Electronics Research Laboratory
University of California
Berkeley, California

NEURAL PULSE FREQUENCY MODULATED CONTROL SYSTEMS

by

T. Pavlidis

Supported in part by
the Air Force Office of Scientific Research,
under Grant AF-AFOSR-117-63.

April 19, 1963

ACKNOWLEDGMENT

The author gratefully acknowledges the many helpful discussions and encouragement of Professor E. I. Jury during the reported research. He also expresses his thanks to Mr. C. Galtieri for the discussions on the filtering analysis.

ACKNOWLEDGMENT

The author gratefully acknowledges the many helpful discussions and encouragement of Professor E. I. Jury during the reported research. He also expresses his thanks to Mr. C. Galtieri for the discussions on the literature analysis.

ABSTRACT

A scheme of pulse frequency modulation is considered, based on Blair's theory of neural excitation. Its application in control systems is studied and the results (sampling theorem and stability) show that this scheme of P. F. M. has certain advantages over some of the conventional systems.

TABLE OF CONTENTS

I. INTRODUCTION

Although many types of discrete systems have been studied quite extensively,[1] the use of pulse frequency modulation has been given relatively little attention.

However, pulse frequency modulation is the basic way of transmission of information through neurons.[2] Therefore, it is a tempting problem to try to apply this physiological pattern to control systems. C. C. Li[3] and A. U. Meyer[4+] considered a class of such systems, i.e., integral pulse frequency modulated systems. However, their pattern presents little resemblance to the actual biological way of pulse frequency modulation.[++] A better model for the physiological systems has been proposed by Blair[6] (p. 379). An even more general form of frequency modulation could be described as follows: let $S(t)$ be a stimulus applied at time $t=0$. Then an "exciter" $p(t)$ is generated according to the equation

$$D_1 p(t) = D_2 S(t)$$

where D_1, D_2 are operators of any analytic form, e.g.,

$$D_1 = a_2 \frac{d^2}{dt^2} + a_1 t \frac{d}{dt} + a_o$$

$$D_2 = b_o + b_{-1} \int_0^t dt$$

[+] Meyer considered also some more general schemes (see footnote of p. 2).

[++] The same is true with the so-called delta modulation,[5] which has been developed for telecommunication practice purposes. In this case pulses are transmitted in time intervals which are integer multiples of a basic time interval. This restriction is undesirable for the stability of a closed loop system, as will become evident from the development of Secs. VIII and IX.

We always assume $p(0) = 0$. As soon as $|p(t)|$ reaches a critical value r, then a pulse is emitted of standard height and duration and the "exciter" returns to the zero value. The sign of the pulse is the same as $p(t)$ at the time of the emission.

In this report we will consider the case of

$$D_1 = \frac{d}{dt} + c \qquad \text{and} \qquad D_2 = 1$$

The resulting pattern of P. F. M. will be called neural pulse frequency modulation (N. P. F. M.) because it is the same as the one proposed by Blair for neurons.+ A basic difference, however, will be that we are going to consider double-signed (D. S.) pulses, while neurons generally carry only single-signed (S. S.) pulses. C. C. Li[3] considered both cases; however, the case of S. S. pulses results in more complicated systems. This is one instance in which it seems that engineering is able to design a more flexible system than nature.

The basic properties of N. P. F. M. control systems, such as transient response, stability, sampling theorem and elementary statistical analysis, will be studied. Due to the existence of a "strength-duration" law, this class of systems appears to have good filtering and stabilizing properties which may lead to useful applications.

II. DESCRIPTION OF THE MODULATOR

We proceed now to consider the modulator, which is described by the equation:

+ This is also the same pattern as that called "Relaxation P. F. M. " by Meyer (Ref. 4, pp. 13-17, 76-80). However, the name "neural" is preferred, to show the importance of borrowing ideas from biology. Meyer did not actually study the system, but indicated that it can be studied as an I. P. F. M. system with minor loops. However, such an approach is very complicated.

$$\frac{dp(t)}{dt} + c\,p(t) = S(t) \tag{2.1}$$

where c is a constant of the system.

Whenever $|p(t)|$ reaches a critical value r, a pulse of height h and duration t_h is emitted. Its sign is the sign of p(t). Thereafter the emission of the pulse p(t) equals zero and the whole process is repeated. Electronic implementations of such a modulator are discussed in Appendix A.

The solution of (2.1) for different simple inputs is shown in Table I. For a step input S_o, the firing time (i.e., when a pulse is emitted) is given by

$$t_f \triangleq \frac{1}{c} \ln \frac{|S_o|}{|S_o| - R} \tag{2.2}$$

where R = c r. Obviously, a necessary condition for firing is

$$|S_o| > R \tag{2.3}$$

So we are justified in calling R the threshold of the system.

Condition (2.3) is sufficient for a step input but not for a pulse of duration τ (Table I, No. 4) where we have the additional condition:

$$\tau > \frac{1}{c} \ln \frac{|S_o|}{|S_o| - R} \tag{2.4}$$

This shows that inputs of "too short" duration are not sensed, and therefore the impulse response of the modulator is not really meaningful.

The relation

$$\tau = \frac{1}{c} \ln \frac{|S_o|}{|S_o| - R} \tag{2.5}$$

is defined as the strength-duration curve, in analogy with the physiological findings (Ref. 2, pp. 39-40) (see Fig. 2.1 for plot of 2.5).

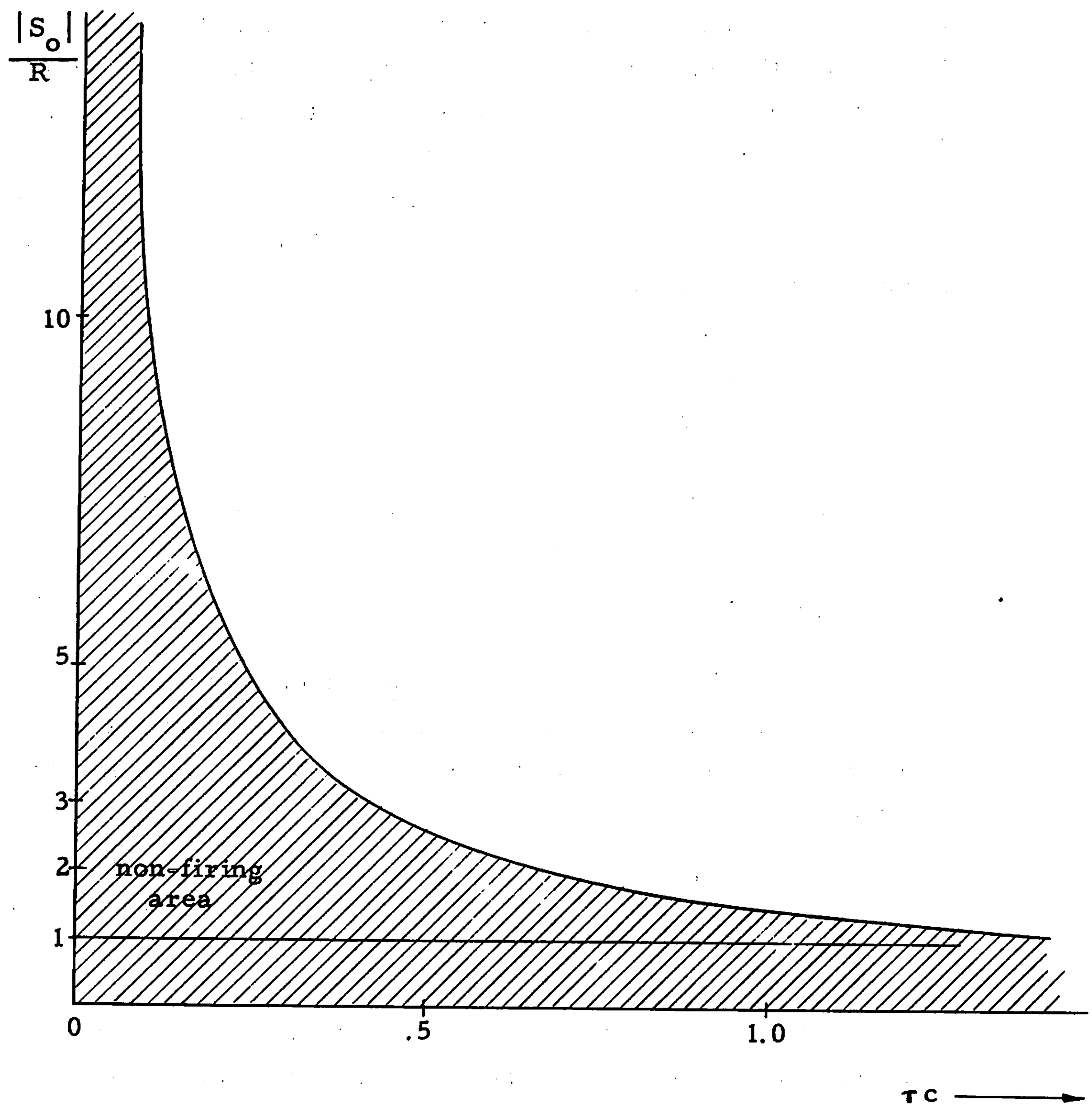


Fig. 2.1

"Strength-duration" curve.

III. RESPONSE TO PULSE TRAINS

In Table 1 the response to a pulse of height S_o and duration τ has been found to be

$$p_1(t) = \frac{S_o}{c} (1 - e^{-c\tau}) e^{-c(t-\tau)} \tag{3.1}$$

for $t > \tau$ and $\tau < \frac{1}{c} \ln \frac{S_o}{S_o - R}$ or

$$| S_o | < \frac{R}{1-e^{-c\tau}} \tag{3.2}$$

Suppose that we have a train of one-signed pulses with period T (Fig. 3.1). Then the response to n of them (assuming that they do

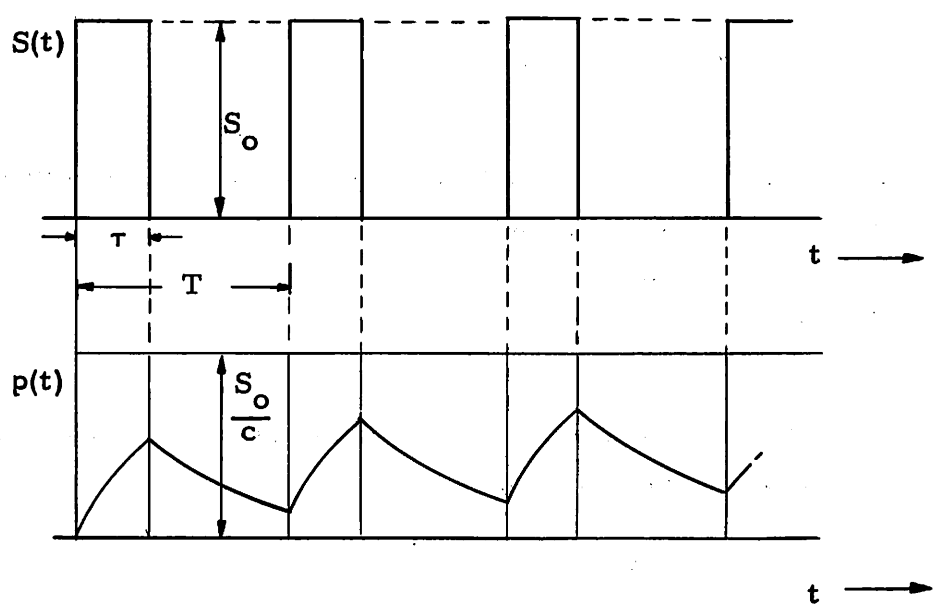


Fig. 3.1
Pulse train response of modulator.

not cause firing) is

$$P(t, n) = \sum_{k=1}^{n} p_k(t) = \sum_{k=1}^{n} \frac{S_o}{c}(1 - e^{-c\tau})\, e^{-c[t-\tau-(k-1)T]} \quad (3.3)$$

or

$$P(t, n) = \frac{S_o}{c}(1 - e^{-c\tau})\, e^{-c(t-\tau)} \sum_{k=1}^{n} e^{c(k-1)T} \quad (3.4)$$

The maximum value of $P(t, n)$ occurs at the end of the n^{th} pulse, i. e., for $t = (n-1)T + \tau$. Therefore,

$$\max P(n) = \frac{S_o}{c}(1 - e^{-c\tau})\, e^{-c(n-1)T} \frac{e^{cnT}-1}{e^{cT}-1} \quad (3.5)$$

or

$$\max P(n) = \frac{S_o}{c}(1 - e^{-c\tau}) \frac{1 - e^{-cnT}}{1 - e^{-cT}} \quad (3.6)$$

$$\lim_{n \to \infty} \max P(n) = \frac{S_o}{c} \frac{1-e^{-c\tau}}{1-e^{-c\tau}} \quad (3.7)$$

From (3. 7) it is obvious that in order that the train of pulses cause firing it is necessary that

$$| S_o | > R \frac{1-e^{-cT}}{1-e^{-c\tau}} \quad (3.8)$$

If (3. 8) holds then the minimum number of pulses required for firing can be found from (3. 6) as

$$n = \frac{1}{cT} \ln \frac{| S_o |}{| S_o | - R \frac{1-e^{-cT}}{1-e^{-c\tau}}} \quad (3.9)$$

By expanding e^{-cT} and $e^{-c\tau}$ we can see that $\frac{1-e^{-cT}}{1-e^{-c\tau}} \approx \frac{T}{\tau}$.

So there is an apparent increase in threshold by approximately T/τ. If the pulses are not all of the same sign then the "apparent threshold increase" is obviously even higher.

This fact indicates that the proposed modulator will have good filtering properties in certain cases. A more systematic discussion of this will follow later.

IV. RESPONSE TO SINUSOIDAL INPUT

From Table I (No. 5) we see that for an input $S_o \sin \omega t$ ($S_o > o$) the response is

$$p(t) = \omega S_o \left[\frac{e^{-ct}}{c^2+\omega^2} + \frac{\sin(\omega t-\psi)}{\omega \sqrt{c^2+\omega^2}} \right] \tag{4.1}$$

where $\psi = \tan^{-1} \omega/c$.

$p(t)$ is a periodic function and we may expect that for high frequencies no firing will occur. To establish conditions for firing we proceed by computing the maximum of $p(t)$:

$$\frac{dp(t)}{dt} = \frac{\omega S_o}{\sqrt{c^2+\omega^2}} \left[\frac{-ce^{-ct}}{c^2+\omega^2} + \cos(\omega t-\psi) \right] = 0$$

for $e^{-ct} \cos\psi = \cos(\omega t-\psi)$. (4.2)

This equation cannot be solved algebraically with respect to t. Assume, however, that it is satsified for $t = t^*$. Then

$$e^{-ct^*} = \frac{\cos(\omega t^*-\psi)}{\cos \psi} = \sqrt{\omega^2 + c^2}\ \frac{\cos(\omega t^*-\psi)}{c}$$

and substituting in (4.1) we obtain

$$p(t^*) = \omega S_o \left[\frac{\cos(\omega t^* - \psi)}{c\sqrt{\omega^2 + c^2}} + \frac{\sin(\omega t^* - \psi)}{\omega\sqrt{\omega^2 + c^2}} \right]$$

or

$$p(t^*) = \frac{S_o}{c} \sin \omega t^* \qquad (4.3)$$

The condition for firing is

$$S_o > \frac{R}{\sin \omega t^*} \qquad (4.4)$$

Relation (4.4) is equivalent to the <u>sampling theorem</u> after substituting t^* from Eq. (4.2) (excluding, of course, the trivial solution $t^* = 0$). In this way we can plot a family of curves (for different values of c) of ω as function of S_o/r.

For $c = 0$, Eq. (4.1) reduces to

$$p(t) = \frac{S_o}{\omega} (1 - \cos \omega t)$$

and it reaches its maximum for $\cos \omega t = -1$, hence

$$\max p(t) = \frac{2S_o}{\omega}$$

Therefore, we have as a condition

$$\omega < \frac{2S_o}{r} \qquad (4.5)$$

(Compare results in Ref. 3.)

Note that (4.2) can be written as

$$e^{-\frac{\omega t c}{c}} \cos \psi = \cos (\omega t - \psi)$$

and by defining $\frac{\omega}{c} = y$ and $\omega t = z$

$$e^{-yz} \cos \psi = \cos (z - \psi) \tag{4.2'}$$

In the same way (4. 4) is written as

$$x \sin z > 1 \tag{4.4'}$$

by defining $x = S_o/R$.

So the sampling theorem is expressed by plotting y as a function of x (compare Eq. 6.15).

A solution of (4. 2) by analog computer is described in Appendix B. However in Sec. V we are going to show that this is not quite necessary because the "sampling theorem" curves can be plotted by using pulses for a test signal instead of sinusoidal waves. The sampling theorem derived there is in good agreement with (4. 2) and (4. 4).

V. MODULATOR FOLLOWED BY INTEGRATOR-EQUIVALENT GAIN

Suppose a modulator is firing pulses $h \cdot \tau_h$ at instants, $t_1, t_2, \ldots, t_n \ldots \ldots$. The integral of its output is given as

$$y(t) = (k-1)h\tau_h + h(t-t_k) \tag{5.1a}$$

for $t_k < t \leq t_k + \tau_h$ and

$$y(t) = kh\tau_h \tag{5.1b}$$

for $t_k + \tau_h < t_{k+1}$. By approximating the output pulse to an impulse of area $\delta = h\tau_h$ we have

$$y(t) = k\delta \tag{5.2}$$

for $t_k \leq t_{k+1}$.

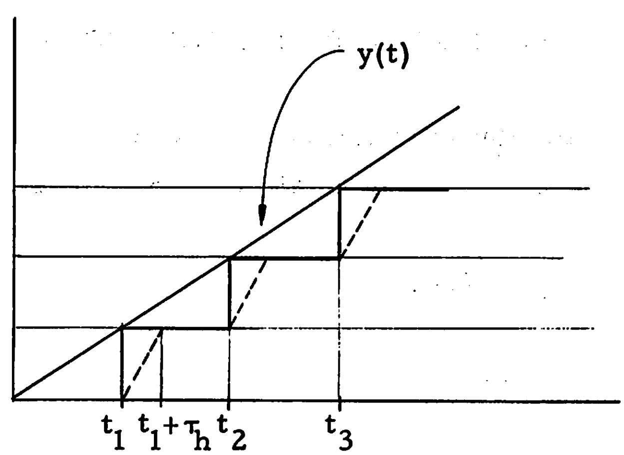
If the input of the modulator is constant S_o then from (2. 2) we have

$$t_{k+1} - t_k = \frac{1}{c} \ell n \frac{S_o}{S_o - R} \qquad [\text{for all } k] \tag{5. 3}$$

by combining (5. 2) and (5. 3) we found the equation of the envelop of the staircase response as:

$$Y(t) = \frac{\delta t}{\frac{1}{c} \ell n \frac{S_o}{S_o - R}} \tag{5. 4}$$

(see Fig. 5. 1). We define as <u>equivalent gain of the modulator,</u> K_m,



<u>Fig. 5. 1. Response of integrator to constant input and its envelop.</u>

the slope of Y(t) divided by S_o. By simple manipulations we obtain:

$$K_m = \frac{\delta}{r} \frac{1}{x \ell n \frac{x}{x-1}} \qquad \text{if } x > 1$$

$$K_m = 0 \qquad \text{if } x \leq 0 \tag{5. 5}$$

where $x \triangleq S_o/R$.

It can be easily verified that $\lim_{x \to \infty} x \, \ell n \frac{x}{x-1} = 1$.

Because of the existence of the "strength-duration" curve, the use of the equivalent gain is rather limited in any analysis of a system using this modulator. Its only value seems to be in comparison of a N. P. F. M. system with other forms. For this purpose we will use the $\lim_{x \to \infty} K_m = \delta/r$, which we will denote by the term "equivalent DC gain." A plot of the function represented by (5.5) is shown in Fig. 5.2.

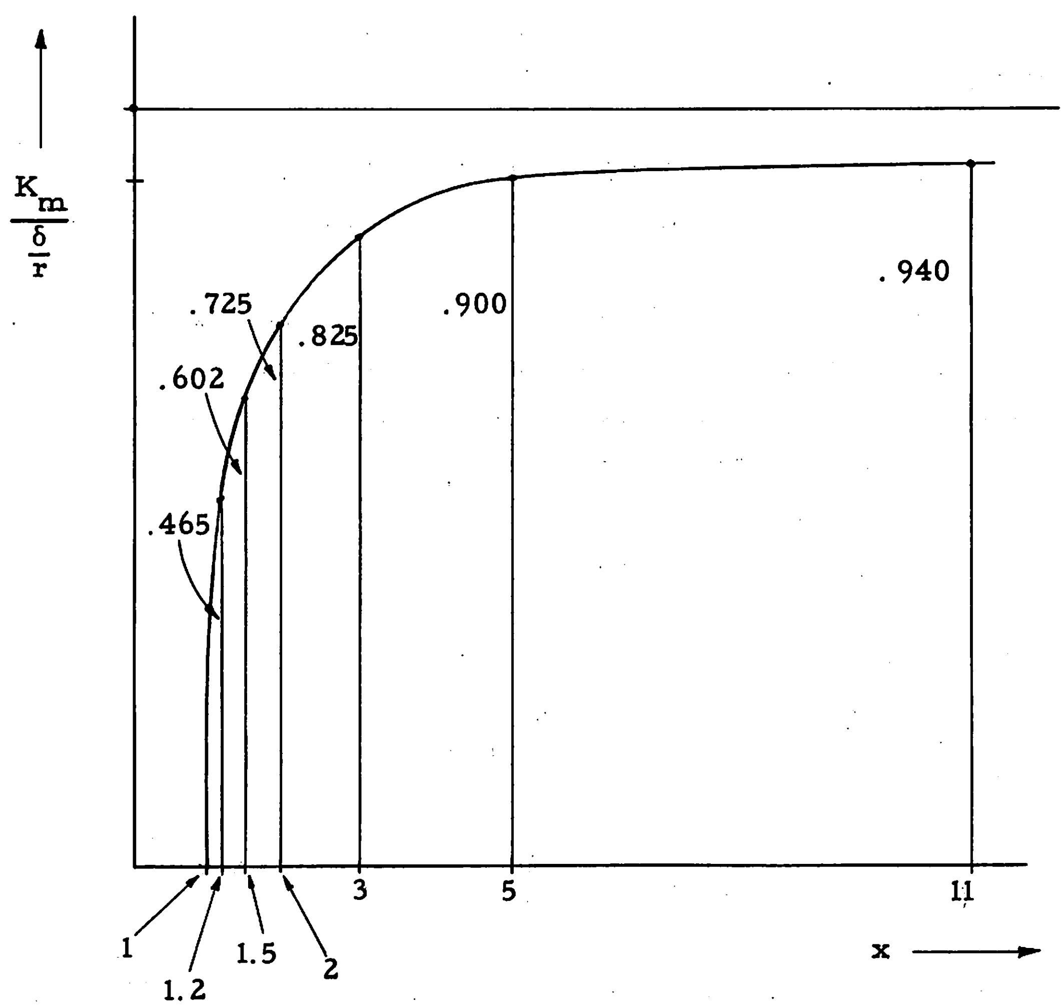


Fig. 5.2

Equivalent gain of modulator for constant input $S_o = R x$

If we assume finite duration of pulses τ_h, then the modulator reaches saturation when

$$S_o = \frac{R}{1-e^{-c\tau_h}} \triangleq S_s \tag{5.6}$$

Obviously, this relation holds only for constant input. However, for characterization we define S_s as the <u>saturation level of the modulator.</u>

The output of the integrator will be then

$$Y(t) = ht \tag{5.7}$$

We call <u>saturation frequency</u>, $\underline{f_s}$ (under a certain input) the inverse $1/\tau_h$ of the pulse duration. This quantity is obviously meaningful under all circumstances.

VI. DERIVATION OF A SAMPLING THEOREM THROUGH WALSH FUNCTIONS

In Sec. II we found the response of the modulator to a pulse. We will consider now the case of a doublet (a positive pulse followed by a negative), with duration of each half τ. Suppose that firing occurs in the first half at time $\tau_1 < \tau$ defined by

$$p_1(\tau_1) = \frac{S_o}{c}\left(1 - e^{-c\tau_1}\right) = r \tag{6.1}$$

The response in the second half will be:

$$p_2(t) = -\frac{S_o}{c}\left(1 - e^{-c(t-\tau)}\right) + \frac{S_o}{c}\left(1 - e^{-c(\tau-\tau_1)}\right)e^{-c(t-\tau)}$$

or

$$p_2(t) = -\frac{S_o}{c}\left(1 - e^{-c(t-\tau)}\right) + \frac{S_o}{c}\left(e^{-c(t-\tau)} - e^{-c(t-\tau_1)}\right) \tag{6.2}$$

The second term is always positive because $e^{-c(t-\tau)} > e^{-c(t-\tau_1)}$;

therefore, no firing will occur at $t = \tau + \tau_1$.

Consider, however, the moment 2τ. Then we have

$$p_2(2\tau) = -\frac{S_o}{c}(1 - e^{-c\tau}) + \frac{S_o}{c}\left(e^{-c\tau} - e^{-c(2\tau-\tau_1)}\right) \qquad (6.3)$$

or

$$p_2(2\tau) = -\frac{S_o}{c}\left[1 - 2e^{-c\tau} + e^{-c(2\tau-\tau_1)}\right] \qquad (6.4)$$

The term in brackets is positive; therefore,

$$|p_2(2\tau)| = \frac{S_o}{c}\left(1 - 2e^{-c\tau} + e^{-c(2\tau-\tau_1)}\right) \qquad (6.5)$$

Equation (6.5) can be written as

$$|p_2(2\tau)| = \frac{S_o}{c}\left[1 - e^{-c\tau_1}\left(2e^{-c(\tau-\tau_1)} - e^{-2c(\tau-\tau_1)}\right)\right] \qquad (6.6)$$

But $2e^{-a} - e^{-2a}$ is less than 1 for any $a \neq 0$, therefore, the term in brackets is larger than $(1 - e^{-c\tau_1})$ and from (6.1) we have

$$|p_2(2\tau)| > r \qquad (6.7)$$

(The case $a = 0$ corresponds to $\tau = \tau_1$ and this is the case when firing occurs at the end of both half periods). Hence, whenever firing occurs at the first half, it also occurs at the second, although with some delay. Obviously, whenever firing does not occur at the first half, it does not occur at the second. This discussion shows that the square wave of amplitude S_o input "passes" through the modulator if and only if its half period is larger than

$$\frac{1}{c}\,\ell n\,\frac{S_o}{S_o - R}$$

or the critical frequency which causes response is:

$$\omega_c = \frac{\pi c}{\ell n \dfrac{S_o/r}{\dfrac{S_o}{r} - c}} \qquad (6.8)$$

where $\omega_c = 0$ for $S_o/r < c$. For $c = 0$ we find that

$$\lim_{c \to 0} \frac{1}{c} \ln \frac{S_o/r}{S_o/r - c} = \frac{r}{S_o}$$

and therefore

$$\omega_o = \frac{\pi S_o}{r} \tag{6.9}$$

To compare with (4.5) we must note that the mean half-wave value of a sinusoidal wave of peak S_o is $(2/\pi)S_o$; therefore (4.5) becomes

$$\omega < \frac{\pi S_{av}}{r} \tag{4.5'}$$

which is in absolute agreement with (6.9). This shows that for $c = 0$, a doublet and a sinusoidal wave "obey" the same sampling theorem. This is not surprising because a doublet is a special case of Walsh functions. The Walsh functions constitute a complete set of orthogonal functions[7] and therefore can be used as a test signal instead of sinusoidal waves. These functions are defined on an interval $0, T$ as follows:

$$\phi(0,0;x) = 1 \qquad 0 \le x \le T$$

$$\phi(1,1;x) = \begin{cases} 1 & 0 \le x < \frac{T}{2} \\ -1 & \frac{T}{2} \le x < T \end{cases}$$

$$\phi(2,1;x) = \begin{cases} 1 & 0 \le x < \frac{T}{4} \qquad \frac{T}{2} < x < \frac{3T}{4} \\ -1 & \frac{T}{4} \le x < \frac{T}{2} \qquad \frac{3T}{4} < x \le T \end{cases}$$

$$\left.\begin{aligned}
\phi(2, 2;x) &= \begin{cases} 1 & 0 \le x \;\frac{T}{4} \qquad \frac{3T}{4} < x \le T \\ -1 & \frac{T}{4} < x \; \frac{3T}{4} \end{cases} \\
\phi(n+1;2k-1;x) &= \begin{cases} \phi(n, k; 2x) & 0 \le x < \frac{T}{2} \\ (-1)^{k+1}(n, k; 2x-1) & \frac{T}{2} < x \le T \end{cases} \\
\phi(n+1, 2k;x) &= \begin{cases} \phi(n, k; 2x) & 0 \le x < \frac{T}{2} \\ (-1)^{k}\phi(n, k; 2x-1) & \frac{T}{2} < x \le T \end{cases}
\end{aligned}\right\} \qquad (6.10)$$

where $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, 2^{n-1}$.

These functions consist of nonsymmetric doublets, as one can easily see (or see Fig. 1 in Ref. 7); therefore, in order that a Walsh function may pass "undistorted" through the modulator, firing must occur during its shortest pulse. The duration T_d of this is, as one can see from (6.10),

$$\begin{aligned}
T_d &= T \quad \text{for} \quad \phi(0, 0) \\
T_d &= \frac{T}{2} \quad \text{for} \quad \phi(1, 1) \text{ and } \phi(2, 2) \\
T_d &= \frac{T}{4} \quad \text{for} \quad \phi(2, 1) \text{ and } \phi(3, 4) \\
T_d &= \frac{T}{8} \quad \text{for} \quad \phi(3, 1),\ \phi(3, 2),\ \phi(3, 3),\ \text{etc.}
\end{aligned} \qquad (6.11)$$

Therefore, we may associate any Walsh function with a doublet frequency π/T_d.

Therefore, a necessary and sufficient condition that a Walsh function may pass through the modulator undistorted (i. e., with firing occurring at least once for each one of its pulses) is that its associated frequency is less than the one given by (6.8). Equation (6.8) is plotted in Fig. 6.1 with c as a parameter. We see that for $S_o/r = c$, $\omega_c = 0$ and moreover, we can find the asymptote of each curve as follows:

$$\text{Slope} = \lim_{S_o/r \to \infty} \frac{\omega_c}{S_o/r} = \lim_{x \to \infty} \frac{\pi}{x \ln \frac{x}{x-1}} = \pi \tag{6.12}$$

$$\text{Ordinate at the origin} = \lim_{S_o/r \to \infty} \left[\omega_c - \pi \frac{S_o}{r}\right] = \pi c \lim_{x \to \infty} \left(\frac{1}{\ln \frac{x}{x-1}} - x\right)$$

$$= -\frac{\pi c}{2} \tag{6.13}$$

$(x \triangleq \frac{S_o/r}{c})$.

Therefore, a simplified relation can be written as

$$\begin{aligned} \omega_c &= 0 \quad \text{for} \quad \frac{S_o}{r} \leq c \\ \omega_c &= \pi \left(\frac{S_o}{r} - \frac{c}{2}\right) \quad \text{for} \quad \frac{S_o}{r} > c \end{aligned} \tag{6.14}$$

This is shown by dotted lines in Fig. 6.1. It holds with good approximation for $S_o/r > 2c$.

Equation (6.8) can be also plotted in a simpler form by defining $y = \omega/\pi c$ and $x = S_o/rc$; then we have

$$y = \frac{1}{\ln \frac{x}{x-1}} \tag{6.15}$$

and the plot is shown in Fig. 6.2.

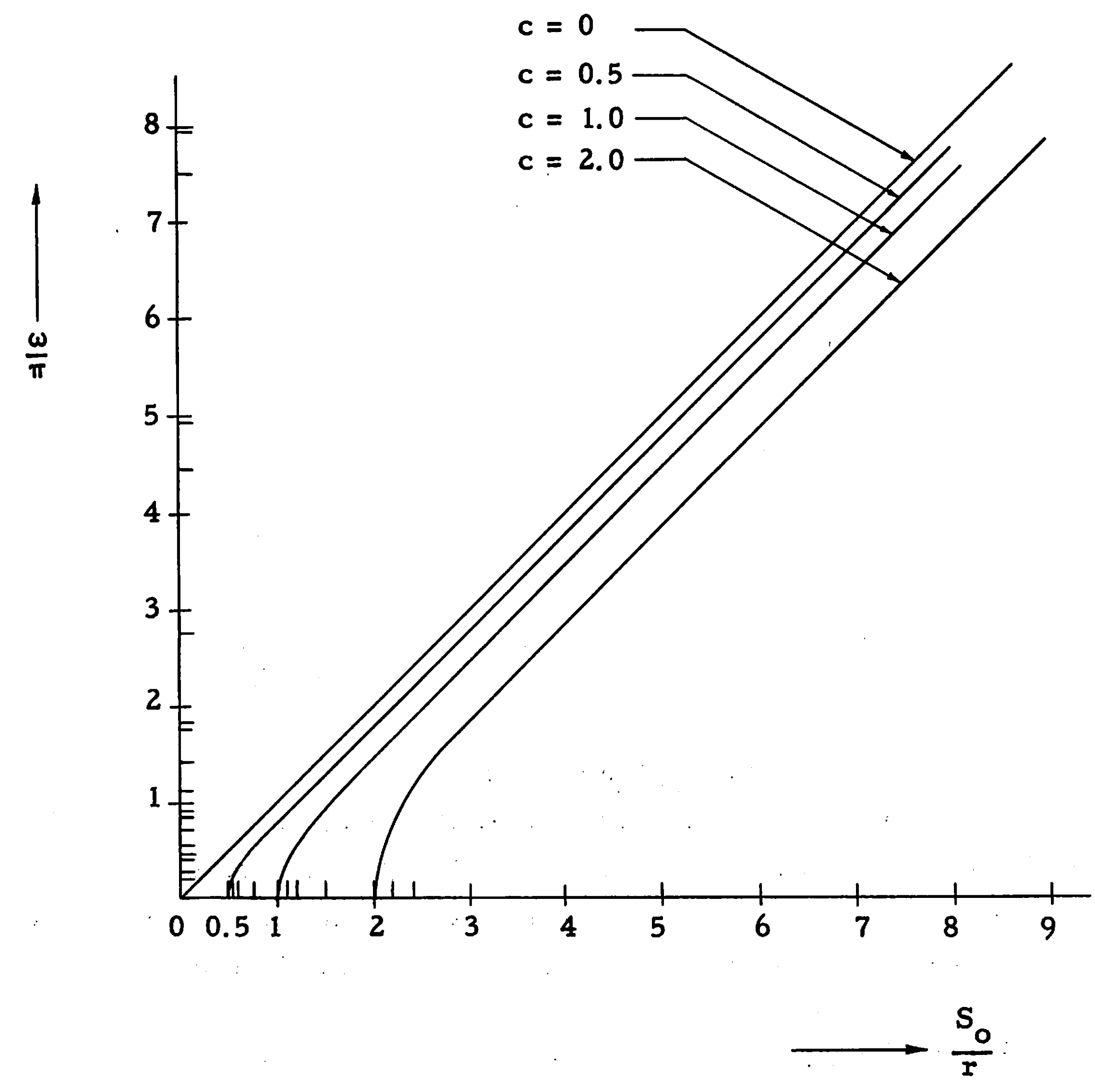


Fig. 6.1

Sampling theorem curves.

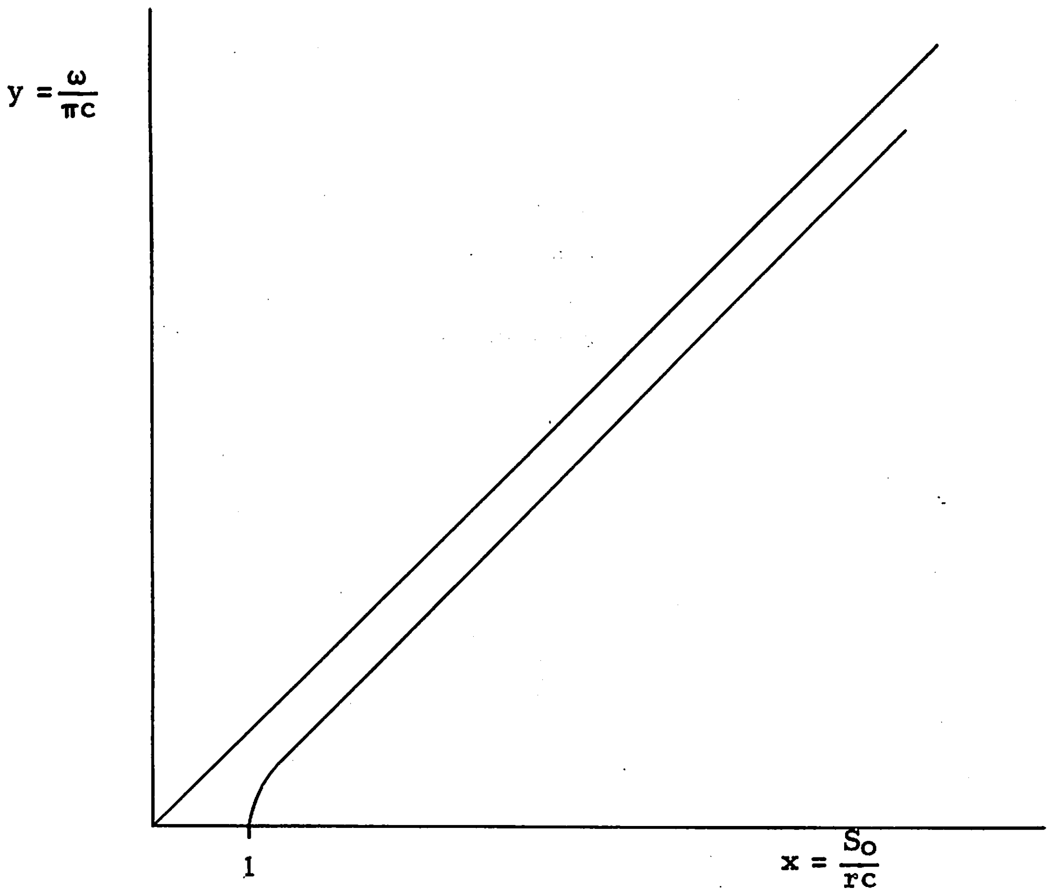


Fig. 6.2

Sampling theorem curve.

This curve can be referred to as the sampling curve and Eq. (6.8) as the sampling theorem for N. P. F. M. systems.

It is interesting to verify that these results are in agreement with the sampling theorem derived for sinusoidal inputs:

Numerical Example

For $x = 2$ $y = \omega/\pi c = 1.44$ (from (6.15)) or $\omega/c = 4.52$. To apply (4.4) we take $S_o/R = \pi/2 \cdot x = 3.14$. Therefore, $\sin \omega t = 1/3.14$ and $\omega t = 161.5^o = 2.819$. $\psi = \tan^{-1} 4.52 = 77.8^o$, $\cos \psi = 0.212$, $\cos(\omega t - \psi) = 0.110$. $ct = \frac{\omega t}{\omega/c} = 0.624$ and $e^{-ct} = 0.535$. Therefore, $e^{-ct} \cos \psi = 0.113$ and (4.2) is satisfied. (Difference 0.003 due to the use of slide rule.) The result depends only on the ratios ω/c and S_o/R, as it should.

This agreement was expected because the sinusoidal wave is expandable on Walsh functions.

VII. QUASI-DESCRIBING FUNCTION

Since Walsh functions constitute a complete basis, one may try to approximate a function by their "fundamental", i.e., $\phi(1,1)$, which actually is a square wave (6.10). If a function has odd-symmetry [i.e., $f(t + T/2) = -f(t)$], then obviously it has no DC component. The coefficient of the expansion corresponding to $\phi(1,1)$ can be found as

$$a_1 = \frac{\int_0^T y(t),\ \phi(1,1,t)dt}{\int_0^T \phi(1,1,t)\ \phi(1,1,t)dt} = \frac{\int_0^{T/2} y(t)dt - \int_{T/2}^T y(t)\ dt}{\int_0^T 1 \cdot dt}$$

or

$$a_1 = 2/T \int_0^{T/2} y(t)dt \qquad (7.1)$$

Therefore, the amplitude of the square wave will be equal to the average value of the function over a half period. For example, if $y(t)$ is a sine wave with unit peak, $a_1 = 2/\pi$. Or, if $y(t)$ is a triangular wave with unit peak, $a_1 = 1/2$.

We proceed now to study the response of the modulator to a square wave of amplitude $S_o(S_o > R)$. The firing rate will be given by

$$t_f = \frac{1}{c} \ell n \frac{S_o}{S_o - R} \qquad \text{(from Table I, No. 1).}$$

The number n of pulses per half period $\frac{T}{2}$ will satisfy the following inequality $(\omega = 2\pi/T)$.

$$\frac{c\pi}{\omega} \frac{1}{\ln \frac{S_o}{S_o - R}} - 1 < n \leq \frac{c\pi}{\omega} \frac{1}{\ln \frac{S_o}{S_o - R}} \tag{7.2}$$

We consider two extreme cases.

(a) $$n = \frac{c\pi}{\omega} \frac{1}{\ln \frac{S_o}{S_o - R}}$$

Then, if the modulator is followed by an integrator, the response at the end of a half wave will be

$$y_m(t) = n\,\delta = \frac{c\pi\delta}{\omega} \frac{1}{\ln \frac{S_o}{S_o - R}} \tag{7.3}$$

On the other hand, the unmodulated response would have been

$$y(t) = \frac{S_o T}{2} = \frac{S_o \pi}{\omega} \tag{7.4}$$

and the equivalent gain of the modulator is (in analogy with Sec. 5)

$$K_m = \frac{y_m(t)}{y(t)} = \frac{c}{S_o} \frac{1}{\ln \frac{S_o}{S_o - R}} = \frac{\delta}{r} \frac{1}{\frac{S_o}{r} \ln \frac{S_o/R}{S_o/R - 1}} \tag{7.5}$$

We see that this equals exactly the expression given by (5.5).

(b) $$n = \frac{c\pi}{\omega} \frac{1}{\ln \frac{S_o}{S_o - R}} - 1$$

In this case the equivalent gain will be

$$K'_m = \frac{\dfrac{c\pi\delta}{\omega}\,\dfrac{1}{\ell n \dfrac{S_o}{S_o - R}} - \delta}{\dfrac{S_o \pi}{\omega}} = K_m - \frac{\omega\delta}{S_o\pi} \tag{7.6}$$

Note that

$$\frac{\omega\delta}{S_o\pi} = \frac{\delta}{r}\,\frac{\omega}{\dfrac{S_o}{rc}\pi c} = \frac{\delta}{r}\,\frac{\omega}{\dfrac{S_o}{R}\pi c} \tag{7.7}$$

By defining $x \triangleq S_o/R$, Eqs. (7.5), (7.6), and (7.7) can be combined to give the following inequality for K_m.

$$\frac{\delta}{r}\left[\frac{1}{x\ell n \dfrac{x}{x-1}} - \frac{\omega}{x\pi c}\right] < K_m \leq \frac{\delta}{r}\,\frac{1}{x\ell n \dfrac{x}{x-1}} \tag{7.8}$$

As an extension of the describing function we define as quasi-describing function the mean value of the limits of the inequality (7.8), i.e.,

$$K(x, \omega) = \frac{\delta}{r}\left[\frac{1}{x\ell n \dfrac{x}{x-1}} - \frac{1}{2}\,\frac{\omega}{x\pi c}\right] \tag{7.9}$$

This can be used in an approximate analysis of N. P. F. M. systems.

Note that any memoryless nonlinearity following the modulator appears as linear gain because of the constant amplitude of the pulses. Also, with square wave as test signal, nonlinearities before the modulator can be treated in a very simple way.

Attention also must be paid to the fact that in (7.9) an integrator has been already included and the linear plant transfer function must be modified in accordance. Another fact which must be noted is that $K(x, \omega)$ includes a half-period lag plus a quantity which we proceed to compute. If

$$n = \frac{c\pi}{\omega} \frac{1}{\ell n \frac{S_o}{S_o - R}}$$

then $p(T/2) = 0$ and the firing time for the first negative impulse will be $t_f = 1/c \cdot \ell n \left[S_o/(S_o - R) \right]$. Half of it corresponds to the extra lag, as one can see from a simple sketch of the response. Therefore, a lag

$$\frac{1}{2c} \ell n \frac{x}{x-1}$$

is introduced and in the phase angle this total lag corresponds to

$$\phi'_m = \frac{\pi}{2} + \frac{\omega}{2c} \ell n \frac{x}{x-1} \tag{7.10}$$

If

$$n = \frac{c\pi}{\omega} \frac{1}{\ell n \frac{S_o}{S_o - R}} - 1$$

it can be easily seen that the extra lag is about the same. Therefore, we finally have

$$\phi(x, \omega) = \frac{\pi}{2} + \frac{\omega}{2c} \ell n \frac{x}{x-1} \tag{7.11}$$

The complex-valued quasi-describing function is given by

$$K_c = K(x, \omega) \, e^{-j\phi(x, \omega)} \tag{7.12}$$

with $K(x, \omega)$ from (7.9) and $\phi(x, \omega)$ from (7.11) (for $x \leq 1$, $K_c = 0$).

VIII. CLOSED LOOP SYSTEM—STABILITY

We are going to consider for a while the more general system, i.e., where the "exciter" p(t) is related to the "stimulus" S(t) by the relation:

$$\frac{P(s)}{S(s)} = G_1(s) \tag{8.1}$$

where $G_1(s)$ is any rational function.

Again, whenever p(t) reaches r we have the emission of a pulse of the same sign as p(t) and then the initial conditions, i.e., p(o), p'(o), p''(o), etc., are set to zero.

Such a system is shown in Fig. 8.1.

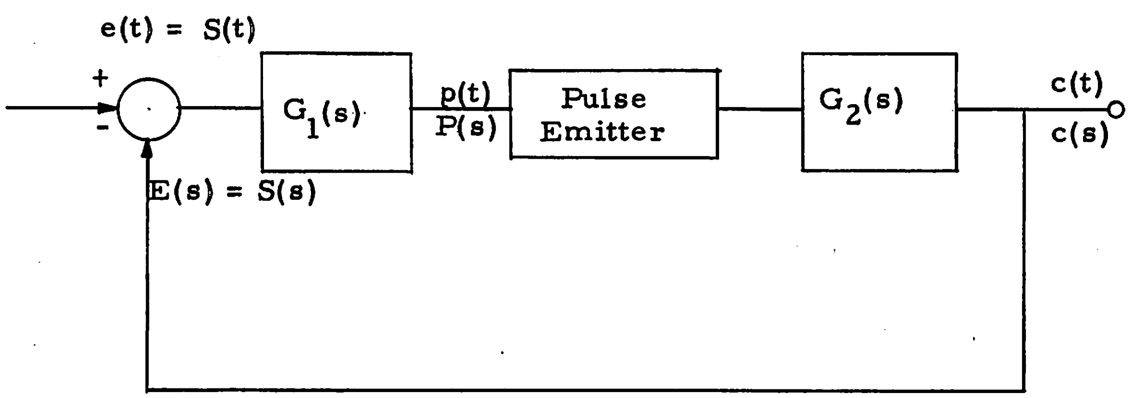


Fig. 8.1

The impulse response of such a system is not very meaningful for reasons explained in Sec. II. So we assume as testing signal a pulse emitted by the modulator.

We define a system as stable if the response p(t) to an impulse emitted by the modulator takes a subthreshold value in finite time and stays there. (It is understood that the system has been in equilibrium before.) This definition corresponds to the case of asymptotic stability.[8]

For simplification we approximate the pulses by impulses of the same area, δ. Then the response of the "plant" after n pulses will be:

$$c(t) = K\,\delta \sum_{i=1}^{n} \epsilon_i g_2(t-t_i)\, u(t-t_i) \tag{8.2}$$

where

$$\begin{aligned} \epsilon_i &= 1 \text{ if } p_{i-1} > r \\ \epsilon_i &= 0 \text{ if } |p_{i-1}| < r \\ \epsilon_i &= -1 \text{ if } p_{i-1} < -r \end{aligned} \tag{8.3}$$

or taking the Laplace transform

$$C(s) = K\,\delta\, G_2(s) \sum_{i=1}^{n} \epsilon_i\, e^{-st_i} \tag{8.4}$$

for zero input:

$$S(s) = E(s) = -C(s) \tag{8.5}$$

Therefore,

$$P(s) = K\, G_1(s)\, G_2(s) \sum_{i\pm 1}^{n} \epsilon_i\, e^{-st_i} \tag{8.6}$$

Normalizing $P(s)$ with respect to r and noting that $\frac{\delta}{r}$ is the equivalent gain of the modulator K_m we have

$$P_n(s) = K\, K_m\, G_1(s)\, G_2(s) \sum_{i=1}^{n} \epsilon_i\, e^{-st_i} \tag{8.7}$$

We want $p_n(t) < 1$ for all t. This can be obtained by choosing low enough gain if the linear system $G_1(s)\ G_2(s)$ is stable. Note that this is too strict a requirement because the quantity

$$\sum_{i=1}^{n} \epsilon_i\, e^{-st_i}$$

may be possibly equal to zero. However, this cannot be checked unless the transient response of the system is studied in detail (see next paragraph). So we conclude that the system of Fig. 8.1 is <u>structurally stable</u> (i. e., can be made stable by choosing low enough gain K^9) whenever $G_1(s) G_2(s)$ is stable.

We consider now different special cases.

(a) <u>Integral Pulse Frequency Modulation.</u>

Here, $G_1(s) = 1/s$; hence

$$\frac{G_2(s)}{s}$$

must be stable. If $G_2(s) = 1/s$, then we have a double pole at the origin. In this case the response of the system is simply bounded but presents sustained oscillations. This checks with earlier results (Ref. 3, p. 82), that whenever the linear plant is stable, the I. P. F. M. system has bounded response.

(b) <u>Neural Pulse Frequency Modulation.</u>

Here

$$G_1(s) = \frac{1}{s+c}$$

Therefore, $G_2(s)$ must be stable, so whenever the linear plant is stable the N. P. F. M. system is structurally stable.

This general stability criterion does not give much information about the design parameters of the system, so in the next few paragraphs we are going to examine in more detail certain low order systems.

IX. STABILITY OF LOW ORDER SYSTEMS

A. Consider an impulse of area δ emitted in the system of Fig. 9.1

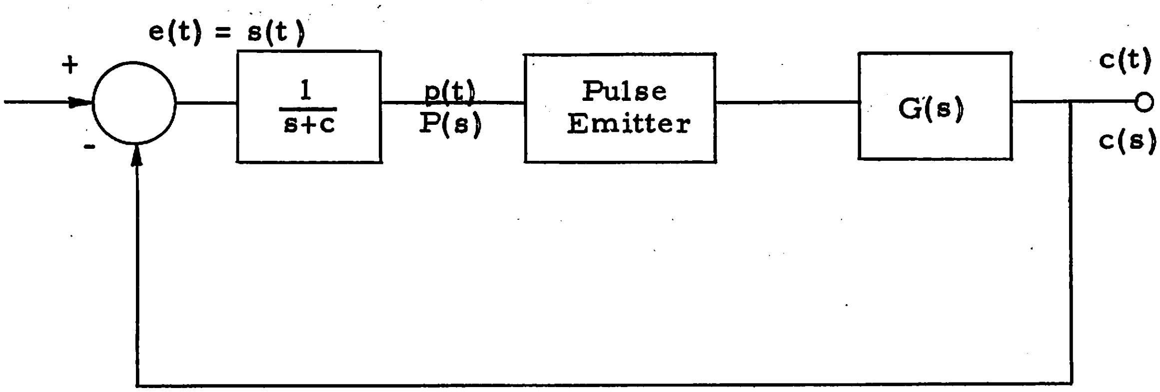


Fig. 9.1

with $G(s) = K/s$. The output of the system will be

$$c(t) = K\delta \tag{9.1}$$

$$p(t) = -\frac{K\delta}{c}(1 - e^{-tc}) \tag{9.2}$$

If

$$\frac{K\delta}{c} < r$$

no other impulse will be emitted. This condition of "quietness" is described by:

$$KK_m < c \tag{9.3}$$

If $K\delta > r$, then after some time t_r, a negative impulse will be emitted and the output will be zero.

B. Consider the system of Fig. 9.1 with $G(s) = K/S^2$. Then after the emission of a pulse we have

$$c_o(t) = K\delta t \tag{9.4}$$

$$p_o(t) = -\frac{K\delta}{c}t + \frac{K\delta}{c^2}(1 - e^{-ct}) \tag{9.5}$$

After a certain time t_1 a negative impulse is emitted and

$$c_1(t) = K\delta t - K\delta(t - t_1) = K\delta t_1 \tag{9.6}$$

$$p_1(t) = -\frac{k\delta t_1}{c}(1 - e^{-c(t-t_1)}) \tag{9.7}$$

It is simple to check that $\frac{k\delta t_1}{c} > r$ (see Fig. 9.2). Thus, another negative impulse will be emitted after a certain time t_2, and then

$$c_2(t) = K\delta t_1 - K\delta(t - t_2)$$

$$p_2(t) = -\frac{K\delta\, t_1}{c}\,(1 - e^{-c(t-t_2)}) + \frac{K\delta}{c}\,(t - t_2) - \frac{K\delta}{c^2}\,(1 - e^{-c(t-t_2)}) \tag{9.8}$$

or

$$p_2(t) = -\left[\frac{K\delta}{c}\;\; t_1 + \frac{1}{c}\left(1 - e^{-c(t-t_2)}\right)\right] + \frac{K\delta}{c}\;(t - t_2) \tag{9.9}$$

We compute the maximum of $p_2(t)$

$$\frac{dp_2(t)}{dt} = -\,K\delta\,(t_1 + \frac{1}{c})\,e^{-c(t-t_2)} + \frac{K\delta}{c} = 0$$

for $e^{-c(t-t_2)} = \frac{1}{1 + ct_1}$ or

$$t - t_2 = \frac{1}{c}\,\ell n(1 + ct_1)$$

Substituting these values into (7.9) we obtain

$$p_2 \max = -\frac{K\delta\, t_1}{c} + \frac{K\delta}{c^2}\,\ell n\,(1 + ct_1)$$

But from (7.5) we see that

$$\frac{K\delta\, t_1}{c} = \frac{K\delta}{c^2}\,(1 - e^{-ct_1}) + r$$

Therefore,

$$p_2 \max = \frac{K\delta}{c^2}\;\left[-\,(1 - e^{-ct_1}) + \ell n\,(1 + ct_1)\right] - r \tag{9.10}$$

But $(1 - e^{-ct_1}) < \ln(1 + ct_1)^{+}$; hence the term in brackets is always positive and p_2 max > -r. Hence no firing of negative impulse occurs.

However, a firing of a positive impulse occurs at time t_3. From Fig. 9.2 (or Eq. 9.9) we easily see that $c(t) < R$ and therefore the system settles down with subthreshold response. Therefore, it is always asymptotically stable.

The fact that a system with only a double pole at the origin is asymptotically stable shows the advantage of the N. P. F. M. The conditions derived in Sec. VIII are too restrictive. Indeed, in (8.7) $\epsilon_i = 0$ for $i > 4$, $\epsilon_i = 1$, $\epsilon_2 = -1$, $\epsilon_3 = -1$, $\epsilon_4 = +1$. Therefore,

$$p_n(s) = K K_m \frac{1}{s^2(s+c)} \left[1 - e^{-st_1} - e^{-st_2} + e^{-st_3}\right]$$

$$\lim_{s \to o} s p_n(s) = K K_m \frac{1}{c} \lim_{s \to o} \frac{1 - e^{-st_1} - e^{-st_2} + e^{-st_3}}{s}$$

$$= K K_m \frac{1}{c} \lim_{s \to o} \frac{1 - 1 + st_1 + [s^2] - 1 + st_2 + [s^2] + 1 - st_3 + [s^2]}{s}$$

$$= K K_m \frac{1}{c}(t_1 + t_2 - t_3) \qquad ([s^2] \triangleq \text{terms with factor } s^2)$$

Therefore, the final value of $p(t)$ is finite and coincides with

$$\frac{c_3(t)}{rc} = \frac{1}{c}[K\delta t_1 - K\delta(t - t_2) + K\delta(t - t_3)] = \frac{1}{c} K \frac{\delta}{r}(t_1 + t_2 - t_3)$$

$^{+}$ $\ln(1 + x) = 1 - e^{-x} = 0$, for $x = 0$. Taking the derivative we see that $\frac{d\ln(1+x)}{dx} = \frac{1}{1+x} > 0$ for $x > 0$ and $\frac{d}{dx}(1 - e^{-x}) = e^{-x} > 0$ for $x > 0$. But $\frac{1}{1+x} > e^{-x}$ hence $\ln(1 + x)$ increases faster than $1 - e^{-x}$.

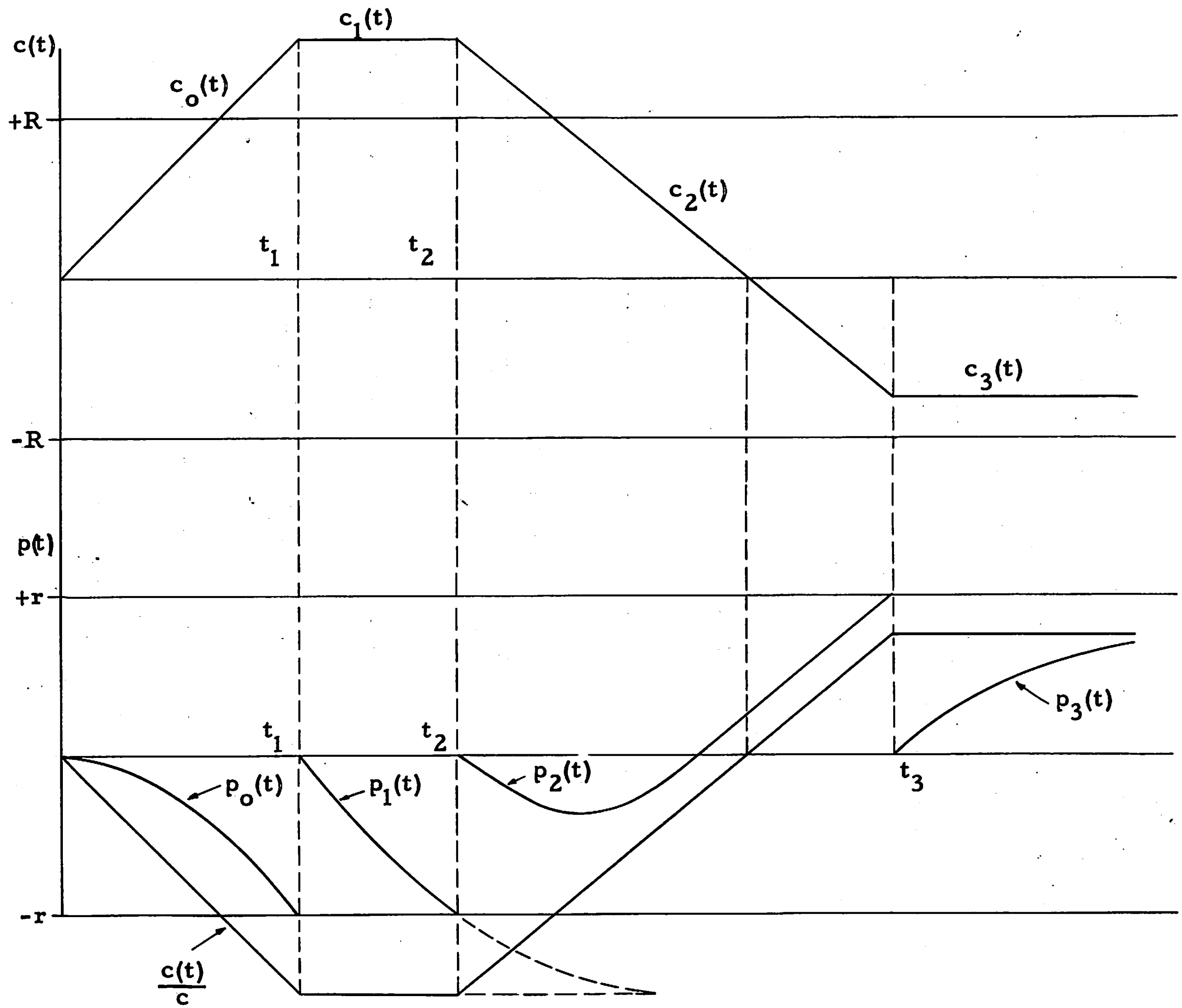


Fig. 9.2

Response of double integrator closed loop system to a pulse.

In a similar way the stability of any first or second order system can be checked.

We proceed furthermore to prove that the system with open loop transfer function K/s^2 is asymptotically stable in the large. The response to a step input is shown in Fig. 9.3. Assume

$$y_k = y_{k+1} > r \tag{9.11}$$

Then

$$y_k = y_{k-1} + \delta(t_k - t_{k-1}) \tag{9.12}$$

$$y_{k+2} = y_{k+1} - \delta(t_k - t_{k-1}) \tag{9.13}$$

Therefore,

$$y_{k+2} = y_{k-1} + \delta[(t_k - t_{k-1}) - (t_{k+2} - t_{k+1})] \tag{9.14}$$

Obviously $t_{n+2} - t_{n+1} > t_n - t_{n-1}$ because $y(t)$ is increasing in its value in the first case and decreasing in the second. Therefore,

$$y_{k+2} < y_{k-1} \tag{9.15}$$

With a similar argument we can generally see that

$$y_{k+m+1} < y_{k-m} \tag{9.16}$$

Assume now that $y_{k-m} < r$ (first impulse as shown in Fig. 9.3). Then the last impulse will occur at most at y_{k+m}. Therefore, the system emits $k - (k-m) = m$ impulses before "leveling" and $(k + m) - (k + 1) = m - 1$ after. Therefore, the slope after each

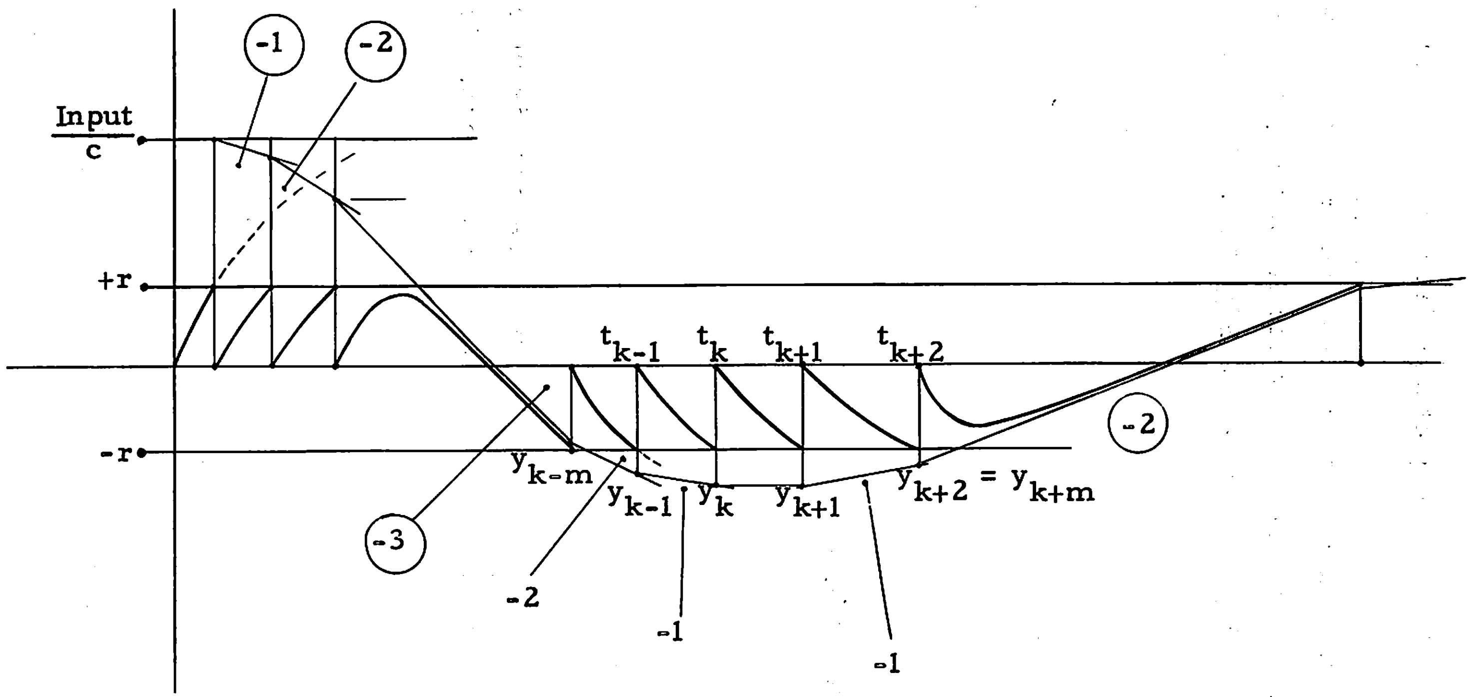


Fig. 9. 3

Response of a double integrator closed loop system to a step input.

maximum (or minimum) is decreased by one and the oscillations eventually will stop. Hence, the system is asymptotically stable in the large.

X. RAMP INPUT RESPONSE

Consider the system of Fig. 9.1 with $G(s) = K/s$ and a ramp input $S_o t$. Assume the n^{th} impulse is emitted at time t_n. Then the input to the modulator will be

$$S_o t - nK\delta = S_o(t - t_n) + S_o t_n - nK\delta \tag{10.1}$$

After a pulse is emitted at time t_{n+1} the input will be

$$S_o(t - t_{n+1}) + S_o t_{n+1} - (n + 1) K\delta \tag{10.2}$$

The system obviously will reach a constant rate of firing when

$$S_o t_{n+1} - (n + 1) K\delta = S_o t_n - nK\delta \tag{10.3}$$

or

$$S_o(t_{n+1} - t_n) = K \delta \tag{10.4}$$

The slope of the envelope of the output in this will be

$$\frac{\Delta c}{\Delta t} = \frac{K\delta}{K/S_o} = S_o \tag{10.5}$$

Therefore, the system follows exactly the input. This is a verification that the system reproduces time-varying inputs.

The time between two firings is, from (10.4),

$$\Delta t = \frac{K}{S_o} = K K_m \frac{r}{S_o} \tag{10.6}$$

The time for the first firing is found from Table I, No. 2, to be approximately

$$(\Delta t)_o \cong \sqrt{\frac{2r}{S_o}} \tag{10.7}$$

In a similar way the response to various inputs of a given system can be found.

XI. ELEMENTARY ANALYSIS OF FILTERING PROPERTIES

In any P. F. M. system one may consider the injection of noise before or after the modulator. C. C. Li[3] (pp. 134-193) studied the second case in an I. P. F. M. and found that the system is relatively immune to noise. This is not a surprising result, and can be generalized to any pulse-frequency-modulated system,[4] or even to any system transmitting information in discrete form. Of course, this advantage is offset by the reduction in the bandwidth of the system. Therefore, we will not consider this case further. The more interesting (and more difficult to solve) problem is the injection of noise at the input of the modulator.

For the purpose of elementary analysis (for the sake of comparing various systems), we proceed with the following assumptions:

(a) both signal and noise are in the form of pulses (this is a reasonable assumption in view of the discussions of Secs. VI and VII);

(b) both are stationary processes and their distributions are such that the expectations of both the time duration T and the absolute value of amplitude H at the signal are much greater than those of the noise (Fig. 11.1). (This is not a very good assumption, because in any case such a distribution does not present a difficult filtering problem.)

We define as <u>first filtering figure of merit</u> of a system the

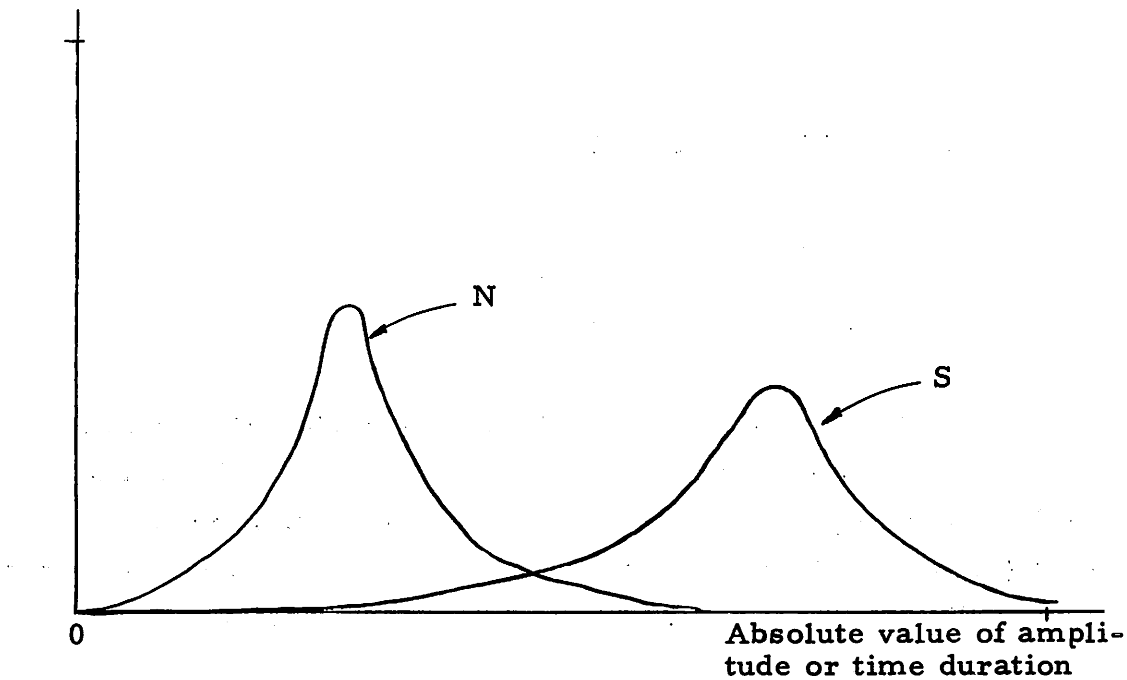


Fig. 11.1
Probability density distributions of signal (S) and Noise (N)

equivalent gain "Ψ" for a pulse input, as defined in Sec.VII (extension to other types of systems is obvious). Under the assumption (b) a system has good filtering properties whenever $\Psi \longrightarrow 0$ if either $H \longrightarrow 0$ or $T \longrightarrow 0$. This is a rather rough estimation and we are going to define later a more accurate distinction based on the second filtering figure of merit. However, we proceed to compute first the value of Ψ for various systems (in discrete systems we will always assume an integrator following).

(1) Continuous system: Obviously, $\Psi = 1$.

(2) Continuous system with dead zone $\pm$ D :

$\Psi = 1$ if $H > D$

$\Psi = 0$ if $H \leq D$

(3) Continuous system with Saturation S :

$\Psi = 1$ if $H \leq S$

$\Psi = \frac{S}{H}$ if $H > S$

(4) Pulse amplitude modulation (Classical Sampled-Data System) (Assumption of unit gain). For pulse duration $\tau > T$ (sampling rate) or $\omega < \omega_r/2 = \pi/T$ we have $\Psi = f(\omega)$. If $\omega > \omega_r/2$, then $\Psi = 0$ (or if $\tau < T$, probability of response $= \tau/T$. Values of gain are 1 or 0, therefore $E(\Psi) = \tau/T$). If $\tau > T$ then there will be n outcoming impulses where n is given as

$$\frac{\tau}{T} - 1 < n < \frac{\tau}{T}$$

$$\Psi = \frac{nHT}{H\tau} = n\frac{T}{\tau}$$

Therefore,

$$\Psi_{max} = \frac{T}{\tau} \frac{\tau}{T} = 1$$

and

$$\Psi_{min} = \frac{T}{c}\left(\frac{\tau}{T} - 1\right) = 1 - \frac{T}{\tau}$$

and equivalent gain (the average)

$$\Psi = 1 - \frac{T}{2\tau} \text{ for } \tau > T$$

(5) Pulse-width modulation: The output pulse width is given by

$$\Delta t = \frac{H}{H_o} T \quad \text{if} \quad H < H_o$$

$$\Delta t = T \quad \text{if} \quad H \geq H_o$$

This is true provided $\tau > \Delta t$. The integral will be (if h height of pulses) $h\Delta t$ at the end of the pulse, or $H/H_o \cdot T \cdot h$. If n pulses are emitted then

$$\Psi = \frac{n \frac{H}{H_o} T h}{\tau H} = \frac{h}{H_o} n \frac{T}{\tau}$$

Again,

$$\frac{\tau}{T} - 1 < n \leq \frac{\tau}{T}$$

Therefore, $\Psi = 1 - T/2\tau$ for $\tau > T$ and $H \leq H_o$. If $H > H_o$, then $\Psi = \frac{nTh}{H\tau}$ and $\Psi_{max} = H_o/H$ $\Psi_{min} = H_o/H - H_o/H \cdot T/\tau$ of $\Psi = H_o/H \cdot (1 - T/2\tau)$ for $\tau > T$ and $H \geq H_o$.

The condition $\tau > T$ can be essentially substituted by $\tau > \Delta t$.

If $\tau <$ t we have only one pulse as output of area $H_o \Delta t = H T$ and hence

$$\frac{HT}{H\tau} = \frac{T}{\tau} > 1$$

(6) Integral P. F. M: For threshold r and output area δ the system fires whenever $H\tau > r$. The number of pulses n is given by

$$\frac{H\tau}{r} - 1 < n \leq \frac{H\tau}{r}$$

$\Psi = \frac{n\delta}{H\tau}$, therefore $\Psi_{max} = \frac{H\tau}{r} \frac{\delta}{H\tau} = \frac{\delta}{r} = 1$. By proper choice of δ and r

$$\psi_{min} = \left(\frac{H\tau}{r} - 1\right)\frac{\delta}{H\tau} = \frac{\delta}{r} - \frac{\delta}{H\tau} = 1 - \frac{r}{H\tau}$$

We define the equivalent gain

$$\psi = 1 - \frac{r}{2H\tau} \qquad \text{for } H\tau > r$$

$$\psi = 0 \qquad \text{for } H\tau < r$$

(7) Neural P. F. M.: In Sec. VII we found

$$\psi = \frac{1}{\frac{H}{R}\ln\frac{H}{H-R}} - \frac{1}{2\frac{H}{R}\tau c} \qquad \text{for } H > R, \quad \tau > \frac{1}{c}\ln\frac{4}{H-R}$$

$$\psi = 0 \quad \text{if} \quad H < R \quad \text{or} \quad \tau < \frac{1}{c}\ln\frac{H}{H-R}$$

(ψ has been normalized by putting $\delta/r = 1$.)

These results plotted in each case are shown in Figs. 11. 2 and 11. 3. From them we see that the N. P. F. M. system has the better filtering properties based on the first filtering figure of merit. As second filtering figure of merit we may define the ratio of the average value of the equivalent gain over the range where the signal lies with probability P with respect to the corresponding magnitude for the noise.

This becomes a useful design criterion for the appropriate choice of parameters of the system. (The value of P can be chosen, for example, at 50%.) We are not going to proceed in further examination of this method in this report because this would require a more accurate notion of the distributions of the noise and the signal. However, from Figs. 11. 2 and 11. 3 it is clear that the P. F. M. offers more possibilities than the other schemes.

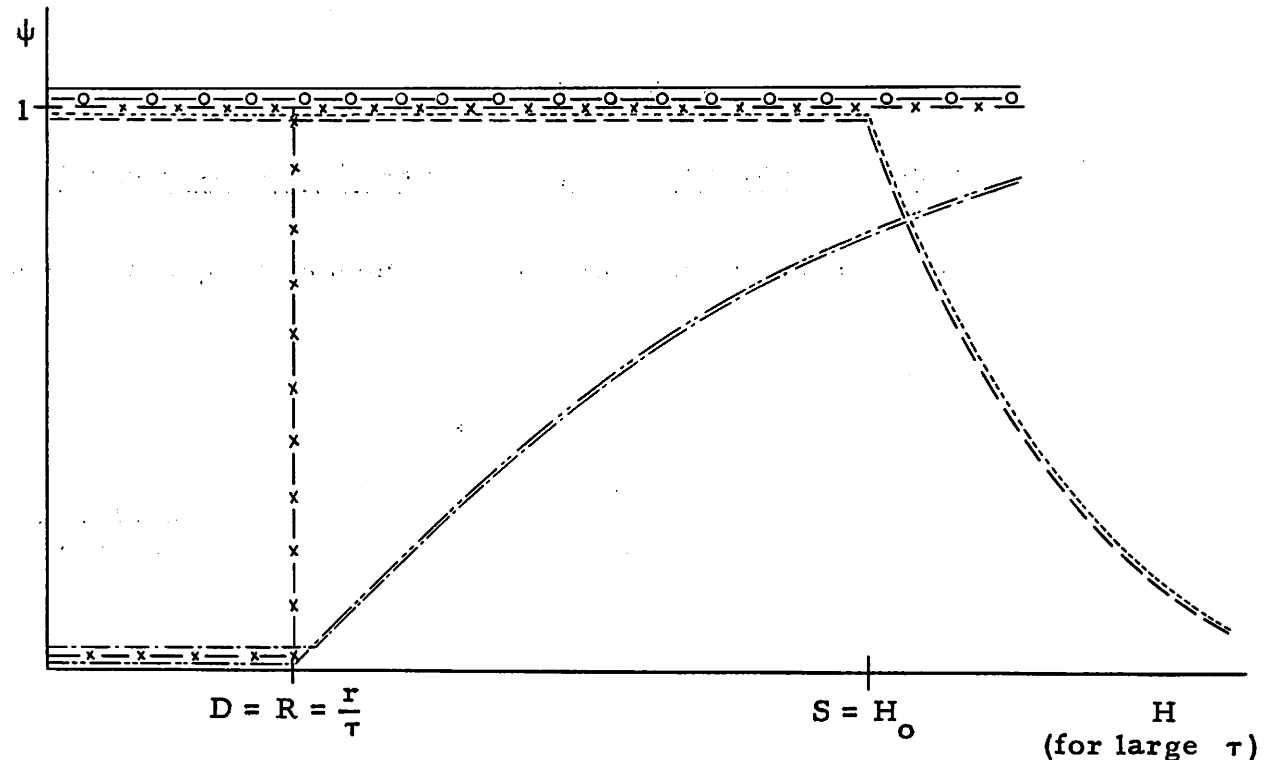


Fig. 11. 2
Equivalent gain for different systems under pulse inputs.

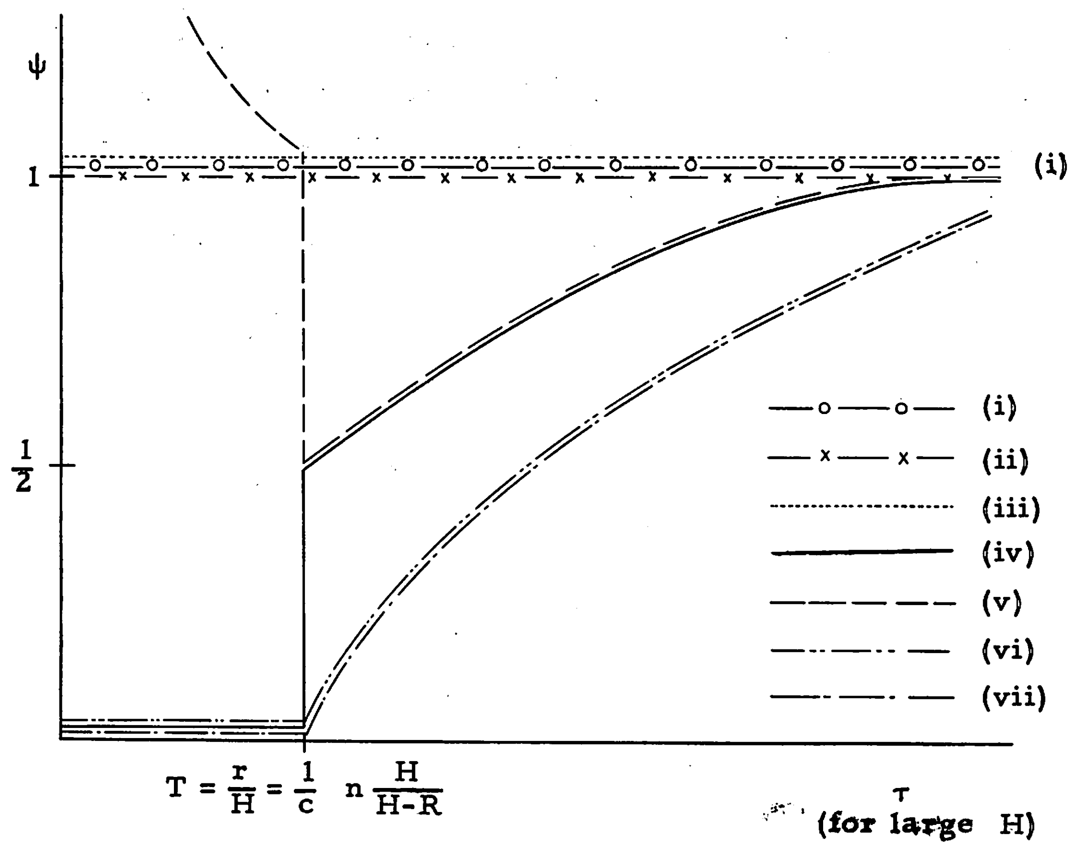


Fig. 11. 3
Equivalent gain for different systems under pulse inputs.

APPENDIX A

Electronic Implementation of Proposed Pulse Frequency Modulator

In Fig. A.1 is shown a possible implementation by using neon

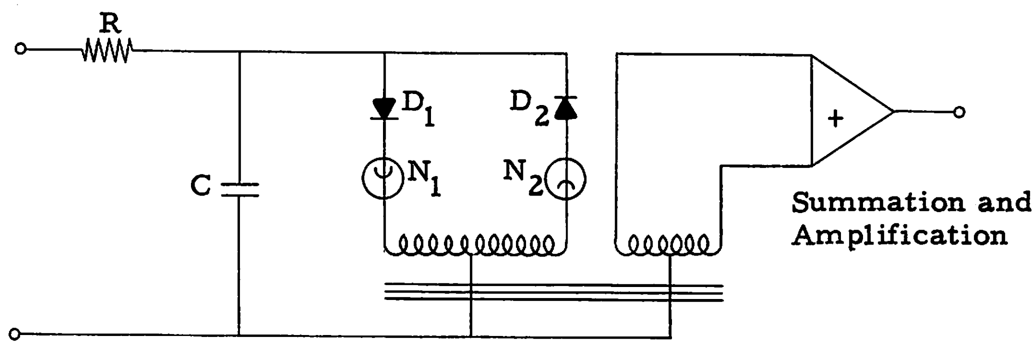


Fig. A.1

tubes (N_1 and N_2). The constant $c = 1/RC$. It is easy to see that this implementation adds an additional gain (pre-modulator) of value $1/RC$ and this should be taken into consideration in any analog computer simulation of a given system.

A variation is shown in Fig. A.2. The contacts a', b' close

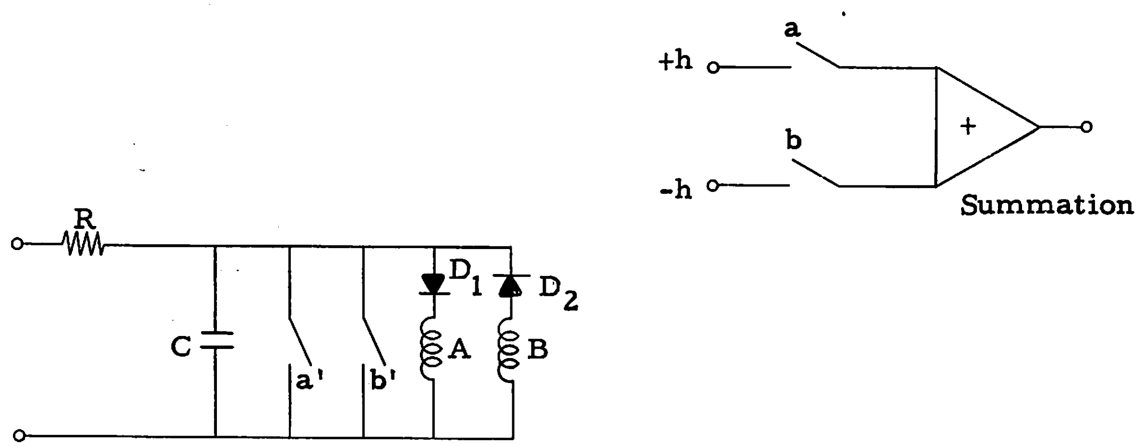


Fig. A.2

after the constants a, b. This form has the advantage that we can actually control the duration of the pulse, while this is not so easy with the circuit of Fig. A. 1.

APPENDIX B

Solution of a Transcendental Equation on the Analog Computer

We will solve Eq. (4. 2) with respect to t, namely

$$e^{-ct} \cos \psi = \cos(\omega t - \psi) \qquad (B.1)^{+}$$

Define the function

$$x(t) = e^{ct} \frac{\cos(\omega t - \psi)}{\cos \psi} \qquad (B.2)$$

Then the solution of (B. 1) coincides with time t, when $x(t) = 1$.

However, (B. 2) is the solution of the differential equation

$$\frac{d^2x}{dt^2} - 2c \frac{dx}{dt} + (c^2 + \omega^2)\, x = 0 \qquad (B.3)$$

with initial conditions

$$x(0) = 1 \qquad \left.\frac{dx}{dt}\right|_{t=0} = \frac{c^2 + \omega^2}{c}$$

This can be verified very easily. Therefore, using the circuit of Fig. B. 1 we obtain as output x(t). The diode D is used to determine the exact position of $x(t) = 1$. Moreover, special attention must be paid during the measurements because the system is unstable ($c > 0$).

[+] Obviously, $t = 0$ is a solution, but trivial.

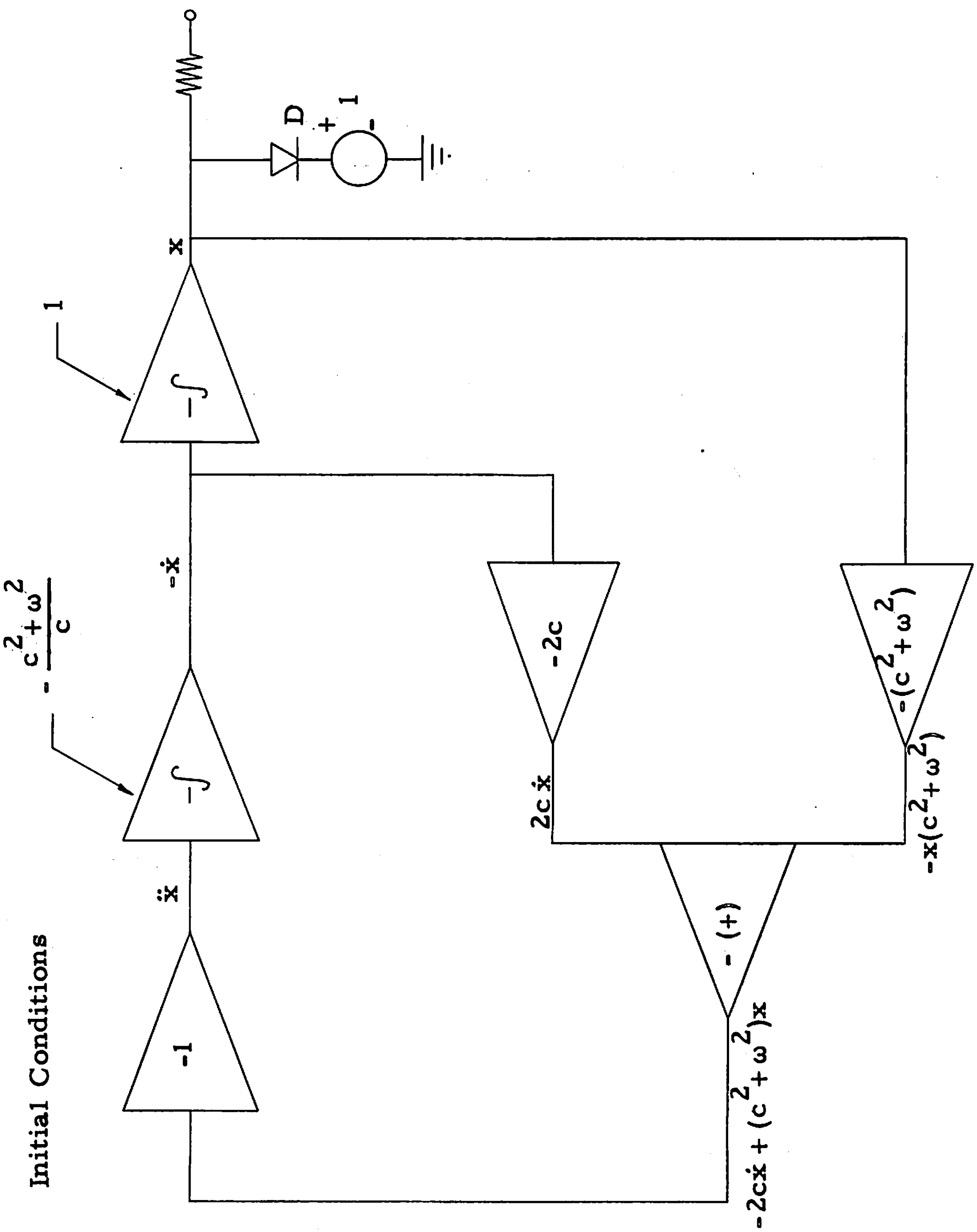


Fig. B.1
Analog computer set up for solving Eq. B.3

Actually, one has to take only one value of c because of (4. 2') and (4. 4').

Table I. Solutions of $\frac{dp}{dt} + cp = S$

	Input: S(t)	Response: p(t)	t when $p(t) = r$, $R \triangleq rc$	t for $p_{max}(t)$
1	$S_o u(t)$	$\frac{S_o}{c}\left(1 - e^{-ct}\right)$	$t = \frac{1}{c} \ln \frac{S_o}{S_o - R}$	∞
2	$S_o t\, u(t)$	$\frac{S_o}{c}\left[t - \frac{1}{c}\left(1 - e^{-ct}\right)\right]$	$t \approx \sqrt{\frac{2R}{cS_o}}$ (*)	∞
3	$S_o t^2 u(t)$	$\frac{S_o}{c}\ t^2 - \frac{2t}{c} + \frac{2}{c^2}\ 1 - e^{-ct}$	$t \approx \sqrt[3]{\frac{3R}{cS_o}}$ (**)	∞
4	$S_o\left[u(t) - u(t-\tau)\right]$	$\frac{S_o}{c}\left(1 - e^{-c\tau}\right) e^{-c(t-\tau)}$	$t = \frac{1}{c} \ln \frac{S_o}{S_o - R}$, $t \leq \tau$	τ
5	$S_o \sin \omega t\, u(t)$	$\omega S_o \frac{e^{-ct}}{c^2+\omega^2} + \frac{\sin(\omega t - \psi)}{\omega\sqrt{c^2+\omega^2}}$, $\psi = \tan^{-1}\frac{\omega}{c}$		
6	$A + B(1 - e^{-at})$	$\frac{A+B}{c}\left(1 - e^{-ct}\right) - \frac{B}{a-c}\left(e^{-ct} - e^{-at}\right)$		

Notes $u(t)$ = unit step For non-zero intial conditions add $p(0)\,e^{-ct}$

(*) approx. $e^{-ct} = 1 - ct + \frac{c^2t^2}{2}$ (**) approx. $e^{-ct} = 1 - ct + \frac{c^2t^2}{2} - \frac{c^3t^3}{6}$

REFERENCES

1. Jury, E. I., "Some aspects of sampling schemes in feedback control systems," Regelungstechnik, Vol. 3, pp. 98-107; March 1963.
2. Ruch, R. C. and Fulton, J. F. (editors), Medical Physiology and Biophysics, Saunders, 1960.
3. Li, C. C., Integral Pulse Frequency Modulated Systems, Ph. D. Dissertation, Northwestern University, 1961.
4. Meyer, A. U., Pulse Frequency Modulation and its Effect in Feedback Systems, Ph. D. Dissertation, Northwestern University, 1961.
5. Schouten, J. F., de Sager, F., and Greefkes, J. A., "Delta modulation, a new modulation for telecommunication," Philips Technical Review, Vol. 13, No. 9, pp. 237-245; March 1952.
6. Rashevsky, N., Mathematical Biophysics, 3rd ed., Dover, 1960; Vol. 1.
7. Hammond, Jr., J. L. and Johnson, R. S., "A review of orthogonal square-wave functions and their application to linear networks," J. Franklin Inst., Vol. 273, No. 3, pp. 211-225; March 1962.
8. Kalman, R. E. and Bertram, J. E., "Control systems analysis and design via the "Second Method" of Lyapunov," Journal of Basic Engineering, Vol. , pp. 371-393; June 1960.
9. Pyatnitskii, E. S., "On the structural stability of single-loop control systems in the presence of lag," Automation and Remote Control, Vol. 23, No. 7, pp. 787-797; July 1962.