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ON THE CHARACTERIZATION OF CONTINUOUS GAUSSIAN PROCESSES

by

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1. Introduction

In the discussion of problems connected with radar detection and ranging it is often useful to characterize a continuous, real valued gaussian process through expressions of the form [1,2]

$$p(s(t)) = C\exp\left[-\frac{1}{2}\int_T\int_T x(t)R^{-1}(s,t)x(t)\,ds\,dt\right] \tag{1.1}$$

where $x(t)$ is a time function on some closed interval T of the real line and $R^{-1}(s,t)$ is some positive semi-definite function on the square $T\times T$. Such expressions are usually referred to as "probability functionals."

Unfortunately, as is well known, such expressions are not well-defined, except in the case in which the covariance function of the process has a finite set of eigenfunctions.

We will show in what follows that we may write expressions <u>similar</u> to (1.1) which are well-defined in the general case and such that they contain all the information we need.

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Since there is no difficulty in extending the results presented here to the complex-valued, multidimensional case, we will limit ourselves to the consideration of the real-valued, one-dimensional case.

2. Formulation of the Problem

Let $\{X(t), t \in T\}$ be a gaussian random process with expectation m(t) and covariance function R(s, t). We will assume that R(s, t) is positive definite and continuous over the square TXT.

Consider any complete countable set of square integrable functions on T, $\left\{\phi_k(t), t \in T\right\}_1^\infty$. Let $\left\{\psi_k(t) \; t \in T\right\}_1^\infty$ be the reciprocal basis, i.e.

$$\int_T \psi_j(t)\, \phi_k(t) \phi t = \delta_{jk} \qquad k = 1, 2, \ldots \tag{2.1}$$

Every sample function x(t) of the process X(t) may be represented by

$$x(t) = \sum_{k=1}^{\infty} x_k \phi_k(t) \tag{2.2}$$

where

$$x_k = \int_T \psi_k(t) x(t)\, dt \tag{2.3}$$

The random variables X_k, k=1, 2, . . . , defined by

$$X_k = \int_T \psi_k(t) X(t) dt \tag{2.4}$$

form a set of jointly normal random variables such that for any finite n

$$p(x_1, \ldots, x_n) = N_n(\underline{m}_n, \underline{S}_n) \tag{2.5}$$

where $N_n(\underline{m}_n, \underline{S}_n)$ indicates the n-variate normal density with mean $\underline{m}_n$ and covariance matrix $\underline{S}_n$. We have

$$\underline{m}_n = (m_1, \ldots, m_n) \tag{2.6}$$

$$m_k = \int_T \psi_k(t) m(t) dt \tag{2.7}$$

$$\underline{S}_n = \| \sigma_{ij} \| \qquad i, j = 1, \ldots, n \tag{2.8}$$

$$\sigma_{ij} = \int_T \psi_i(s) \, R(s, t) \psi_j(t) \, ds dt = \sigma_{ji} \tag{2.9}$$

We will have

$$R(s, t) = \sum_{i, j=1}^{\infty} \sigma_{ij} \, \phi_i(s) \phi_j(t) \tag{2.10}$$

The Karhünen-Loève expansion theorem[3] says that there is a countable set of orthonormal functions $\left\{ \tilde{\phi}_k \right\}_1^{\infty}$ such that the above expansion is diagonal and consequently the r. v. X_k are independent. Let $\left\{ \mu_k \right\}_1^{\infty}$ be the corresponding elements of the diagonal of the (infinite dimensional) covariance matrix.

Whatever basis we use we would like to define

$$p(x(t)) \triangleq p(x_1, x_2, \ldots) \tag{2.11}$$

and we would like to write something of the form

$$p(x(t)) = N_{\infty}(\underline{m}_{\infty}, \underline{S}_{\infty}) \tag{2.12}$$

Unfortunately we are unable to associate a well-defined meaning to the symbols in the right-hand side of Eqs. (2.11) and (2.12). In the following section we present a way of avoiding this difficulty.

3. Reappraisal of the Problem

The problems in which we are interested are decision problems; that is, we have a set (often finite) of possible alternatives and we want to choose the "best," in some sense. The techniques involved in the case of a finite number $n \geq 2$ of alternatives, do not differ appreciably from those used for $n = 2$. For simplicity of notation we will therefore consider only the last case. In particular we will analyze the case of simple detection; i.e.

H_o: no signal present

H_1: a specific signal present

In this case we have that under H_o and H_1 the covariance function of the process remains the same and

$$m^{(o)}(t) \triangleq E[\,x(t)|H_o] \equiv 0$$

$$m^{(1)}(t) \triangleq E[\,x(t)|H_1] = s(t)$$

where $s(t)$ is the given signal which may have been transmitted (assumed to be of integrable square).

For any given basis $\left\{\phi_k\right\}_1^{\infty}$ consider the sequence of gaussian random processes $\left\{x_n(t),\ t\epsilon T\right\}$ having expectation $m_n(t)$ and covariance function $R_n(s,t)$ given by

$$m_n(t) = \sum_{k=1}^{n} m_k \phi_k(t) \tag{3.1}$$

$$R_n(s,t) = \sum_{i,j=1}^{n} \sigma_{ij}\phi_i(s)\phi_j(t) \tag{3.2}$$

If for any finite n we indicate by $x_n(t)$ a sample function of $X_n(t)$, we may define (i=0, 1)

$$p(x_n(t)|H_i) \triangleq p(x_1, \ldots, x_n|H_i) = N_n(m_n^{(i)}, \underline{S}_n) \tag{3.3}$$

In the problem we are considering (simple hypothesis vs. simple alternative) we know that for each n the only admissible decision procedures are those for which we accept H_o whenever

$$\frac{p(x_n(t)|H_o)}{p(x_n(t)|H_i)} > \lambda \tag{3.4}$$

for some appropriately chosen λ, $0 \leq \lambda \leq \infty$. We can always write (i=0, 1)

$$p(x_n(t) \mid H_i) = p(x_1|x_2, \ldots, x_n, H_i)\, p(x_2, \ldots x_n|H_i) \tag{3.5}$$

If the random process is such that $p(x_2, \ldots, x_n|H_i)$ is independent of i, then the ratio in Eq. (3.4) will depend only on the conditional probability densities $p(s_1|x_2, \ldots, x_n, H_i)$. Therefore, insofar as our decision is concerned we may restrict our attention to the conditional densities.

Since the set of orthonormal functions $\left\{\tilde{\phi}_k\right\}_1^{\infty}$ (introduced in Section 2) is complete, there is at least one k such that

$$\int \tilde{\phi}_k(t)\, s(t)\, dt \neq 0 \tag{3.6}$$

Without any loss of generality assume k=1. Define

$$\phi_1(t) = s(t) \tag{3.7}$$

$$\phi_k(t) = \tilde{\phi}_k(t) \qquad k \neq 1 \tag{3.8}$$

Expand $\phi_1(t)$ with respect to the "natural" basis $\left\{\tilde{\phi}_k\right\}_1^{\infty}$. We get

$$\phi_1(t) = \sum_{k=1}^{\infty} \beta_k \tilde{\phi}_k(t) \tag{3.9}$$

where

$$\beta_k = \int_T \tilde{\phi}_k(t)\phi_1(t)\, dt \tag{3.10}$$

with $\beta_1 \neq 0$. The reciprocal basis $\left\{\psi_k\right\}_1^{\infty}$ is given by

$$\psi_1(t) = \frac{1}{\beta_1} \tilde{\phi}_1(t) \tag{3.11}$$

$$\psi_k(t) = -\frac{\beta_k}{\beta_1} \tilde{\phi}_1(t) + \tilde{\phi}_k(t) \qquad k > 1 \tag{3.12}$$

We have

$$m_1^{(1)} = 1 \qquad m_k^{(1)} = 0 \qquad k \neq 1 \tag{3.13}$$

Therefore for all n we have

$$p(x_2, \ldots, x_n | H_i) = p(x_2, \ldots, x_n | H_o) \tag{3.14}$$

and according to our previous remarks we may restrict our attention to $p(x_1 | x_2, \ldots, x_n, H_i)$. The advantage of this approach is that while the probability densities $p(x_1, \ldots, x_n | H_i)$, $p(x_2, \ldots, x_n | H_i)$ continue increasing in dimensionality as n increases, the dimensionality of the conditional densities remains constant. This implies that as $n \to \infty$ the conditional densities remain meaningful.

We remark also that the above procedure is equivalent to representing the sample space of the process as the direct sum of two subspaces, one of which is spanned by the signal while the projection of any sample function on the other is independent of the signal.

With our choice of basis we obtain the following relations

$$\sigma_{11} = \mu_1/\beta_1^2 \tag{3.15}$$

$$\sigma_{kk} = \mu_k + \beta_k^2 \mu_1/\beta_1^2 \qquad k \neq 1 \tag{3.16}$$

$$\sigma_{1k} = -\beta_k \mu_1/\beta_1^2 \qquad k \neq 1 \tag{3.17}$$

$$\sigma_{jk} = \beta_j \beta_k \mu_1/\beta_1^2 \qquad j \neq k,\ j \neq 1,\ k \neq 1 \tag{3.18}$$

It is simple to prove that

$$p(x_1 | x_2, \ldots, x_n, H_i) = N_1(m_1^{(i)} - \sum_{k=2}^{n} \sigma_n^2 \frac{\beta_k}{\mu_k} x_k, \sigma_n^2) \tag{3.19}$$

where

$$\sigma_n^2 = \left[\sum_{j=1}^{n} \frac{\beta_j^2}{\mu_j} \right]^{-1} \tag{3.20}$$

Define

$$R_n^{-1}(s, t) \triangleq \sum_{i, j=1}^{n} \alpha_{ij}^{(n)} \psi_i(s) \psi_j(t) \tag{3.21}$$

where the $\alpha_{ij}^{(n)}$ are determined by the equation

$$\int_T R_n^{-1}(s, \tau) R_n(\tau, t) d\tau = \sum_{i=1}^{n} \psi_i(s) \phi_i(t) \tag{3.22}$$

It is easy to show that

$$\alpha_{11}^{(n)} = 1/\sigma_n^2 \tag{3.23}$$

$$\alpha_{kk}^{(n)} = 1/\mu_k \qquad k \neq 1 \tag{3.24}$$

$$\alpha_{1k}^{(n)} = \beta_k \mu_k \qquad k \neq 1 \tag{3.25}$$

$$\alpha_{jk}^{(n)} = 0 \qquad j \neq k,\ j \neq 1,\ k \neq 1 \tag{3.26}$$

We draw attention to the fact that except for $\alpha_{11}^{(n)}$ the $\alpha_{ij}^{(n)}$'s do not depend on n, so that the superscript may be dropped.

Using the definition of $R_n^{-1}(s,t)$ in Eq. (3.19) we may write

$$p(x_1|x_2,\ldots,x_n,H_i) = \frac{1}{\sqrt{2\pi}\,\sigma_n} \exp\left\{-\frac{1}{2\sigma_n^2}\cdot\left[\sigma_n^2 \int_T\int_T \phi_1(t)\, R_n^{-1}(t,\tau)\, x(t)\, dt d\tau - m_1^{(i)}\right]^2\right\} \tag{3.27}$$

As $n \to \infty$ we obtain (recalling the definition of $\phi_1(t)$)

$$p(x_1|x_2,\ldots,H_i) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2\sigma^2}\cdot\left[\sigma^2 \int_T\int_T s(t) R^{-1}(t,\tau)\, x(\tau) dt d\tau - m_1^{(i)}\right]^2\right\} \tag{3.28}$$

where

$$\sigma^2 = \left[\sum_{j=1}^{\infty} \frac{\beta_j^2}{\mu_j}\right]^{-1} \tag{3.29}$$

$$R^{-1}(s,t) = \sum_{i,j=1}^{\infty} \alpha_{ij}\, \psi_i(s)\psi_j(t) \tag{3.30}$$

$$\alpha_{11} = 1/\sigma^2 \tag{3.31}$$

We also have

$$\int_T R^{-1}(s,\tau)\, R(\tau,t)d\tau = \sum_{i=1}^{\infty} \psi_i(s)\phi_i(t) = \delta(s-t) \tag{3.32}$$

It is apparent that since

$$\sum_{i=1}^{\infty} \psi_i(s)\, \phi_i(t) = \delta(s-t) \tag{3.33}$$

for any basis $\left\{\phi_i\right\}_1^{\infty}$, the expression in Eq. (3.28) is independent of the particular choice of the basis (except for the first coordinate). The particular choice we used was helpful only in determining explicitly all the quantities involved. Furthermore, since $\beta_1 \neq 0$, we have $\sigma < \infty$. It is possible for σ to vanish (at least in principle); in this case the normal distribution is degenerate and the value of x_1 under H_i is equal to $m_1^{(i)}$ with probability one. This agrees exactly with the "singular" case discussed by Grenander[4] and also by Reed et al.[5]

It is clear that the quantity

$$d = \sigma^2 \int_T\int_T s(t)\, R^{-1}(t,\tau)x(\tau)dt d\tau \tag{3.34}$$

represents a minimal sufficient statistic for the problem.

4. White Noise

The "white noise" gaussian process $\{W(t), t\in T\}$ may be defined as the limit of a sequence of gaussian random processes $\{W_n(t),\ t\in T\}$ having covariance function

$$Q_n(s,t) = \frac{N_o}{2} \sum_{k=1}^{n} \tilde{\phi}_k(s)\,\tilde{\phi}_k(t) \tag{4.1}$$

where $\left\{\tilde{\phi}_k\right\}_1^{\infty}$ is <u>any</u> orthonormal basis on T. If we pick

$$\tilde{\phi}_1(t) = \frac{s(t)}{\sqrt{E_s}} \tag{4.2}$$

where

$$E_s = \int_T s^2(t)\,dt \tag{4.3}$$

we will have

$$\beta_1 = \sqrt{E} \tag{4.4}$$

$$\beta_i = 0 \qquad i \neq 1 \tag{4.5}$$

$$m_1^{(1)} = 1 \tag{4.6}$$

$$m_i^{(1)} = 0 \qquad i \neq 1 \tag{4.7}$$

while

$$\mu_i = \frac{N_o}{2} \qquad i=1,2,\ldots \tag{4.8}$$

Equation (3.19) becomes

$$p(x_1 | x_2, \ldots, x_n, H_i) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2\sigma^2}\left[\frac{1}{E_s}\int s(t)x(t)dt - m_1^{(i)}\right]^2\right\} \tag{4.9}$$

where $\sigma^2 = N_o/2E$. The right-hand side is <u>independent of n</u> and therefore as $n \to \infty$ the same expression will hold. The standard deviation is exactly the signal-to-noise (voltage) ratio $R = \sqrt{N_o/2E}$, as expected.

5. Conclusions

The main interest of the approach presented above lies not in its conclusions, which are well known, but in the fact that, in a reasonably simple and straightforward way, it puts in evidence the quantities which are really significant to the problem.

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