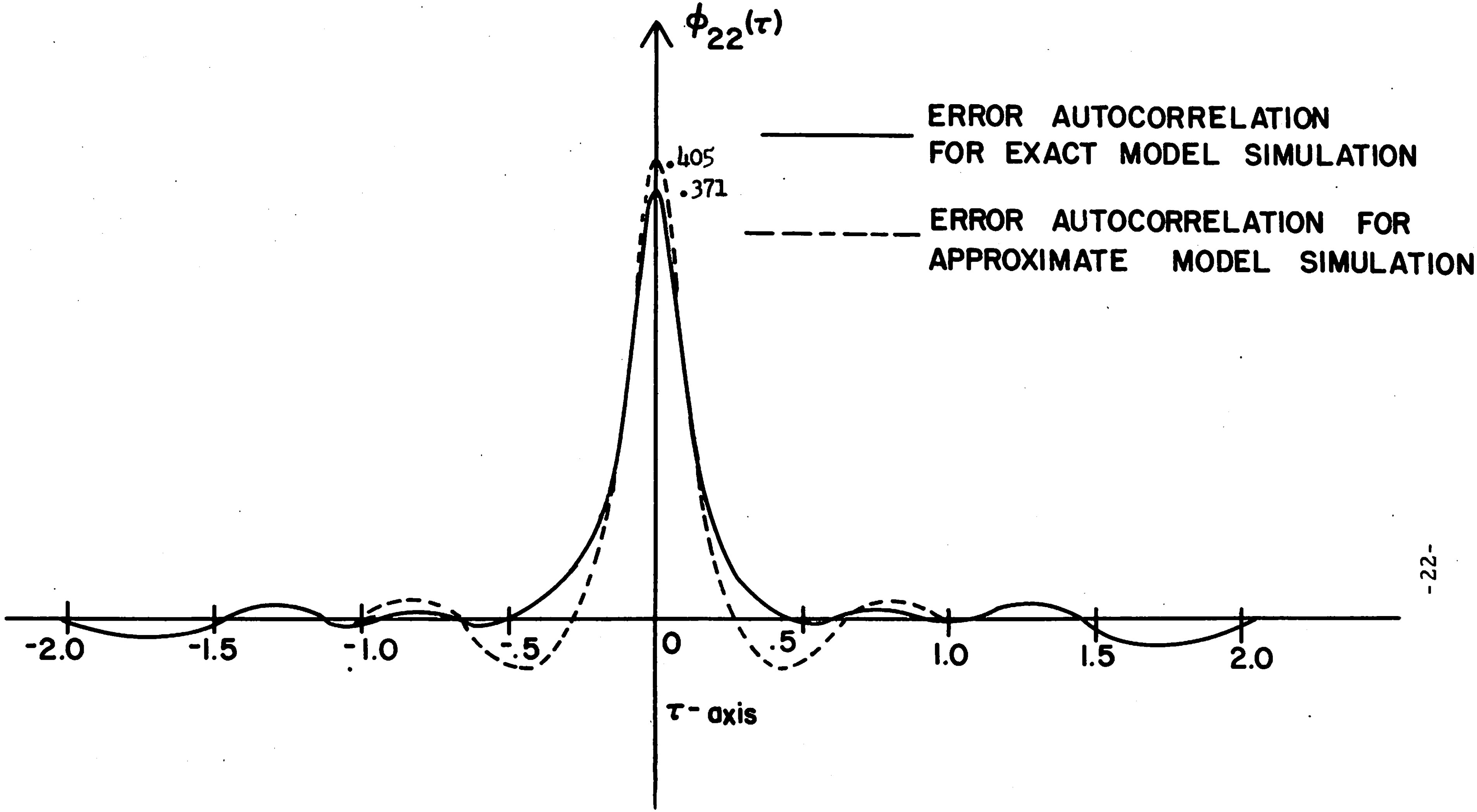

FIGURE 8

ERROR AUTOCORRELATION FUNCTION OF THE EXACT AS WELL AS THE APPROXIMATE MODEL OF PWM SYSTEM BY SIMULATION

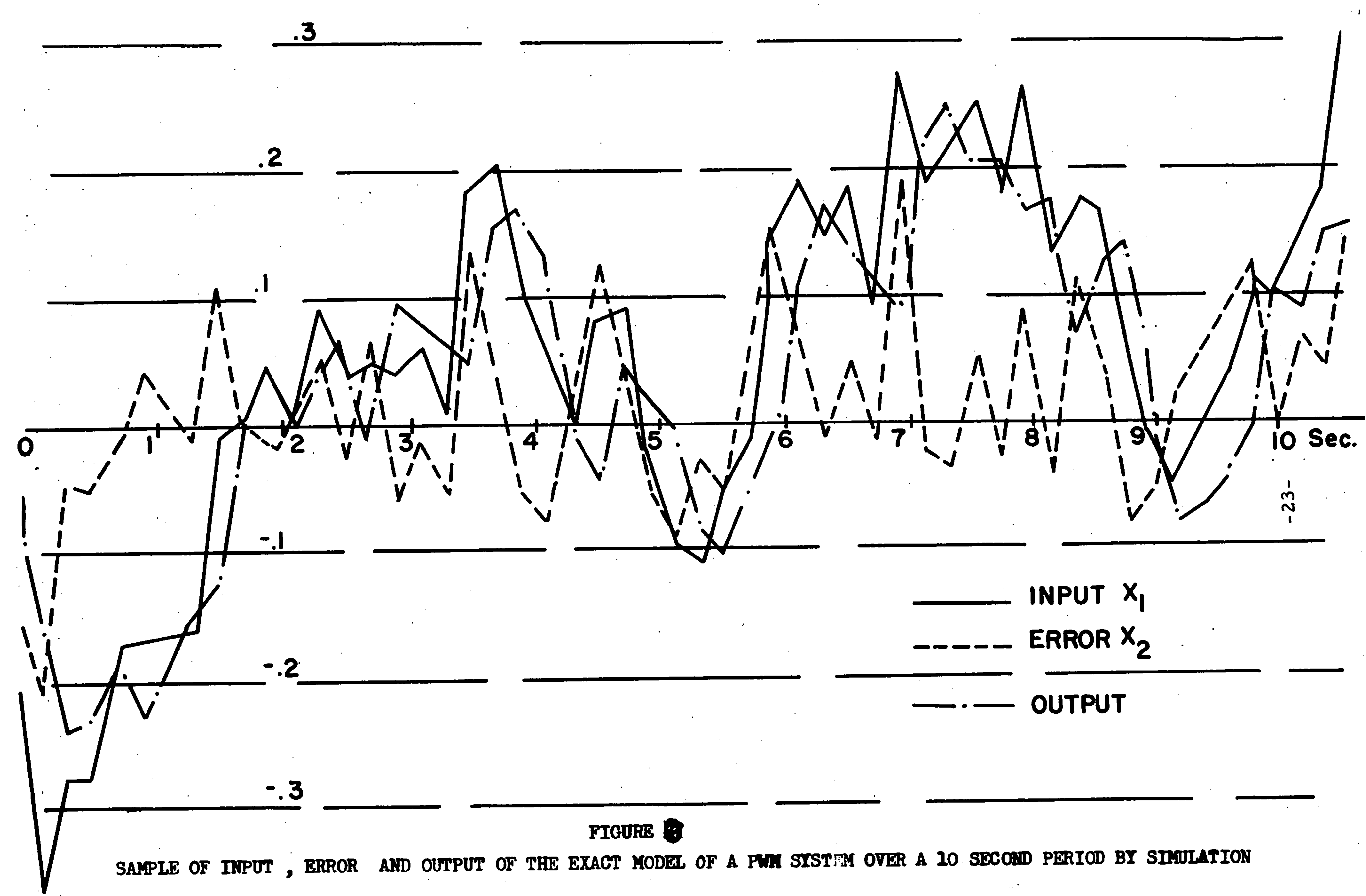

FIGURE [illegible]

SAMPLE OF INPUT , ERROR AND OUTPUT OF THE EXACT MODEL OF A PWM SYSTEM OVER A 10 SECOND PERIOD BY SIMULATION

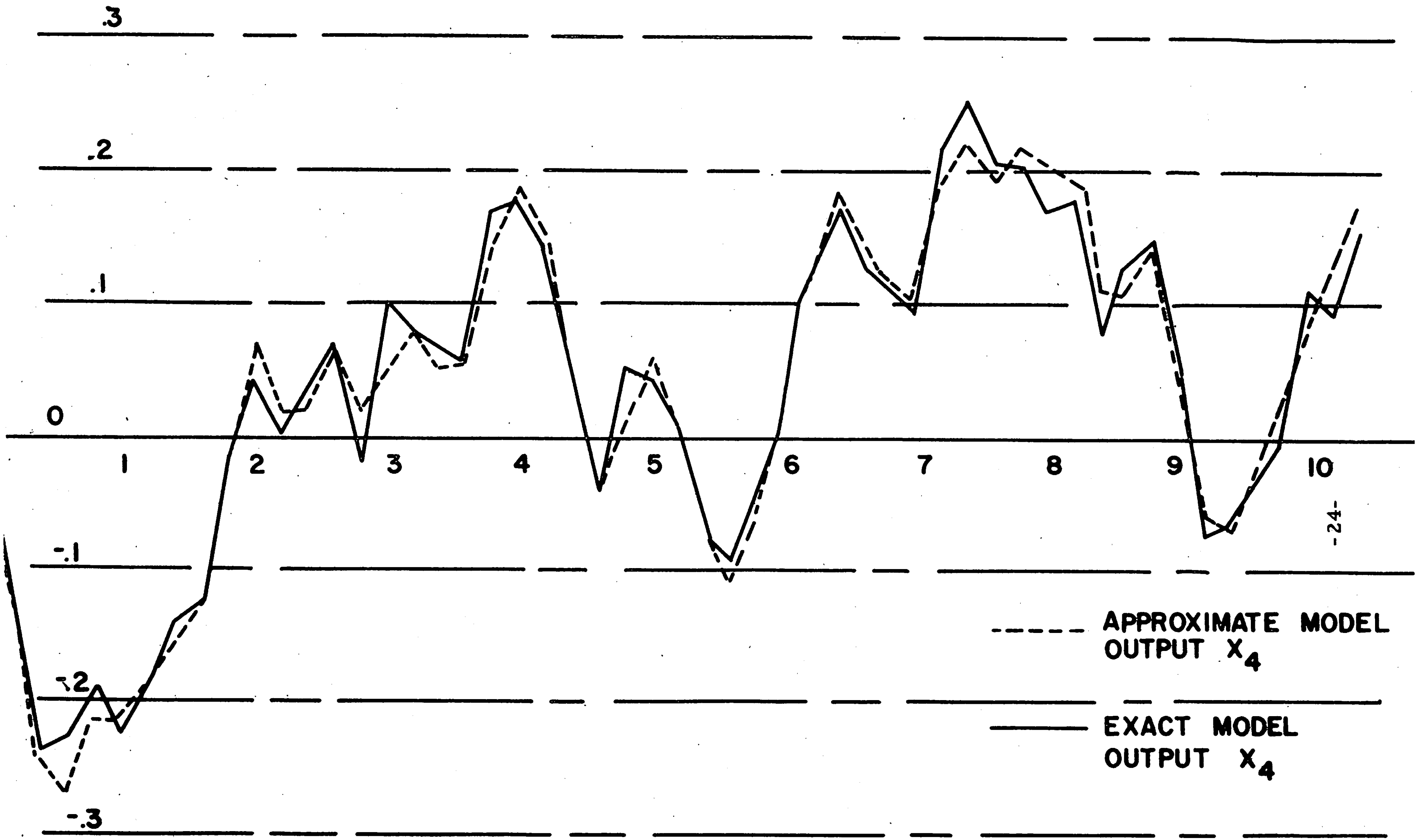

FIGURE 10
SAMPLE OF OUTPUT X_4 FOR EXACT AND APPROXIMATE MODEL OF A PWM SYSTEM OVER A 10 SECOND PERIOD BY SIMULATION

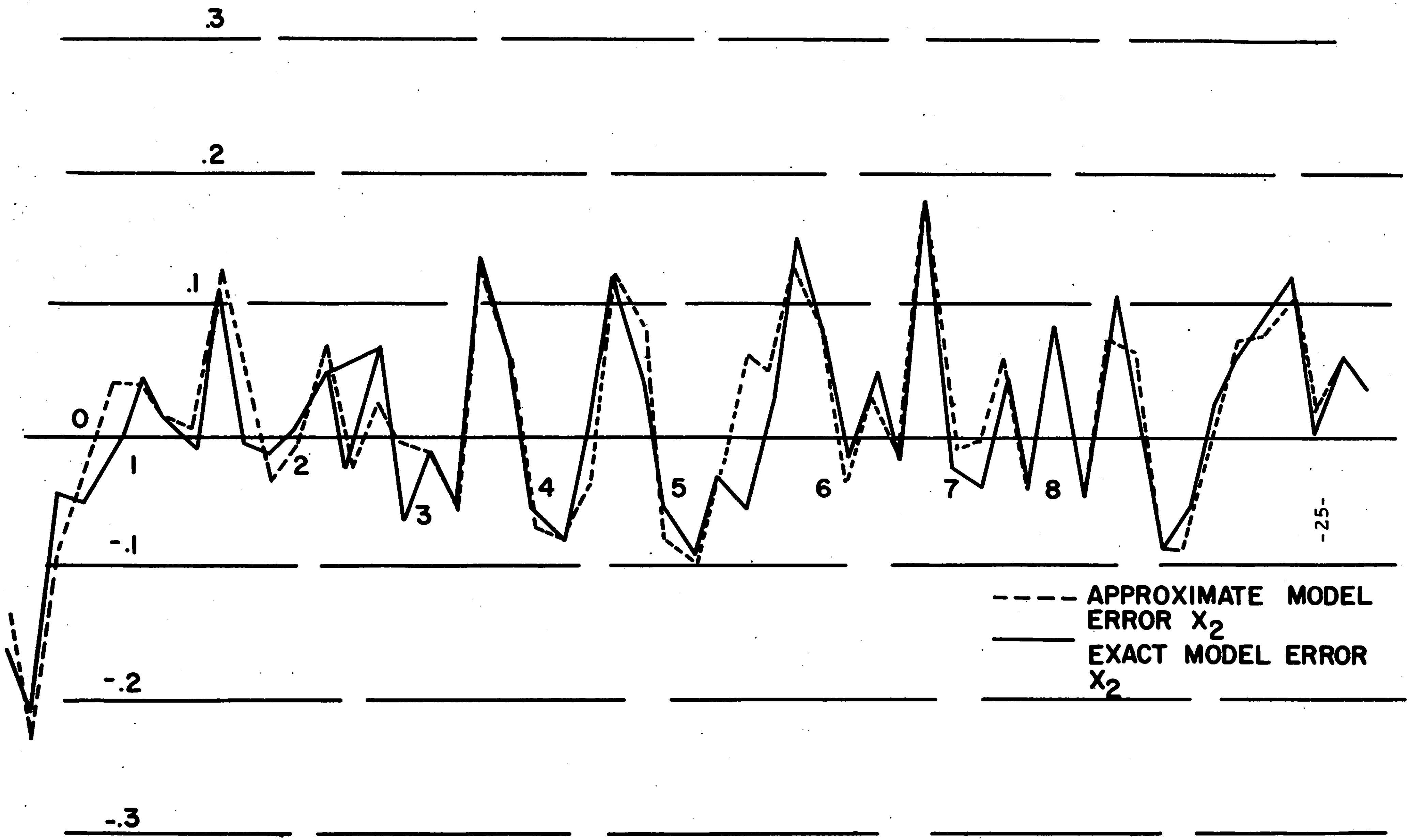

FIGURE 11

SAMPLE OF ERROR X_2 FOR EACH AND APPROXIMATE MODEL FOR A PWM SYSTEM OVER A 10 SECOND PERIOD BY SIMULATION

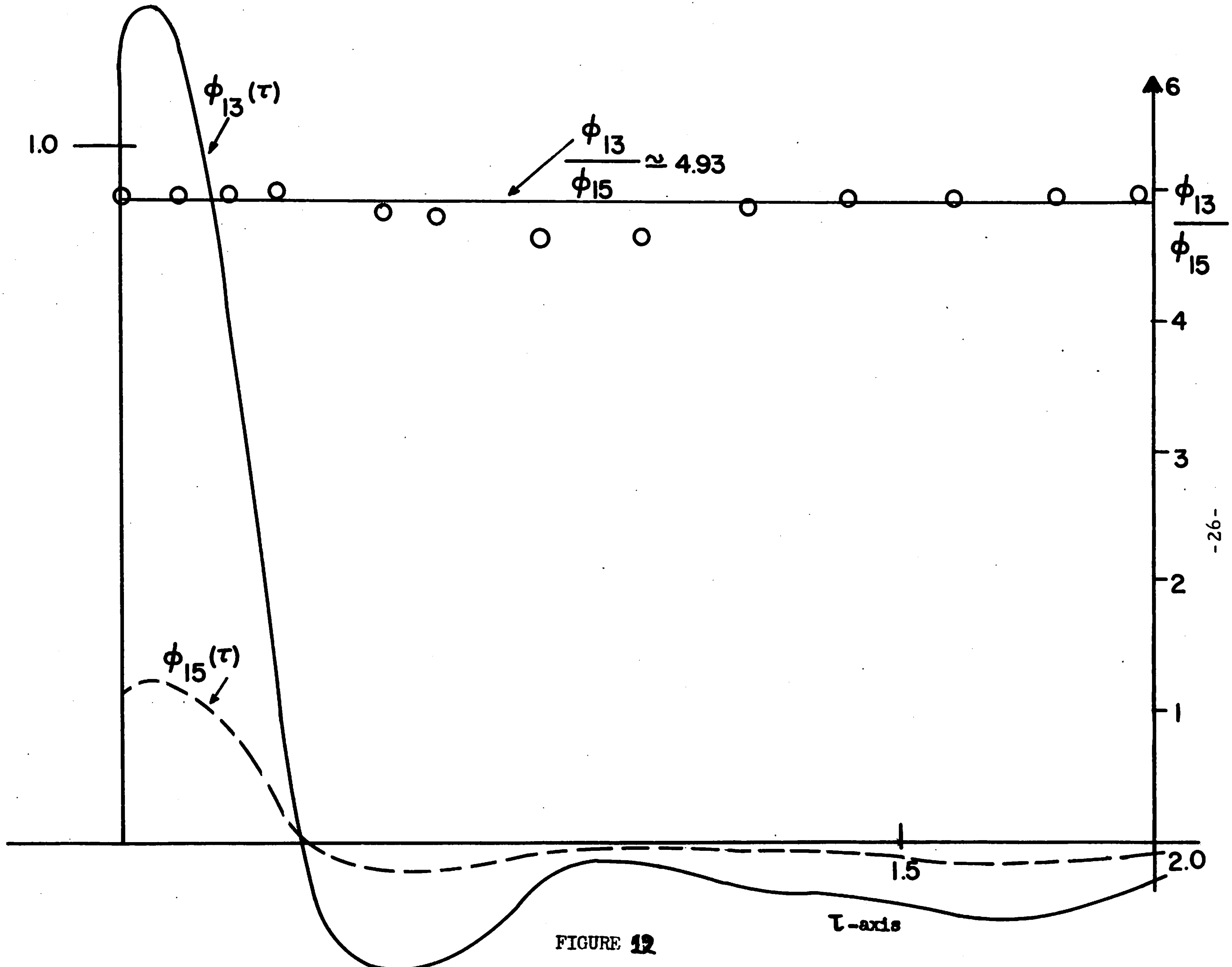

FIGURE 12

RELATIONSHIP $\phi_{13} = f[\phi_{15}]$ IS OBTAINED BY PLOTTING ϕ_{13} , ϕ_{15} , ϕ_{13}/ϕ_{15} VERSUS τ

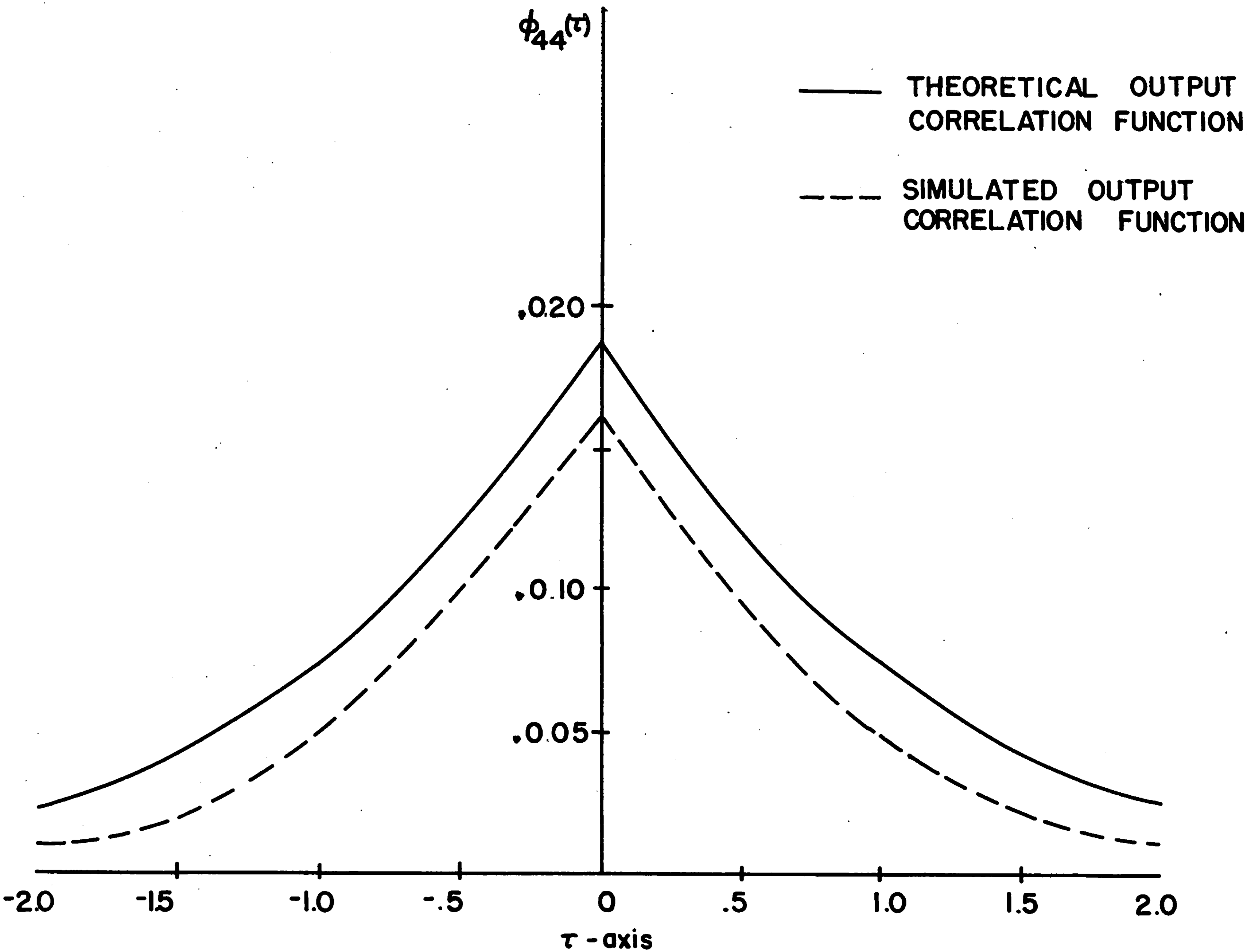

FIGURE 13

OUTPUT CORRELATION FUNCTION FOR APPROXIMATE MODEL

References

1. Gupta, S. C., and E. I. Jury, "Statistical Study of Pulse-Width Modulated Control Systems," Journal of Franklin Institute, Vol. 273, No. 4, April 1962, pp. 292-321.

2. Kramer, R., "Effects of Quantization on Feedback Systems with Stochastic Inputs," E.S.L., Report 7967 R2, M.I.T., July 1959.

3. Costas, J. P., "Periodic Sampling of Stationary Time Series," Research Lab. of Electronics, Report 156, M.I.T., 1950.

4. Andeen, R. A., "The Principle of Equivalent Areas," Trans. AIEE Applications and Industry, Vol. 79, Part II, pp. 332-336 (1960).

5. Kadota, T. T., "Analysis of Non-linear Sampled-Data Systems with Pulse Width Modulators," Ph.D. Thesis, University of California, Berkeley, 1960.

6. Nuttal, A. N., "Theory and Application of a Separable Class of Random Processes," D.Sc. Thesis, M.I.T., 1960.

7. Bussgang, J. J., "Cross-Correlation Functions of Amplitude Distorted Signals," R.L.E., Report 216, M.I.T., 1952.