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A STUDY OF LIMITATIONS ON THE CHARACTERISTICS OF NETWORK FUNCTIONS

by

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ABSTRACT

In this report the general form of the limitations imposed on driving point and transfer immittances when some part of the input circuit of a network is fixed is investigated. The limitations are found to be bounds on the real and imaginary parts of the functions at points in the right half plane. At certain specific frequencies determined by the fixed part of the net work, the values of the immittance function and some of its derivatives are fixed.

These limitations on the values of immittance functions at points in the right half plane are then expressed in terms of integrals of the function along the imaginary axis by means of Cauchy's integral formula.

The limitations on integrals are utilized to find limitations on the characteristics of ideal immittance functions. Several examples are given to illustrate the use of these limitations.

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I. INTRODUCTION

Electronic circuits such as filters or amplifiers invariably contain parasitic elements such as interelectrode capacitance or lead inductance. It is well known that these parasitic elements impose certain limitations on the overall characteristics of the circuit. Bode[1] and others have investigated the limitations imposed by a single shunt capacitance on network response functions. Fano[2] has considered limitations on the reflection coefficient of a given passive impedance. However the general form of network limitations has not been investigated. In this report we consider the general limitations imposed on the response functions of linear, lumped, time-invariant networks by fixed elements which the network is required to contain. We first consider limitations on driving point and transfer immittance functions which are imposed by fixing part of the input circuit of the network. Next limitations on integrals of these immittance functions are considered. Finally some examples are given in which limitations on the bandwidth and magnitude of ideal transfer functions are found.

II. LIMITATIONS ON DRIVING POINT AND TRANSFER IMMITTANCE FUNCTIONS

Previous investigations of this subject have been concerned almost entirely with the effects of a single shunt capacitance. In this section the general restrictions imposed on driving point and transfer immittance functions by fixed input circuits of any degree of complexity have been considered.

The term "immittance" is used to indicate either impedance or admittance. A driving point immittance is an immittance function for which the input and output are measured at the same terminal pair while for a transfer immittance the input and output may be measured at different terminal pairs.

The general notation K_{ij} rather than Z_{ij} or Y_{ij} has been used to indicate an immittance function with excitation at the j^{th} terminal pair and response at the i^{th} terminal pair whenever it was not important to specify either impedance or admittance.

A driving point immittance function is said to be passive at a point s_o in the right half plane of the complex frequency variable s if

$$|\operatorname{Arg} K(s_o)| \leq |\operatorname{Arg} s_o| \qquad (2:1)$$

for

$$|\operatorname{Arg} s_o| \leq \frac{\pi}{2}$$

If this condition is satisfied in the entire right half plane then K is said to be a positive real or "p. r." function.[3] We shall refer to an RLC network whose driving point immittances are p. r. as a passive network. The transfer immittance of a passive network has no poles in the right half plane but may have zeros anywhere in the plane.

A driving point immittance function is said to be active at a point s_o in the right half plane if its value at s_o is the negative of some passive driving point immittance.[4] There are no restrictions on the transfer function of an active network. Note that a driving point immittance may be neither active nor passive at a point s_o.

General Form of the Limitations on Driving Point Immittance Functions

The general limitations imposed on driving point immittance functions by fixing part of the input circuit of the network take the form of bounds on the real and imaginary parts of the immittance at points in the right half plane of the complex frequency variable.

Referring to Figure 1, N_a represents a fixed two-port network while N_b represents an arbitrary passive network. N denotes the overall network consisting of the fixed two-port N_a terminated in the passive one-port N_b. K_1 is the driving point immittance of N and K_2 is the (p. r.) driving point immittance of the passive network N_b. The fixed network N_a is described by a set of two-port parameters denoted by k_{ij}, (i, j = 1, 2). The k_{ij}'s are used here as a general notation to indicate one of the two-port parameter sets: open circuit impedance parameters, short circuit admittance parameters or either of the hybrid parameter sets. Now for the circuit of Figure 1, the limitations imposed on K_1 by the fixed network N_a are considered.

The admittance K_1 may be expressed in terms of K_2 and one of the sets of k_{ij} parameters as

$$K_1 = k_{11} - \frac{k_{21}\, k_{12}}{k_{22} + K_2} \tag{2.2}$$

or, more convenient for our purposes as

$$K_1 = (k_{11}) \left\{ \frac{\left(\frac{k_{11}\, k_{22} - k_{21}\, k_{12}}{k_{11}}\right) + K_2}{k_{22} + K_2} \right\} \tag{2.3}$$

The k_{ij} are all known functions of the complex frequency variable s since the network N_a is fixed. Then it follows that for a particular value s_o of s the expression for K_1 is a function of K_2 only:

$$K_1 = (a)\left\{\frac{b + K_2}{c + K_2}\right\} \tag{2.4}$$

where a, b, and c are complex constants. This equation has the form of the well know linear fractional transformation.[5]

K_2 is the driving point immittance of a passive network and is therefore a positive real function. Then from the properties of immittances stated previously we have for s_o in the right half plane

$$|\operatorname{Arg} K_2| \leq |\operatorname{Arg} s_o| \tag{2.5}$$

This represents a wedge shaped region in the right half of the K_2 plane, bounded by the lines

$$\operatorname{Arg} K_2 = \pm \operatorname{Arg} s_o \tag{2.6}$$

These lines are then mapped into arcs of circles in the K_1 plane by the transformation (2.4). The region in the K_1 plane bounded by these arcs then represents the allowable values of $K_1(s_o)$. This mapping is illustrated in Figure 2 for a particular choice of N_a. Referring to Figure 2 (a), N_a is conveniently represented by its open circuit impedance parameters:

$$\begin{aligned} z_{11} &= \frac{1}{s+1} = z_{21} = z_{12} \\ z_{22} &= \frac{s+2}{s+1} \end{aligned} \tag{2.7}$$

Consider the limitations imposed on Z_1 by N_a at a frequency $s_o = (1 + j)$ in the right half plane as represented in Figure 2(b). $Z_2(s_o)$ must fall in the wedge shaped region illustrated in Figure 2(c) which is bounded by the lines

$$\operatorname{Arg} Z_2(s_o) = \pm \operatorname{Arg} s_o = \pm \frac{\pi}{4} \tag{2.8}$$

The allowable values of $Z_1(s_o)$ are obtained by mapping this region onto the Z_1 plane via the transformation

$$Z_1(s_o) = \frac{1}{(s_o+1)}\left\{\frac{1 + Z_2(s_o)}{\left(\frac{s_o+2}{s_o+1}\right) + Z_2(s_o)}\right\} \tag{2.9}$$

This mapping is illustrated in Figure 2(d).

Thus it has been shown that the limitations imposed on K_1 by N_a take the form of bounds on the possible values of K_1 at points in the right half of the complex frequency plane. Note that if $k_{22}(s_o)$ is active as defined previously, then K_1 may have a pole at s_o and is therefore unbounded. Also, if N_a is lossless and passive then K_1 may have poles on the imaginary axis and is thus not bounded there.

The Invariants of Driving Point and Transfer Immittance

For certain frequencies s_o in the right half plane, the value of K_1 may be completely independent of the choice of N_b. At these frequencies the input terminals of N_a are "isolated" from the output terminals of N_a and there is no signal transmitted through the fixed network to N_b. This may be demonstrated as follows where N_a is assumed to be passive.

In equation 2.4 it may happen that $b = c$, in which case the transformation becomes degenerate. In other words the entire wedge shaped region of allowable values of $K_2(s_o)$ is mapped into a point in the K_1 plane. It is clear that such a value of K_1 is completely independent of the passive network N_b. Such values of K_1 are referred to as invariants of the driving point immittance. In order to investigate such points in more detail it is convenient to write the transformation in the form of equation 2. 2. Then note that b = c implies either

$$(1) \quad k_{12}(s_o)\, k_{21}(s_o) = 0 \tag{2.10}$$

or

$$(2) \quad k_{22}(s_o) = \infty\,, \quad k_{12}(s_o)\, k_{21}(s_o) < \infty \tag{2.11}$$

In any case, the second term on the right hand side of equation (2. 2) vanishes at these frequencies and we have

$$K_1(s_o) = k_{11}(s_o) \tag{2.12}$$

Now suppose that $k_{12}k_{21}$ has a zero of order m at s_o. Then it follows that in addition to the value of K_1 at s_o, the values of the first(m - 1) derivatives of K_1 at s_o are also independent of the passive network N_b and are given by

$$K_1^{(n)}(s_o) = k_{11}^{(n)}(s_o)\,, \quad n \leqslant (m-1) \tag{2.13}$$

Thus the first m terms of the Taylor series expansion of $K_1(s)$ about the point s_o are invariant quantities and are equal to the first m terms of the Taylor series expansion of $k_{11}(s)$ about s_o.

If N_a is a lossless network and s_o lies on the $j\omega$ axis then k_{11} may have a private pole at s_o, that is, a pole which does not appear in $k_{12}k_{21}$. In this case the invariant quantities are the coefficients of the Laurent series of K_1 about s_o rather than the Taylor series. In the same way k_{22} may have a private pole at $s_o = j\omega_o$ as indicated by equation (2. 11) in which case the first (m + 1) terms of the series are invariant. If k_{22} does not have a private pole at $s_o = j\omega_o$ then $(k_{22} + K_2)$ may have a simple zero there, in which case only $(m - 1)$ terms of the series are invariant.

Consider now the transfer immittance K_{21} of the overall network N of Figure 1 consisting of N_b connected in cascade with N_a. K_{21} can be expressed as

$$K_{21} = \frac{k_{21} T_b}{k_{22} + K_2} \tag{2.14}$$

T_b in this expression represents some transfer function of the network N_b. This may be a transfer immittance or a dimensionless transfer ratio. Now suppose that at a point s_o in the right half plane k_{21} has a zero of order m. Then it can be seen from inspection of equation 2.14 that K_{21} must likewise have a zero of at least order m. The order of the zero can be increased arbitrarily since T_b can have zeros of any order anywhere in the plane but it can only be decreased by one and this only when the zero lies on the $j\omega$ axis.

Thus the first m (or$(m-1)$) terms of the Taylor series expansion of K_{21} about s_o are invariant and are in fact all identically zero

The previous statements concerning invariant quantities are true for analytic functions of immittances as well as immittances. Thus, for example, a reflection coefficient is also subject to limitations of this type. The invariant properties of reflection coefficients have been considered in detail by Fano.[2]

III. INTEGRALS OF DRIVING POINT AND TRANSFER IMMITTANCE FUNCTIONS

It has been shown that fixing part of the input circuit of a network imposes certain restrictions on the driving point and transfer immittances of the network at specific points in the complex frequency plane. From the point of view of the circuit designer however, it is of more interest to know how the behavior of the function along the real frequency (imaginary) axis is restricted. To this end we now consider the limitations on integrals of network functions evaluated along the imaginary axis.

Application of the Cauchy Integral Formula to Network Limitations

Cauchy's integral formula[5] states that if $F(s)$ is analytic within and on a closed curve C and if s_o is any point interior to C then

$$F(s_o) = \frac{1}{2\pi j} \oint_C \frac{F(s)}{s - s_o}\, ds \tag{3.1}$$

where the integral is taken in the positive sense around C. This formula expresses the value of $F(s)$ at any point inside C in terms of its values on C.

The most useful curve C_R for our purposes is illustrated in Figure 3. This curve consists of a portion of the imaginary axis from $-jR$ to $+jR$ and a semicircle in the right half plane of radius R. The small indentations in the path along the imaginary axis are included to avoid any singularities of the integrand which may occur there.

With this choice of C the value of an immittance function at any point in the right half plane can be expressed in terms of an integral along the imaginary axis. If the discussion is restricted to integrands which are such that the integral over the semicircle exists and can be evaluated as R becomes indefinitely large then we have an equation of the form

$$\int_{-\infty}^{\infty} \frac{F(j\omega)}{j\omega - s_o}\, dj\omega \;+\; \lim_{R\to\infty} \int \frac{F(s)}{s - s_o}\, ds = -2\pi j\, F(s_o) \; ^{*} \qquad (3.2)$$

Integrals involving only the real part or imaginary part of a network function are most useful. In order to obtain such integrals we can make use of the fact that on the imaginary axis, the real part of a physically realizable network function is an even function of frequency while the imaginary part is an odd function of frequency. Since the integral is over an interval of the imaginary axis which is symmetric about the origin, the integral of the odd part of the integrand vanishes. Then a function such as $W(s)K(s)$ can be chosen as our integrand where $W(s)$ is either an even function or an odd function which introduces a right half plane singularity at s_o and $K(s)$ is the immittance under consideration. If $W(s)$ is an even function, an integral of the real part of $K(s)$ over the imaginary axis is obtained and if $W(s)$ is odd, an integral of the imaginary part results. This may be illustrated for the example shown in Figure 2. A real part integral will be derived in terms of the possible values of $Z_1(1 + j)$. The function

$$W(s) = \frac{1}{s^4 + 4} \qquad (3.3)$$

is chosen as an even function which introduces right half plane singularities at $s_o = (1 + j)$ and $(1 - j)$. Then integrating yields the result

$$\int_0^{\infty} \frac{R_1(\omega)}{\omega^4 + 4}\, d\omega = \frac{\pi}{16} \operatorname{Re}\left\{ (1 + j)\, Z_1(1 + j) \right\} \qquad (3.4)$$

* The terms due to the paths around the $j\omega$ axis poles go to zero.

In this case no singularities of the integrand can occur on the imaginary axis and the value of the integral around the large semicircle is zero. If the possible values of $Z_1(1 + j)$, illustrated in Figure 2 (d), are now examined for maxima and minima, the following numerical bounds for the integral are obtained.

$$\frac{2\pi}{80} \leqslant \int_o^\infty \frac{R_1(\omega)}{\omega^4 + 4} d\omega \leqslant \frac{3\pi}{80} \tag{3.5}$$

Thus the bounds on Z_1 $(1 + j)$ have been expressed in terms of restrictions on the values of the real part of Z_1 $(j\omega)$.

A general form for such integrals for immittance functions analytic at infinity can be obtained by choosing for the weighting function $W(s)$, the even function

$$W_1(s) = \frac{\left[s^2 - \left(\sigma_o^2 - \omega_o^2\right)\right]}{s^4 - 2\left(\sigma_o^2 - \omega_o^2\right) s^2 + \left(\sigma_o^2 + \omega_o^2\right)^2} \tag{3.6}$$

for real part integrals and the odd function

$$W_2(s) = \frac{s\left[s^2 - \left(\sigma_o^2 - \omega_o^2\right)\right]}{s^4 - 2\left(\sigma_o^2 - \omega_o^2\right) s^2 + \left(\sigma_o^2 + \omega_o^2\right)^2} \tag{3.7}$$

for imaginary part integrals. The resulting integral equations are

$$\int_o^\infty W_1(j\omega)\ \mathrm{Re}K(j\omega)d\omega = \frac{\pi}{2}\ \mathrm{Re}\left\{\frac{K(s_o)}{s_o}\right\} - \pi \sum_\nu k_\nu W_1(j\omega_\nu) \tag{3.8}$$

and

$$\int_0^\infty W_2(j\omega)\,\mathrm{Im}K(j\omega)\,d\omega = \frac{\pi}{2} \quad \mathrm{Re}\left\{K(s_o) - K(\infty)\right\} \tag{3.9}$$

where the k_2's are residues at possible imaginary axis poles at $j\omega_2$ of the integrand. These equations relate the values of the real and imaginary parts of an immittance function along the imaginary axis to the value of the function at any point s_o in the right half plane. It should be noted that these equations have the form of the familiar Hilbert transforms if s_o is on the imaginary axis.

Resistance and Reactance Integral Theorems

Bode[1] and others have investigated the limitations imposed on integrals of driving point immittance functions by a single shunt capacitance across the input terminals of the network. More generally their results have been stated for functions whose power series expansions at zero and infinite frequency are known. These results are the resistance and reactance integral theorems. If the immittance function has power series expansions at zero and infinite frequency given by

$$K(s) = A_o + B_o s - A_1 s^2 - B_1 s^3 + \tag{3.10}$$

and

$$K(s) = A_\infty - \frac{B_\infty}{s} - \frac{A_1'}{s^2} + \frac{B_1'}{s^3} + \tag{3.11}$$

then the integrals of the real and imaginary parts of $K(s)$ on the imaginary axis are given by

$$\int_0^\infty \left[\mathrm{Re}\,K(j\omega) - A\infty\right] d\omega = -\frac{\pi}{2} \quad B_\infty - \frac{\pi}{2}\sum_\nu k_\nu \tag{3.12}$$

and

$$\int_{0}^{\infty} \frac{\operatorname{Im} K(j\omega)}{\omega} \, d\omega = \frac{\pi}{2} \left[A_{\infty} - A_{0} \right] \tag{3.13}$$

where the k_{ν}'s are the residues at any possible poles of the integrand on the $j\omega$ axis. For passive networks the residues are real and non-negative. Bode has also pointed out that integral equations involving higher ordered coefficients of the two series expansions can be derived.

In section II it was shown that fixing part of the input circuit of a network was in general equivalent to fixing the first part of the Taylor or Laurent series expansion of the function about specific points in the right half plane. These points are the right half plane transmission zeros and the private poles of the fixed part of the network. Integrals of the real or imaginary parts of the immittance function along the imaginary axis can be expressed in terms of these fixed Taylor series coefficients by means of Cauchy's integral formula. The remainder of this section is devoted to consideration of some specific real part integral formulas which may be derived. Similar formulas for the imaginary part of an immittance can be derived if desired. The functions under consideration are assumed to be analytic in the right half plane but logarithmic singularities and simple poles with positive real residues are permitted on the imaginary axis. Thus immittances and analytic functions of immittance are included.

<u>Integral Formulas for Complex Points</u>

Suppose the first n terms of the Taylor series expansion of an analytic function about a complex point $s_o = \sigma_o + j\omega_o$ in the right half plane are given as

$$K(s) + A_o + A_1 (s-s_o) + \ldots \tag{3.14}$$

where the A_k's are complex numbers with real and imaginary parts denoted by

$$A_k = A_{rk} + jA_{ik} \tag{3.15}$$

so that $2n$ quantities are specified. For each specified quantity, a real part integral can be written. The appropriate weighting functions for the k^{th} real part integrals are

$$W_{1k}(s) = Ev\left\{\frac{1}{(s-s_o)^{k+1}} + \frac{1}{(s-\bar{s}_o)^{k+1}}\right\} \tag{3.16}$$

for A_{rk} and

$$W_{2k}(s) = Ev\left\{\frac{1}{(s-s_o)^{k+1}} - \frac{1}{(s-\bar{s}_o)^{k+1}}\right\} \tag{3.17}$$

for A_{ik} where Ev denotes "even part of". The resulting real part integral formulas are, for $k=0$

$$\int_o^\infty W_{10}(j\omega)\, ReK(j\omega)\, d\omega = -2\pi A_{ro} + \pi\sigma_o k_\infty - \frac{\pi}{2}\sum_\nu k_\nu W_{10}(j\omega_\nu) \tag{3.18}$$

and

$$\int_0^\infty W_{20}(j\omega) \operatorname{Re} K(j\omega)\, d\omega = -2\pi j A_{io} + \pi j \omega_o k_\infty - \frac{\pi}{2} \sum_\nu k_\nu W_{20}(j\omega_\nu) \qquad (3.19)$$

where k_∞, and the k_ν's represent the residues in possible poles of $K(s)$ on the imaginary axis at infinity and $j\omega_\nu$ respectively. The summation is over all singularities on the finite imaginary axis.

For $k = 1$ the integral formulas are

$$\int_0^\infty W_{11}(j\omega) \operatorname{Re} K(j\omega)\, d\omega = -2\pi A_{r1} + \pi k_\infty - \frac{\pi}{2} \sum_\nu k_\nu W_{11}(j\omega_\nu) \qquad (3.20)$$

and

$$\int_0^\infty W_{21}(j\omega) \operatorname{Re} K(j\omega)\, d\omega = -2\pi j A_{i1} - \frac{\pi}{2} \sum_\nu k_\nu W_{21}(j\omega_\nu) \qquad (3.21)$$

For $k \geqslant 2$ the general formulas are

$$\int_0^\infty W_{1k}(j\omega) \operatorname{Re} K(j\omega)\, d\omega = -2\pi A_{rk} - \frac{\pi}{2} \sum_\nu k_\nu W_{1k}(j\omega_\nu) \qquad (3.22)$$

and

$$\int_0^\infty W_{2k}(j\omega) \operatorname{Re} K(j\omega)\, d\omega = -2\pi j A_{ik} - \frac{\pi}{2} \sum_\nu k_\nu W_{2k}(j\omega_\nu) \qquad (3.23)$$

Integral Formulas for Points on the Real Axis

Suppose the first n terms of the Taylor series expansion of an immittance function about a point σ_o on the positive real axis are given as

$$K(s) = A_o + A_1 (s - \sigma_o) + \ldots \tag{3.24}$$

where the A_K's are real numbers so that for this case only n quantities are specified. Again a real part integral formula can be derived for each specified quantity. For the k^{th} coefficient the appropriate weighting function is given by

$$W_k(s) = Ev\left\{\frac{1}{(s-\sigma_o)^{k+1}}\right\} \tag{3.25}$$

The resulting integral formulas are, for k = o

$$\int_o^{\infty} W_o(j\omega)\, Re\, K(j\omega)\, d\omega = -\pi A_o + \frac{\pi}{2} k_{\infty} \sigma_o - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_o(j\omega_{\nu}) \tag{3.26}$$

for k = 1

$$\int_o^{\infty} W_1(j\omega)\, Re\, K(j\omega)\, d\omega = -\pi A_1 + \frac{\pi}{2} k_{\infty} - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_1(j\omega_{\nu}) \tag{3.27}$$

and for $k \geqslant 2$

$$\int_o^{\infty} W_k(j\omega)\, Re\, K(j\omega)\, d\omega = -\pi A_k - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_k(j\omega_{\nu}) \tag{3.28}$$

where as before k_{∞} and k_{ν} represent the residues in possible imaginary axis poles at infinity and $j\omega_{\nu}$ respectively.

Integral Formulas for Points on the Imaginary Axis

Points on the imaginary axis cannot be dealt with in the same way as points in the right half plane since the weighting functions introduce singularities in the path of integration. This leads in general to non-convergent integrals. However if that part of the input circuit which is fixed is a lossless network then a set of real part integrals can be derived. This follows from the fact that in this case, imaginary axis transmission zeros are also zeros of the real part of the input immittance. Thus the even ordered terms among the fixed Taylor series coefficients are all imaginary while the odd ordered terms are all real. Under these conditions suppose that the first n terms of the Taylor series expansion of an immittance function about a point $j\omega_o$ on the imaginary axis are given as

$$K(s) = A_o + A_1 (s-j\omega_o) + \dots \tag{3.29}$$

where the A_k's are either real or imaginary numbers so that n quantities are specified. Then the appropriate weighting function for the k^{th} coefficient is

$$W_k(s) = Ev\left\{\frac{1}{(s-j\omega_o)^{k+1}}\right\} \tag{3.30}$$

The resulting integral formulas are, for k=0

$$\int_o^{\infty} W_o(j\omega)\,\mathrm{Re}\,K(j\omega)\,d\omega = -\pi A_o - j\omega_o \frac{\pi}{2} k_{\infty} - \frac{\pi}{2}\sum_{\nu} k_{\nu} W_o(j\omega_{\nu}) \tag{3.31}$$

For k=1 the integral formula is

$$\int_{0}^{\infty} W_1(j\omega)\, \text{Re}\, K(j\omega)\, d\omega = -\pi A_1 + \frac{\pi}{2} k_{\infty} - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_1(j\omega_{\nu}) \tag{3.32}$$

For $k > 2$ the general formula is

$$\int_{0}^{\infty} W_k(j\omega)\, \text{Re}\, K(j\omega)\, d\omega = -\pi A_k - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_k(j\omega_{\nu}) \tag{3.33}$$

Again k_{∞} is the residue at a possible pole of $K(s)$ at infinity and the k_2's are the residues at any possible poles on the finite imaginary axis.

Integral Formulas for the Points at Zero and Infinity

Zero and infinity are points on the imaginary axis so that transmission zeros here of the fixed lossless network are also zeros of the real part of the input immittance. Thus the even ordered terms among the fixed Taylor series coefficients are all zero and the odd ordered terms are all real numbers. Then suppose the first $2n$ coefficients of the Taylor series expansion about zero are given by

$$K(s) = A_1 s + A_3 s^3 + \ldots \tag{3.34}$$

where the A_k's are all real numbers. A real part integral formula can be derived for each non-zero coefficient. The appropriate weighting function for the k^{th} coefficient is given by

$$W_k(s) = \frac{1}{s^{k+1}} \tag{3.35}$$

The resulting integral formula is for $k = 1$

$$\int_0^{\infty} W_1 (j\omega) \operatorname{Re} K (j\omega)\, d\omega = - \frac{\pi}{2} A_1 + \frac{\pi}{2} k_{\infty} - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_1 (j\omega_{\nu}) \tag{3.36}$$

and for $k>1$ the general equation is

$$\int_0^{\infty} W_k (j\omega) \operatorname{Re} K (j\omega)\, d\omega = - \frac{\pi}{2} A_k - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_k (j\omega_{\nu}) \tag{3.37}$$

Suppose that the first $2n$ terms of the power series expansion at infinity are given as

$$K(s) = \frac{A_1}{s} + \frac{A_3}{s^3} + \tag{3.38}$$

Then again a real part integral formula can be derived for each non-zero coefficient. The appropriate weighting function for the k^{th} coefficient is

$$W_k(s) = s^{k-1} \tag{3.39}$$

and the resulting integral formulas are for all values of k

$$\int_0^{\infty} W_k (j\omega) \operatorname{Re} K (j\omega)\, d\omega = - \frac{\pi}{2} A_k - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_k (j\omega_{\nu}) \tag{3.40}$$

Note that for $k=0$ this is just Bode's resistance integral theorem with A_{∞} equal to zero.

Integrals Over Paths of Finite Length

All of the integral formulas which have been derived up to this point have been over paths of infinite length. This is a consequence of utilizing Cauchy's integral formula, which requires a closed path of integration, in the derivation. In this section however, limitations on integrals of passive driving point immittance functions over paths of finite length are considered. The driving point immittance is limited, as before, by fixing part of the input circuit of the network. This problem has been considered by Spilker for the case of a single shunt capacitance.[6]

The path of integration is restricted to be in the right half plane so that singularities of the integrand on the path are avoided. The integration is performed between conjugate points in the right half plane as illustrated in Figure 4(a) so that the value of the integral is not a complex number. Of course the value of the integral is independent of the particular right half plane path as long as the end points are fixed.

The circuit to be considered is illustrated in Figure 4(b). Here N_a is a fixed passive two-port network and N_b is an arbitrary passive one-port with p. r. driving point impedance Z_2. N_b is connected to the output of N_a through an ideal transformer of turns ratio n. Suppose N_a is described by its open circuit impedance parameters z_{ij}, and consider the integral of Z_1 given by

$$I(n) = \int_{\bar{s}_o}^{s_o} Z_1(s)\, ds = \int_{\bar{s}_o}^{s_o} \left\{ z_{11} - \frac{z_{21}z_{12}}{z_{22}+Z_2 n^2} \right\} ds \tag{3.41}$$

It was shown in section II that the integrand, Z_1, is bounded at each point of the path of integration and hence the integral must have both upper and lower bounds. Also it is clear that for a particular choice of Z_2 the integral is a function of n only. Therefore for each Z_2 it is reasonable to ask for what values of n the integral attains its extreme values. Differentiating yields

$$I'(n) = \int_{\bar{s}_o}^{s_o} \frac{2z_{21}z_{12}{}^{n}Z_2}{\left[z_{22} + n^2 Z_2\right]^2} \, ds \tag{3.42}$$

Setting the derivative equal to zero gives $n=0$ and $n=\infty$ as solutions for extreme values of the integral. This corresponds physically to terminating N_a in a short circuit or an open circuit. For any other value of n, Z_2 can be chosen such that the integral is not zero, for example $Z_2=z_{22}$. Thus the extreme values of the integral are given by

$$\int_{\bar{s}_o}^{s_o} z_{11}(s) \, ds \quad , \quad \int_{\bar{s}_o}^{s_o} \frac{1}{y_{11}(s)} \, ds \tag{3.43}$$

To illustrate this result consider again the example of Figure 2. For this example the extreme values of the integral of Z_1 are found to be

$$\int_{\bar{s}_o}^{s_o} \frac{ds}{y_{11}(s)} = j \; 2 \tan^{-1} \frac{\omega_o}{2 + \sigma_o}$$

and

$$\int_{\bar{s}_o}^{s_o} z_{11}(s)\, ds = j2 \tan^{-1} \frac{\omega_o}{1+\sigma_o}$$

Thus if s_o is chosen on the imaginary axis, the integral of Z_1 is bounded by

$$\tan^{-1} \frac{\omega_o}{2} \leq \int_0^{\omega_o} R_1(\omega)\, d\omega \leq \tan^{-1} \omega_o$$

IV. LIMITATIONS ON THE TRANSFER FUNCTIONS OF PASSIVE TWO-PORT NETWORKS

This section is devoted to consideration of the limitations imposed by lossless two-port networks on passive transfer functions. The lossless character of the networks allows us to make use of certain simple relationships between driving point and transfer immittances which are based on considerations of power flow in the network.

Limitations on the Transfer Function of Lossless Two-Ports Terminated in a Resistance

In this section we consider the case illustrated in Figure 5 (a) in which N_a is a fixed lossless two-port network. N_a is terminated in a second lossless network N_b which is in turn terminated in a resistance R_o. Thus Z_2 may represent any p. r. driving point impedance realized in the Darlington form. The input to N_a is an ideal current source I_1 and the output of the system is the voltage V_2 measured across R_o. The transfer impedance Z_{21} is the function whose limitations are of interest.

Since N_a and N_b are lossless we can equate the power delivered to the input of N_a to the power dissipated in the load resistance R_o.

$$[\text{Power in}] = |I_1|^2 \,\mathrm{Re}Z_1 = \frac{|V_2|^2}{R_o} = [\text{Power out}] \tag{4.1}$$

Then upon dividing through by $|I_1|^2$ we obtain

$$\mathrm{Re}Z_1 = \frac{1}{R_o}\left|\frac{V_2}{I_1}\right|^2 = \frac{|Z_{21}|^2}{R_o} \tag{4.2}$$

Now it has been shown in section III that from the known properties of the fixed network N_a we can write a set of integral equations of the form

$$\int_o^\infty W_k(j\omega)\ \mathrm{Re}\ Z\,(j\omega)\ d\omega = A_k \tag{4.3}$$

or using (4.2)

$$\int_o^\infty W_k(j\omega)\ \frac{|Z_{21}(j\omega)|^2}{R_o}\ d\omega = A_k \tag{4.4}$$

If the desired form of $|Z_{21}|$ is specified then we can immediately obtain a set of equations giving the limitations on this form of magnitude function. Of course the form specified must contain the transmission zeros of N_a. In order that a solution of the set of equations so obtained exist, it will in general be necessary to allow the driving point immittance to be a non-mimimum function, that is, to have imaginary axis poles. The following simple example illustrates the technique outlined here.

Referring to Figure 5 (b), the fixed network N_a is an LC ladder with three transmission zeros at infinity. This particular example has been chosen in order that the result might be compared with that obtained by Bode[1] for the case of a single shunt capacitance. For this particular network, according to the results obtained in section III, we can write three equations involving integrals of the real part of the driving point impedance Z_1. These are

$$\int_o^\infty R_1(\omega)\, d\omega = \frac{\pi}{2C_1} - \frac{\pi}{2}\sum_\nu k_\nu \tag{4.5}$$

$$\int_o^\infty \omega^2 R_1(\omega)\, d\omega = \left(\frac{\pi}{2C_1}\right)\left(\frac{1}{LC_1}\right) - \frac{\pi}{2}\sum_\nu k_\nu \omega_\nu^2 \tag{4.6}$$

$$\int_o^\infty \omega^4 R_1(\omega)\, d\omega = \left(\frac{\pi}{2C_1}\right)\left(\frac{1}{LC_1}\right)^2 \left(1 + \frac{C_1}{C_2}\right) - \frac{\pi}{2}\sum_\nu k_\nu \omega_\nu^4 \tag{4.7}$$

The k 's represent the residues of any imaginary axis poles of Z_1 which may be present and the ω_ν's represent the locations of such poles. The form of $|Z_{21}|$ can be specified as long as the transmission zeros at infinity are included. The simplest choice for $|Z_{21}|$ is a lowpass function of bandwidth ω_o, equal to a constant K, in the passband and zero outside the passband. With this choice the integrals can be evaluated to give

$$\frac{K^2}{R_o}\,\omega_o = \frac{\pi}{2C_1} - \frac{\pi}{2}\sum_{\nu} k_{\nu} \tag{4.8}$$

$$\frac{K^2}{R_o}\,\frac{\omega_o^{3}}{3} = \left(\frac{\pi}{2C_1}\right)\left(\frac{1}{LC_1}\right) - \frac{\pi}{2}\sum_{\nu} k_{\nu}\omega_{\nu}^{2} \tag{4.9}$$

$$\frac{K^2}{R_o}\,\frac{\omega_o^{5}}{5} = \left(\frac{\pi}{2C_1}\right)\left(\frac{1}{LC_1}\right)^{2}\left(1+\frac{C_1}{C_2}\right) - \frac{\pi}{2}\sum_{\nu} k_{\nu}\,\omega_{\nu}^{4} \tag{4.10}$$

The first of these equations is the same as the result obtained by Bode[1] for the case of a single shunt capacitance. The additional equations represent constraints imposed by the inductance and second shunt capacitance. In order that these equations have a solution it is in general necessary that Z_1 have at lease one pole on the imaginary axis. Such a pole must lie outside the passband and since the network is passive the residue at the pole must be positive. For a pole on the imaginary axis at ω_1, these last two conditions lead to the following inequalities:

$$\omega_1 \geq \omega_o \Rightarrow \frac{K^2}{R_o}\sqrt{\frac{C_1}{L}} > \frac{3\pi}{4}\,\frac{\left(\omega_o\sqrt{LC_1}\right)^{2}-1}{\left(\omega_o\sqrt{LC_1}\right)^{3}} \tag{4.11}$$

$$k_1 \geq 0 \Rightarrow \frac{K^2}{R_o}\sqrt{\frac{C_1}{L}} < \frac{\pi}{2}\,\frac{1}{\left(\omega_o\sqrt{LC_1}\right)} \tag{4.12}$$

If k_1 and ω_1 are eliminated from (4.8), (4.9), and (4.10) then an equation in the following normalized variables is obtained.

$$X = \frac{K^2}{R_o}\sqrt{\frac{C_1}{L}}\ , \qquad Y = \omega_o\sqrt{LC_1} \tag{4.13}$$

This equation, depending on a parameter $\alpha = \frac{C_1}{C_2}$, is

$$\left(\frac{X}{\pi}\right)^2\left(\frac{4Y^6}{45}\right) - \left(\frac{X}{\pi}\right)\left[\frac{Y^5}{10} - \frac{Y^3}{3} + \frac{(1+\alpha)}{2}\ Y\right] + \frac{\alpha}{4} = 0 \tag{4.14}$$

Solutions of this equation for various values of the parameter α are plotted in Figure 6 along with the constraints represented by 4.11 and 4.12.

Limitations on the Transfer Functions of Coupling Networks with Fixed Input and Output Circuits

In this section we consider limitations imposed on the transfer function of a passive network by fixed lossless input and output circuits. The problem is illustrated in Figure 7 (a) where N_a and N_b are fixed lossless two-port networks and N_c is an arbitrary passive coupling network. It is assumed that N_a and N_b have the same transmission zeros and private poles so that equivalent sets of real part integrals can be written for Z_{11} and Z_{22}. Limitations on the transfer impedance Z_{21} are to be considered. The solution to this problem for the case of shunt capacitance at input and output has been given by Bode.[1] This development is intended to show that Bode's technique may be extended to problems involving more complex fixed lossless networks.

Since the three networks are connected in cascade, any transmission zeros of N_a or N_b are zeros of Z_{21} as shown in section II. Thus, because of the assumed equivalence of N_a and N_b, the first m coefficients of the Taylor series expansions of Z_{11}, Z_{22} and Z_{21}

are fixed at a transmission zero of order m of N_a or N_b. These corresponding Taylor series coefficients can be expressed as corresponding integrals of the real parts of the three functions as discussed in section III.

$$\int_0^\infty W(\omega)\, R_{11}(\omega)\, d\omega = A_{11} \tag{4.15}$$

$$\int_0^\infty W(\omega)\, R_{22}(\omega)\, d\omega = A_{22} \tag{4.16}$$

$$\int_0^\infty W(\omega)\, R_{21}(\omega)\, d\omega = 0 \tag{4.17}$$

Now the overall network is passive and this condition can be expressed as

$$R_{11}(\omega) > 0, \quad R_{22}(\omega) > 0, \quad \sqrt{R_{11}(\omega)\, R_{22}(\omega)} \geq |R_{21}(\omega)| \tag{4.18}$$

This can be multiplied by a weighting function and integrated to give

$$\int_0^\infty |W(\omega)\, R_{21}(\omega)|\, d\omega \leq \int_0^\infty \sqrt{W^2(\omega)\, R_{11}(\omega) R_{22}(\omega)}\, d\omega \tag{4.19}$$

The Schwarz inequality can be applied to the integral on the right to give

$$\int_0^\infty \sqrt{W^2(\omega)R_{11}(\omega)R_{22}(\omega)}\, d\omega \leq \sqrt{\int_0^\infty |W(\omega)R_{11}(\omega)|\, d\omega \cdot \int_0^\infty |W(\omega)R_{22}(\omega)|\, d\omega} \tag{4.20}$$

Combining this with the inequality (4.19) we have

$$\int_0^\infty |W(\omega)R_{21}(\omega)|\, d\omega \leq \sqrt{\int_0^\infty |W(\omega)R_{11}(\omega)|\, d\omega \cdot \int_0^\infty |W(\omega)R_{22}(\omega)|\, d\omega} \tag{4.21}$$

If $W(\omega)\, R_{11}(\omega)$ and $W(\omega)\, R_{22}(\omega)$ are non-negative then the two integrals on the right can be evaluated explicitly as A_{11} and A_{22} to give

$$\int_0^\infty |W(\omega)R_{21}(\omega)|\, d\omega \leq \sqrt{A_{11}A_{22}} \tag{4.22}$$

The integral on the left can be evaluated for the desired form of transfer function in terms of the pertinent parameters. The result is the desired limitations. The technique outlined here is illustrated by the following example.

Referring to Figure 7 (b), N_a and N_b are specified as simple L networks. This configuration corresponds to the problem considered by Bode[1] with the added constraint of a series inductance. Because of the fixed transmission zeros at infinity a lowpass transfer function, constant in the passband, is considered. The given networks N_a and N_b lead to six integral equations.

$$\int_0^\infty R_{11}(\omega)\, d\omega = \frac{\pi}{2C_1} - \frac{\pi}{2}\sum_\nu k_{\nu 11} \leq \frac{\pi}{2C_1} \tag{4.23}$$

$$\int_0^\infty \omega^2 R_{11}(\omega)\, d\omega = \frac{\pi}{2L_1C_1^2} - \frac{\pi}{2}\sum_\nu k_{\nu 11}\omega_{\nu 1}^2 \leq \frac{\pi}{2L_1C_1^2} \tag{4.24}$$

$$\int_0^\infty R_{22}(\omega)\, d\omega = \frac{\pi}{2C_2} - \frac{\pi}{2}\sum_\nu k_{\nu 22} \leq \frac{\pi}{2C_2} \tag{4.25}$$

$$\int_0^\infty \omega^2 R_{22}(\omega)\, d\omega = \frac{\pi}{2L_2C_2^2} - \frac{\pi}{2}\sum_\nu k_{\nu 22}\ \omega_{\nu 2}^2 \leq \frac{\pi}{2L_2C_2^2} \tag{4.26}$$

$$\int_0^\infty R_{21}(\omega)\, d\omega = 0 \tag{4.27}$$

$$\int_0^\infty \omega^2 R_{21}(\omega)\, d\omega = 0 \tag{4.28}$$

The ideal form of lowpass transfer function which is constant in the passband and which satisfies (4.27) and (4.28) is given by

$$Z_{21}(s) = \frac{K}{\left[\sqrt{1+\left(\frac{s}{\omega_o}\right)^2} + \frac{s}{\omega_o}\right]^n} \tag{4.29}$$

where n is an odd integer greater than three and K is a real constant. Then it is easily found that for this transfer function

$$\int_0^{\infty} |R_{21}(\omega)|\, d\omega = \frac{\omega_o Kn}{n^2 - 1} \cot \frac{\pi}{2n} \tag{4:30}$$

$$\int_0^{\infty} \omega^2 |R_{21}(\omega)|\, d\omega = \frac{\omega_o^3 K}{4} \left\{ \frac{n}{n^2-1} \cot \frac{\pi}{2n} - \frac{n}{n^2-9} \cot \frac{3\pi}{2n} \right\} \tag{4. 31}$$

From these expressions the limitations on bandwidth and impedance level K are found to be

$$K\omega_o \leq \frac{n^2-1}{n} \tan \frac{\pi}{2n} \cdot \frac{\pi}{\sqrt{2\ C_1 C_2}} \tag{4. 32}$$

$$K\omega_o^3 \leq \frac{2\pi}{\left\{\left(\frac{n}{n^2-1}\right) \cot \frac{\pi}{2n} - \left(\frac{n}{n^2-9}\right) \cot \frac{3\pi}{2n}\right\} C_1 C_2 \sqrt{L_1 L_2}} \tag{4. 33}$$

These limitations for the normalized bandwidth and impedance level with $L_1 = L_2 = L$ and $C_1 = C_2 \doteq C$ are plotted in Figure 8. These equations depend on the parameter n but the variation with n is so slight that it may be ignored for purposes of calculating the curves of Figure 8.

Lossless Two-port Networks With Input and Output Terminated in Resistance

In this section the problem of transmission through a fixed lossless two-port network with arbitrary passive terminations is considered. The problem is illustrated in Figure 9 (a). Here N_a

is the fixed lossless two-port and N_b and N_c are arbitrary lossless two-ports terminated in resistance. It is required to find the limitations on the transducer power gain of the filter. A solution to this problem for the case of a simple capacitive coupler has been given by Baerwald.[7] In this development a different approach to the problem is used and the technique developed can be applied to problems involving any lossless two-port network.

It is clear that since neither N_b nor N_c is required to be finite it will be possible to obtain perfect power transmission, that is, the transducer power gain is unity, over some band(s) of frequencies. This is accomplished by constructing an image matched filter which uses N_a as the basic section. However the bandwidth of perfect power transmission and the location of the passband may be changed to some extent by adding additional elements to the basic section. Of course, the transmission zeros of N_a must always appear as transmission zeros of the overall network. Thus the pertinent problem to consider is that of limitations on the bandwidth of perfect power transmission.

The general procedure to be used is to first bisect N_a thus obtaining two networks with fixed input circuits. Since the transducer power gain is unity in the passband, the driving point immittances K_1 and K_2 (both impedances or both admittances) of the two networks must be conjugates of one another in the passband. A necessary condition for the conjugateness of K_1 and K_2 over the passband is the equality of corresponding integrals over the passband of the real parts of the two functions. A second necessary condition is that corresponding integrals over the imaginary parts of the two functions must be negatives of one another. The solution of the set of equations obtained from the application of these necessary conditions leads to limitations on the bandwidth.

The location of the bisection of N_a is only important in that a judicious choice may simplify the manipulations involved in obtaining the result. If N_a is represented in the Darlington form the bisection should be made so that equal numbers of sections representing a particular transmission zero remain on each side of the cut. In case there is an odd number of such sections then the middle section can be bisected or the cut can be made on one side of the middle section. An example which follows will illustrate the technique outlined here.

Referring to Figure 9(b), N_a is given as an LC pi network. This network has three transmission zeros at infinity. In view of the location of the fixed transmission zeros a lowpass transfer function is considered. The center section of the three is bisected by cutting the inductance as indicated in Figure 9(c). Then each half of N_a has one transmission zero and a private pole at infinity so that two integrals of the real and two integrals of the imaginary part may be written for Y_1 and the same for Y_2. The simplest solution which utilizes the available real and imaginary part areas most efficiently is obtained by letting Y_1 and Y_2 be band-limited minimum susceptive functions, that is, the real parts of the two admittances are identically zero outside the passband. Then for Y_1 the following integral equations can be written, where the irrational factors are used as described by Bode to obtain integrals of the imaginary part of the admittance over the passband

$$\int_{o}^{\omega_o} G_1(\omega)\, d\omega = \frac{\pi}{L} \tag{4.34}$$

$$\int_{o}^{\omega_o} \omega^2 G_1(\omega)\, d\omega \doteq \frac{2\pi}{L^2 (C_1 + C_x)} \tag{4.35}$$

$$\int_0^{\omega_o} \frac{\sqrt{\omega_o^2 - \omega^2}}{\omega} B_1(\omega)\, d\omega \doteq \frac{\pi}{2} \left[\frac{2}{L} - \omega_o G_{01}\right] \tag{4.36}$$

$$\int_0^{\omega_o} \omega\sqrt{\omega_o^2 - \omega^2}\, B_1(\omega)\, d\omega = \frac{\pi}{2} \left[\frac{4}{L^2(C_1 + C_x)} - \frac{\omega_o^2}{L}\right] \tag{4.37}$$

and similarly for Y_2:

$$\int_0^{\omega_o} G_2(\omega)\, d\omega = \frac{\pi}{L} \tag{4.38}$$

$$\int_0^{\omega_o} \omega^2 G_2(\omega)\, d\omega = \frac{2\pi}{L^2(C_2 + C_y)} \tag{4.39}$$

$$\int_0^{\omega_o} \frac{\sqrt{\omega_o^2 - \omega^2}}{\omega} B_2(\omega)\, d\omega = \frac{\pi}{2} \left[\frac{2}{L} - \omega_o G_{02}\right] \tag{4.40}$$

$$\int_0^{\omega_o} \omega\sqrt{\omega_o^2 - \omega^2}\, B_2(\omega)\, d\omega = \frac{\pi}{2} \left[\frac{4}{L^2(C_2 + C_y)} - \frac{\omega_o^2}{L}\right] \tag{4.41}$$

Here C_x and C_y represent any additional capacitance that may be added to C_1 and C_2 as the first elements of N_b and N_c and G_{01} and G_{02} represent the values of the real parts of Y_1 and Y_2 at zero frequency. The solution of the four equations obtained by application of the above mentioned necessary conditions leads to the result

$$\omega_o^2 \leq \frac{4}{LC}$$

where C is the larger of C_1 and C_2.

V. SUMMARY AND CONCLUSIONS

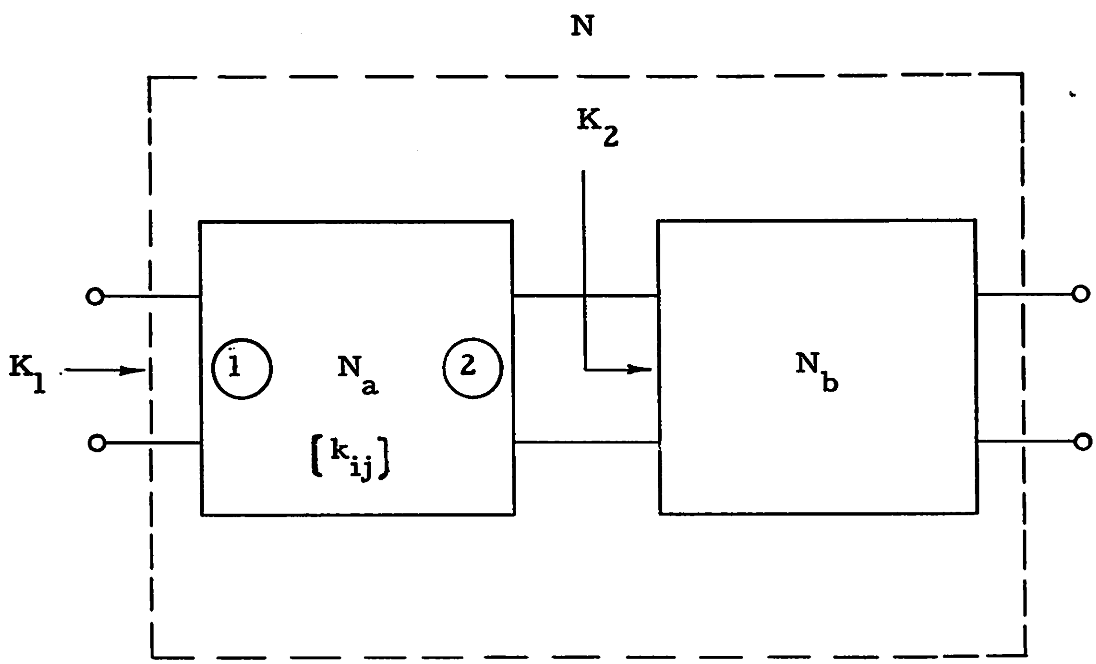
It has been shown that the driving point immittance of a passive network is subject to certain definite limitations when some part of the input circuit of the network is fixed. These limitations take the form of bounds on the real and imaginary parts of the function at points in the right half of the complex frequency plane. At right half plane frequencies where the input terminals of the network are "isolated" from the variable part of the circuit, the value of the input immittances and the values of some of its derivatives are invariant. These frequencies are the "transmission zeros" of the fixed part of the network. These limitations apply to driving point and transfer immittances and to analytic functions of immittance such as reflection coefficients.

The limitations on the values of the functions and its derivatives at points in the right half plane can be related to limitations on the bahavior of the function along the imaginary axis by means of Cauchy's integral formula. A number of specific formulas have been given which correspond to the various types of transmission zeros which may arise. It is shown that Bode's resistance and reactance integral theorems are special cases of such integral formulas.

Upper and lower bounds are derived for integrals of passive driving point immittance over paths of finite length in the right half plane.

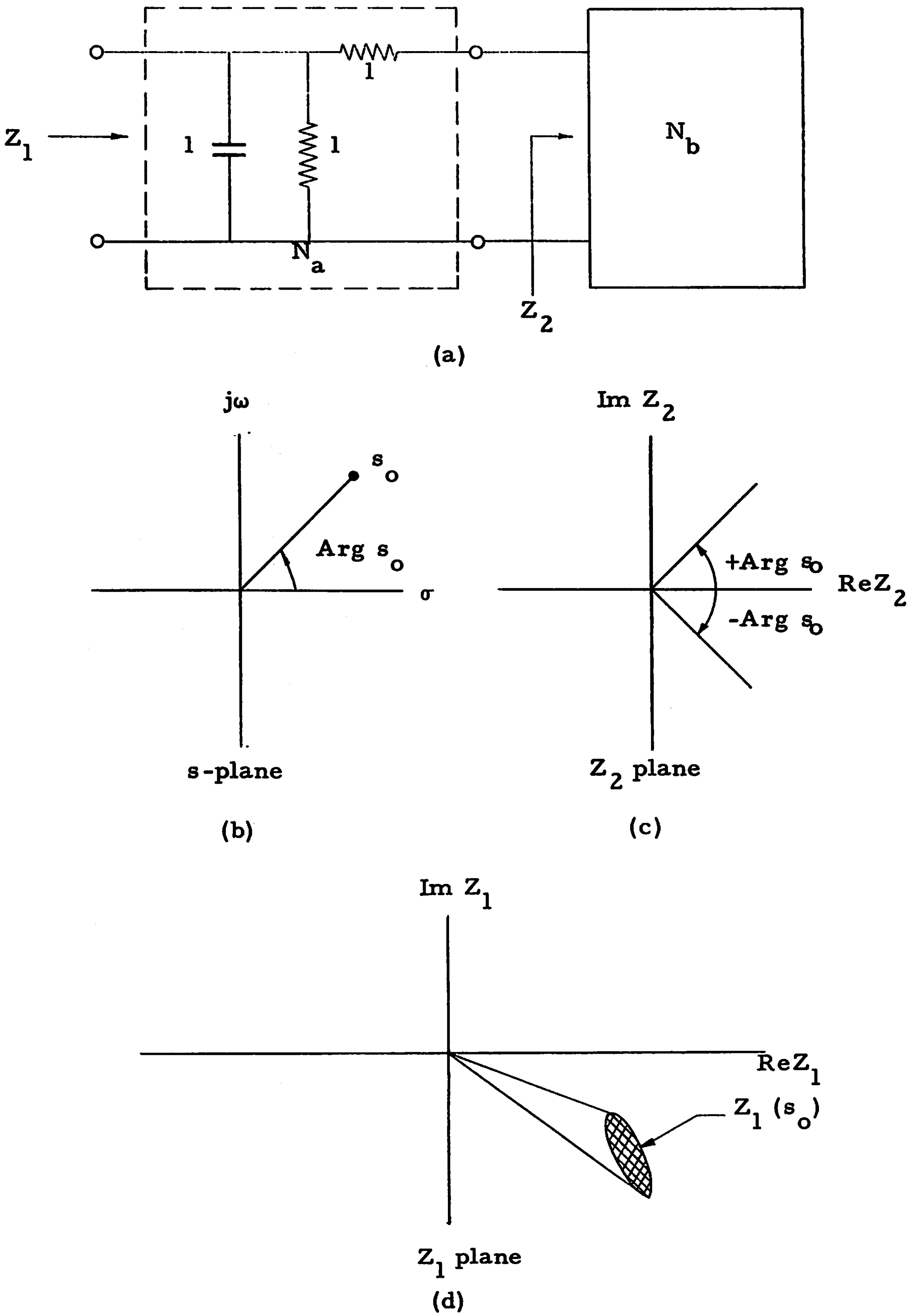
The limitations on immittance functions are applied to derive formulas for restrictions on the magnitude and bandwidth of the transfer functions of passive networks when the fixed part of the network is lossless.

This study has shown that requiring part of a network to be in a particular form places definite restrictions on response functions of the network. These restrictions manifest themselves as limitations on characteristics of the network functions such as magnitude of real or imaginary parts of the function, magnitude of the function, and bandwidth of constant magnitude.



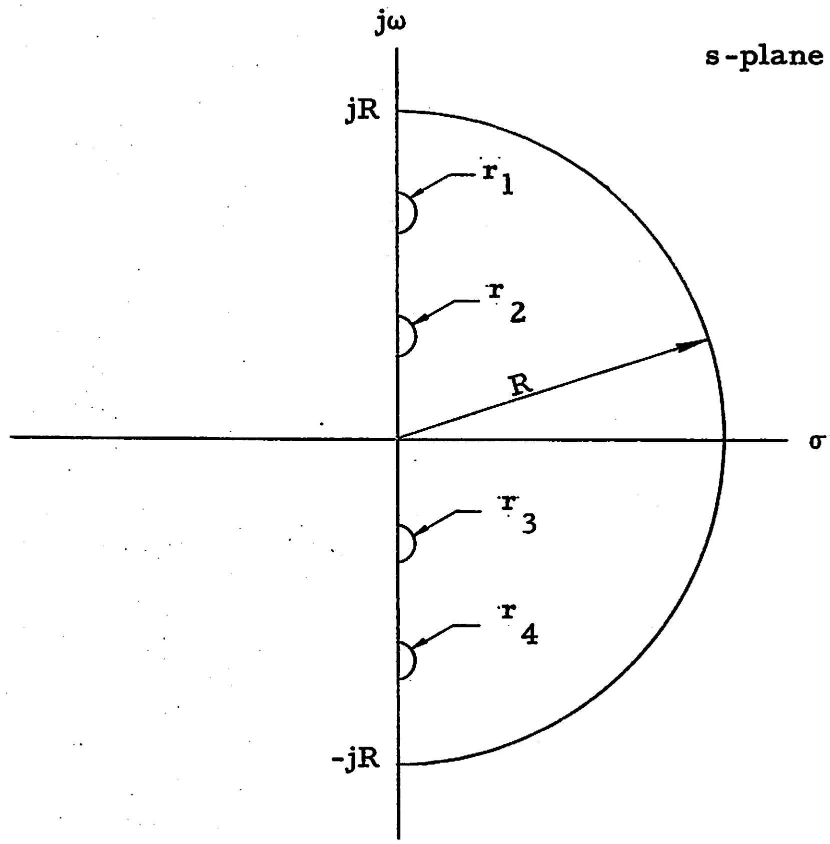
CONSTRAINTS ON DRIVING POINT IMMITTANCE

FIGURE 1



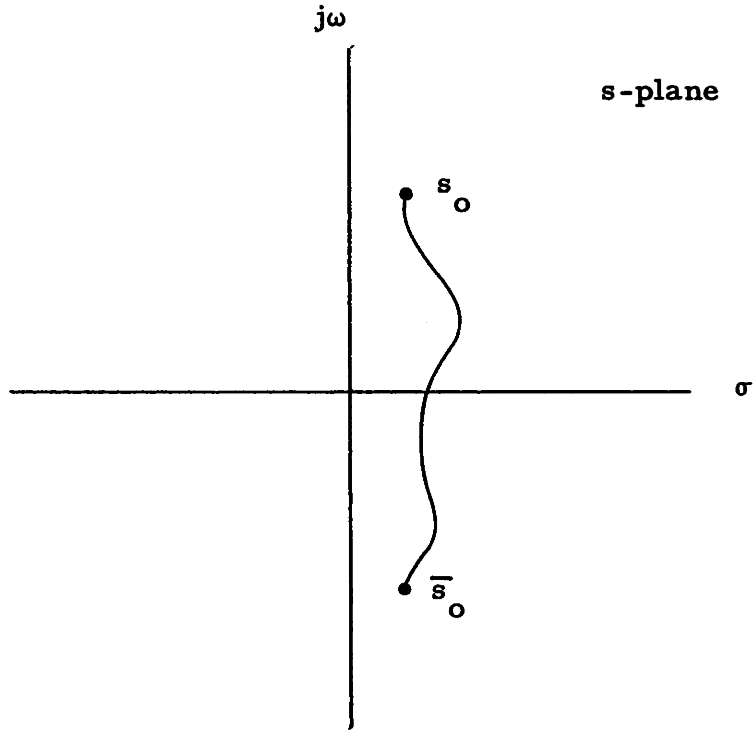
EXAMPLE OF LIMITATIONS ON Z_1

FIGURE 2

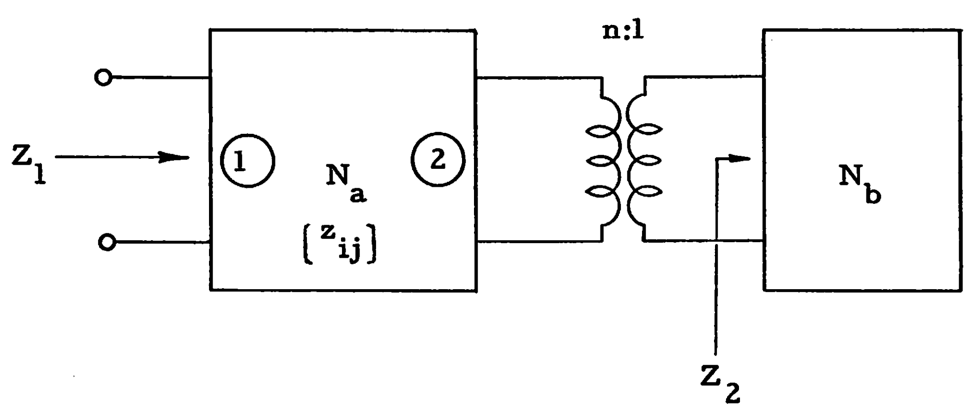


CURVE C_R FOR CONTOUR INTEGRATION

FIGURE 3

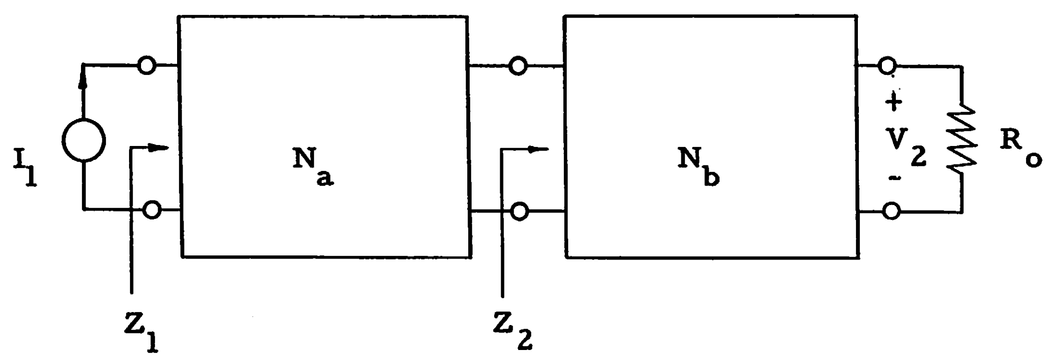


INTEGRATION BETWEEN CONJUGATE POINTS
(a)



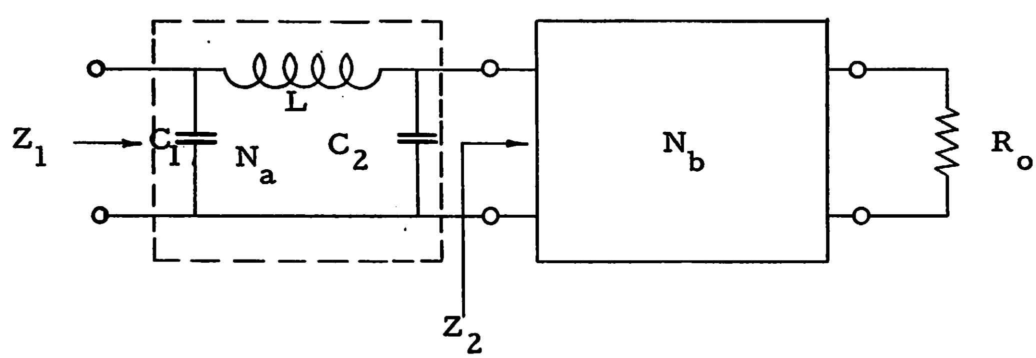
TRANSFORMER COUPLING N_a AND N_b
(b)

FIGURE 4



RESISTANCE TERMINATED LOSSLESS 2-PORT

(a)



LC Pi INPUT CIRCUIT

(b)

FIGURE 5

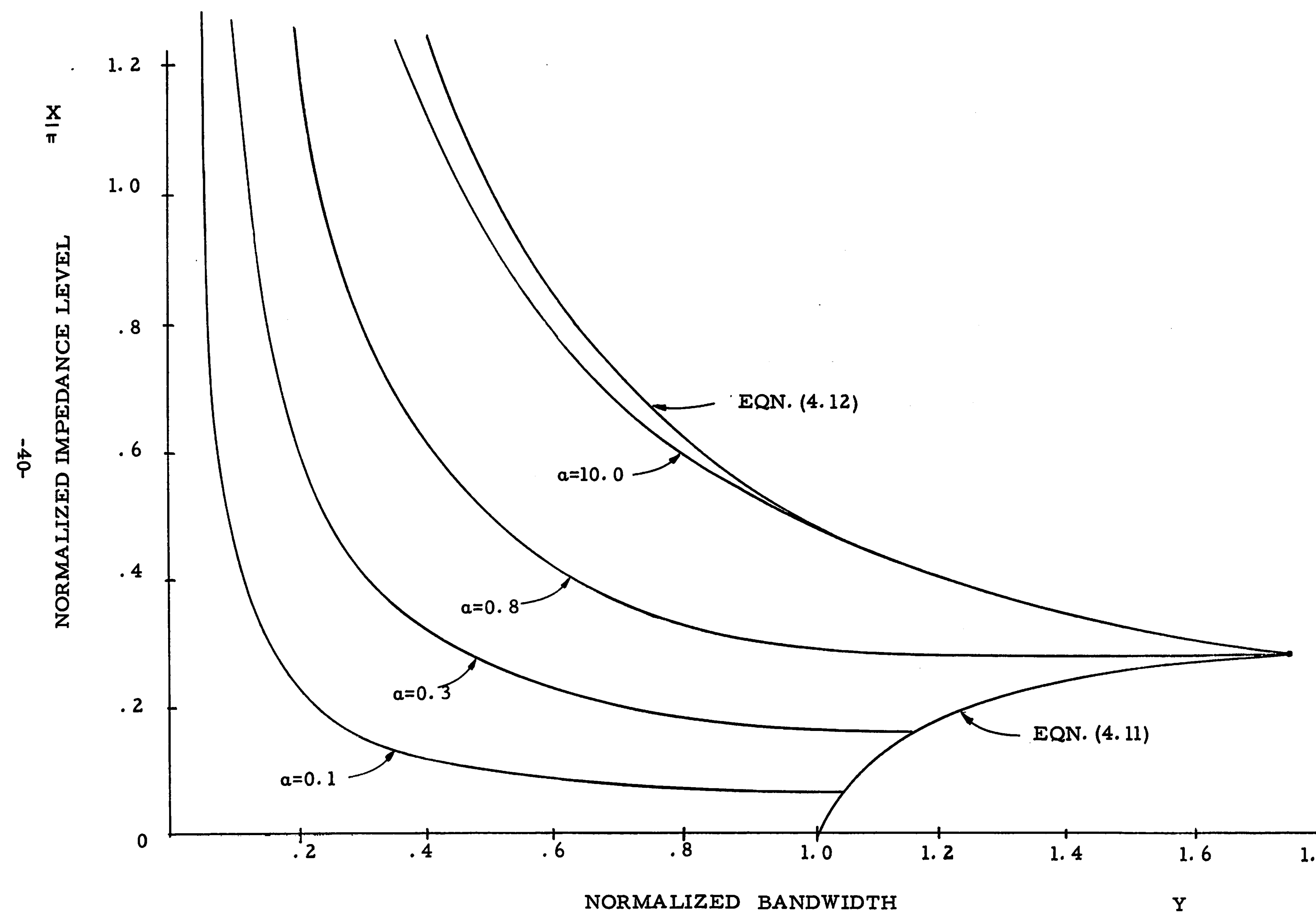
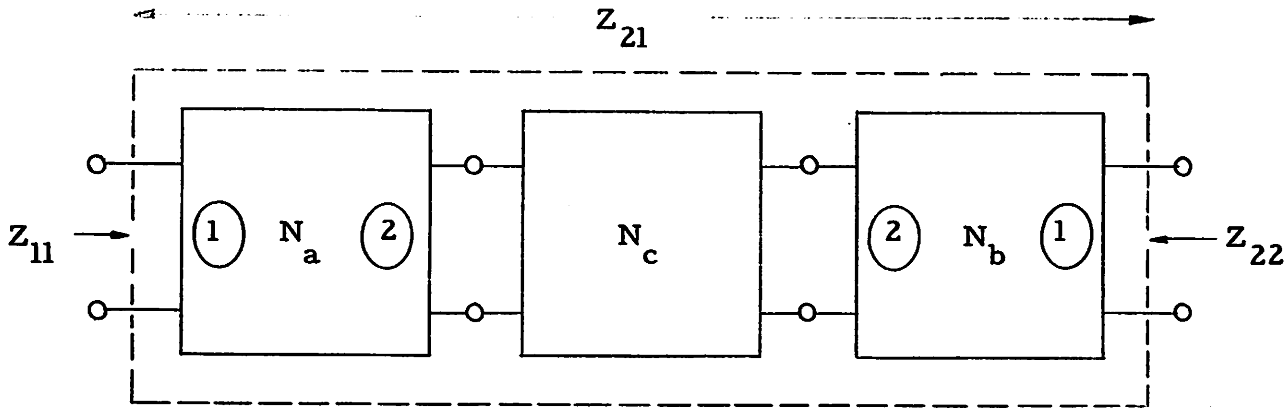
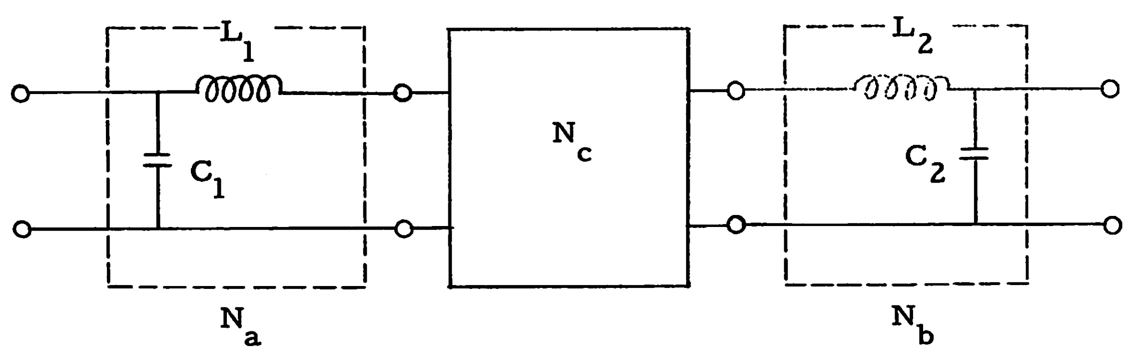


FIGURE 6 PLOTS OF EQUATIONS (4.11) AND (4.12)



FIXED INPUT AND OUTPUT CIRCUITS

(a)



LC INPUT AND OUTPUT CIRCUITS

(b)

FIGURE 7

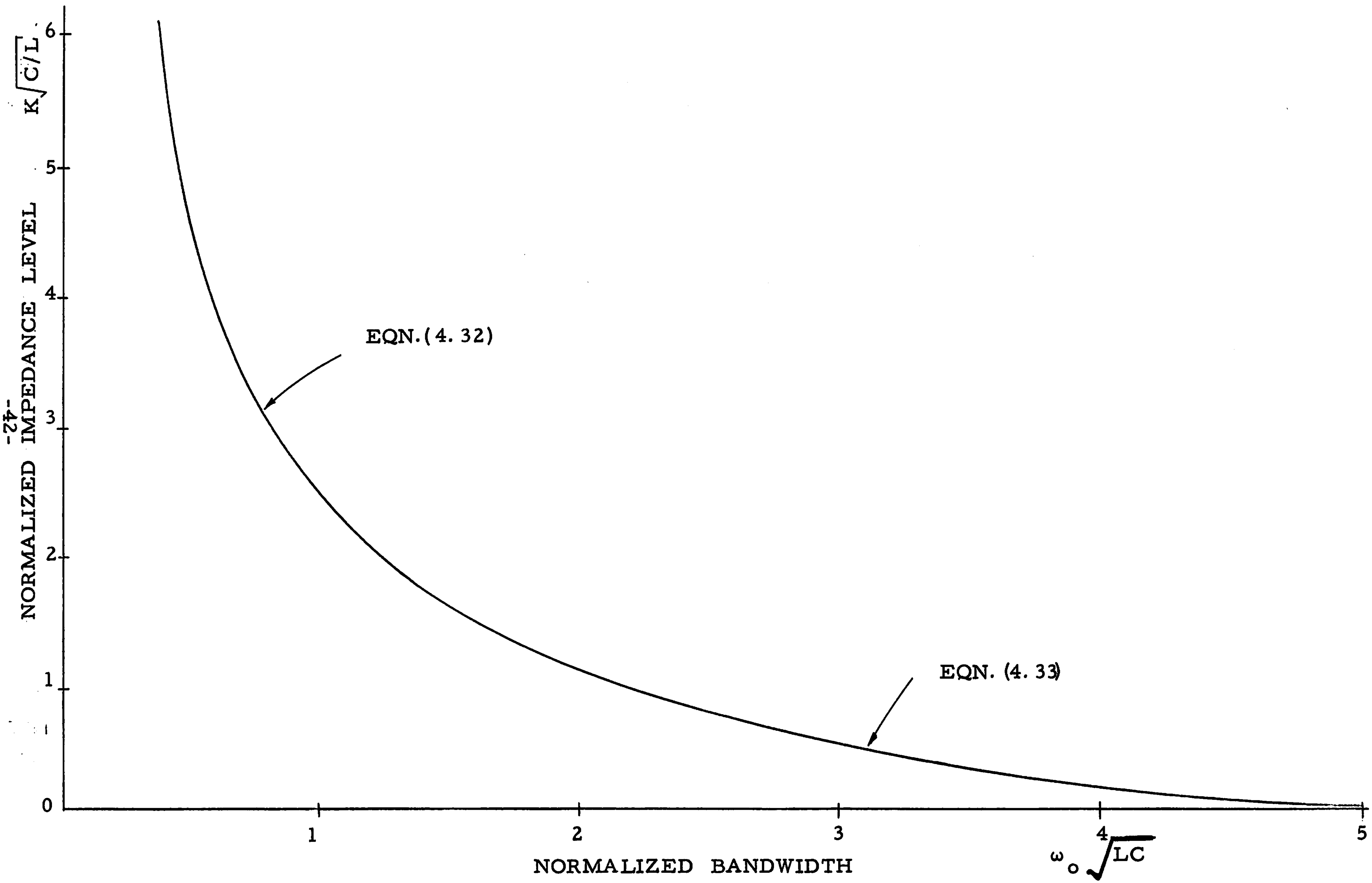


FIGURE 8 PLOTS OF EQUATIONS (4. 32) AND (4.33)

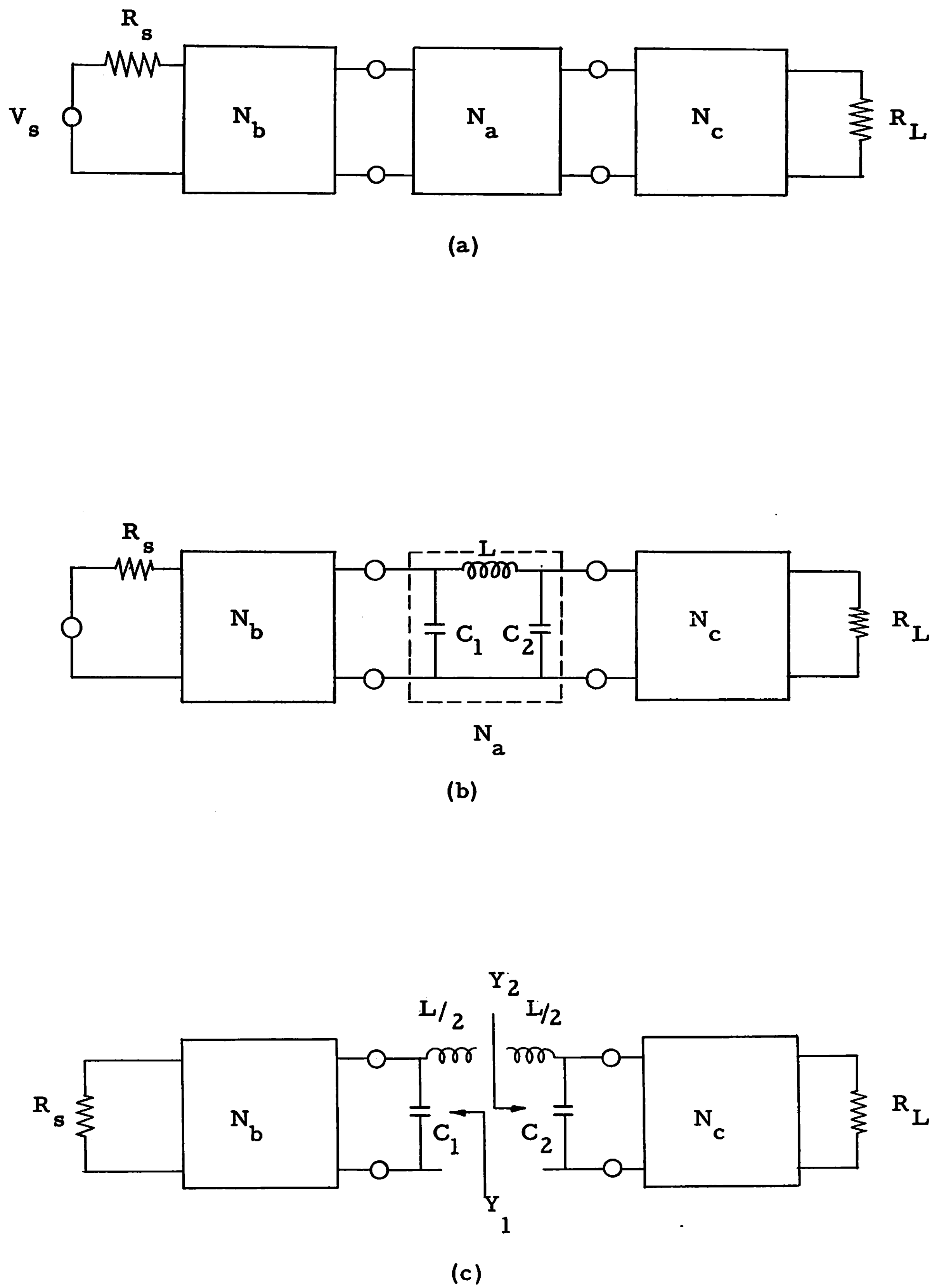


FIGURE 9

LOSSLESS NETWORKS WITH FIXED INPUT AND OUTPUT CIRCUITS

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