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ON AN AXIS-CROSSING PROPERTY OF GAUSSIAN NOISE

by

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I. INTRODUCTION

This note reports a result concerning the distribution of intervals between successive zeros of Gaussian noise. Let x(t) be a zero-mean stationary Gaussian process, with covariance function of the form

$$E\, x(t)\, x(t+\tau) = \rho(\tau) = 1 - \frac{t^2}{2!} + \frac{at^3}{3!} + 0(t^4) \qquad (1)$$

Let ξ be a random variable denoting an interval between two successive zeros of x(t). The problem of finding the probability distribution of ξ is of considerable interest and remains unsolved. (For further references and detailed discussions, see Refs. 5 and 6.)

Let $F(\tau) = \text{Prob}\ \xi < \tau$ be the distribution function of ξ, and $q(\tau) = \frac{dF(\tau)}{d\tau}$ be the density function. It has been shown[2] that $q(0^+) = 0$ if $a = 0$, but $q(0^+)$ is different from zero if a is non-zero. The exact value of $q(0^+)$ is hitherto not known, although very good bounds have been given by Longuet-Higgins.[2] In this note we propose a value for $q(0^+)$. While it is not proved that the proposed value is exact, the fact that it lies within the close bounds given by Longuet-Higgins and the way that it is derived lend credibility to its being so.

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II. TWO-DIMENSIONAL GAUSS-MARKOFF PROCESS

We consider a zero-mean Gaussian process $x(t)$ with covariance function

$$\rho(t) = e^{-\frac{1}{2}at}\left(\cos\ \beta t + \frac{a}{2\beta}\sin \beta t\right), \left(\beta = \sqrt{1 - \frac{a^2}{4}}\,,\right)$$
$$= 1 - \frac{t^2}{2!} + \frac{at^3}{3!} + 0(t^4) \tag{2}$$

It is clear that $\rho(\tau)$ has the form given by (1). In addition, (2) also implies that $x(t)$ and its derivative $\dot{x}(t)$ are the components of a two-dimensional Gauss-Markoff process.[7] This Markoffian property is used below in deriving a set of integral equations.

We define $\varphi(x_o, \dot{x}_o, t)$ to be the conditional probability that $x(\tau)$ has no zero-crossing in $0<\tau<t$ given $x(0) = x_o$ and $\dot{x}(0) = \dot{x}_o$, $(x_o>0)$. Let $p(x, \dot{x}\,|\, x_o, \dot{x}_o t)$ be the conditional probability density function,* i. e.,

$$p(x, \dot{x}\,|\,x_o, \dot{x}_o, t)\ dx\ d\dot{x}\ =$$
$$\mathrm{Prob}\left[x(t)\in (x, x + dx),\ \dot{x}(t) \in (\dot{x}, \dot{x} + d\dot{x})\,\middle|\, x(0) = x_o,\ \dot{x}(0) = \dot{x}_o\right] \tag{3}$$

By considering $x(\tau)$ at its last zero-crossing in the interval $(0, t)$ and using the Markoffian property of $x(\tau)$ and $\dot{x}(\tau)$, we find that for $x_o<0$

$$\int_0^\infty dx \int_{-\infty}^\infty d\dot{x}\ \ p(x, \dot{x}\,|\, x_o, \dot{x}_o, t) =$$
$$\int_0^t d\tau \int_0^\infty d\dot{x}\ \ |\dot{x}|\ \ p(0, \dot{x}\,|\, x_o, \dot{x}_o, \tau)\ \ \varphi(0^+, \dot{x}, t-\tau) \tag{4}$$

similarly for $x_o > 0$

*For an exact expression of $p(x, \dot{x}\,|\, x_o, \dot{x}_o, t)$, see Ref. 7.

$$\varphi(x_o, \dot{x}_o, t) = \int_0^\infty dx \int_{-\infty}^\infty d\dot{x} \;\; p(x, \dot{x} \mid x_o, \dot{x}_o, t)$$
$$- \int_0^t d\tau \int_0^\infty d\dot{x} \;\; |\dot{x}| \;\; p(0, \dot{x} \mid x_o, \dot{x}_o, \tau) \quad (0^+, \dot{x}, t-\tau) \tag{5}$$

These integral equations (4) and (5) are closely related to those given by Siegert[4] and Helstrom.[1] The quantity $p(0, \dot{x} \mid x_o, \dot{x}_o, t) \mid \dot{x} \mid \; dt \; d\dot{x}$ in (4) and (5) is the probability that $x(\tau)$ has a zero for $\tau \in (t, t + dt)$ with $\dot{x}(t) = \dot{x}$ given $x(0) = x_o$ and $\dot{x}(0) = \dot{x}_o$.[3] The probability $P(t)$ that $x(\tau) > 0, \; \tau \in (0, t)$ is clearly given by

$$P(t) = \int_0^\infty dx_o \int_{-\infty}^\infty d\dot{x}_o \;\; p(x_o, \dot{x}_o) \, \varphi(x_o, \dot{x}_o, t)$$
$$= \frac{1}{2\pi} \int_0^\infty dx_o \int_{-\infty}^\infty d\dot{x}_o \quad e^{-\frac{1}{2}(x_o^2 + \dot{x}_o^2)} \, \varphi(x_o, \dot{x}_o, t) \tag{6}$$

With the use of (4) and (5) in (6), we find

$$P(t) = \frac{1}{2} - \int_0^t \int_0^\infty \dot{x} \;\; p(0, \dot{x}) \, \varphi(0^+, \dot{x}, \tau) \, d\dot{x} \; d\tau$$
$$= \frac{1}{2} - \frac{1}{2\pi} \int_0^t \int_0^\infty \dot{x} \quad e^{-\frac{1}{2}\dot{x}^2} \, \varphi(0^+, \dot{x}, \tau) \, d\dot{x} \; d\tau \tag{7}$$

The functions $F(\tau)$ and $q(\tau)$ are easily related to $P(t)$ (see, for example, Ref. 5, Section 1.7). We have

$$1 - F(t) = -2\pi \frac{d}{dt} P(t) \tag{8}$$

$$\text{and } q(t) = 2\pi \frac{d^2}{dt^2} P(t) \tag{9}$$

If the integral equations (4) and (5) can be solved for $\varphi(0^+, \dot{x}, t)$, then $F(t)$, $P(t)$ and $q(t)$ for this case are completely determined.

III. LIMITING BEHAVIOR OF q(t)

Although $\varphi(0^+, \dot{x}, t)$ has not been found, its behavior for t near zero can be estimated. First of all, $\varphi(0^+, \dot{z}, t)$ has the following properties:

$$\varphi(0^+, \dot{x}, t) = 0 \quad , \quad \dot{x} < 0, \quad t > 0$$

$$\varphi(0^+, \dot{x}, 0) = 1$$

and for small t

$$\lim_{\dot{x} \longrightarrow \infty} \varphi(0^+, \dot{x}, t) = 1$$

Now, by setting $x_o = 0^+$ in (5), we find

$$\begin{aligned}\varphi(0^+, \dot{x}_o, t) &= \int_0^\infty dx \int_{-\infty}^\infty d\dot{x} \; p(x, \dot{x} \mid 0^+, \dot{x}_o, t) \\ &\quad - \int_0^t d\tau \int_0^\infty d\dot{x} \; \dot{x} \; p(0, \dot{x} \mid 0^+, \dot{x}_o, \tau) \; \varphi(0^+, \dot{x}, t-\tau)\end{aligned} \tag{10}$$

For small t, the first term on the right-hand side of (10) is approximately

$$\int_0^\infty dx \int_{-\infty}^\infty d\dot{x} \; p(x, \dot{x} \mid 0^+, \dot{x}_o, t) \cong \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{\frac{3}{2at}}\,\dot{x}_o}^\infty e^{-\frac{z^2}{2}} \, dz \tag{11}$$

While as $t \rightarrow 0$ $\varphi(0^+, \dot{x}_o, t)$ does not behave exactly as

$\int_0^{\infty} dx \int_{-\infty}^{\infty} d\dot{x} \;\; p(x, \dot{x} \,|\, 0^+, x_o, t)$, its behavior is not likely to be drastically different. Considerations of this and the properties of $\varphi(0^+, x_o, t)$ suggest the approximation

$$\varphi(0^+, \dot{x}_o, t) \cong \sqrt{\frac{2}{\pi}} \int_0^{\frac{a \dot{x}_o}{\sqrt{t}}} e^{-\frac{1}{2}z^2} \, dz, \quad \dot{x}_o > 0 \tag{12}$$

a being a parameter to be determined.

To proceed, we multiply both sides of (4) by

$p(x_o, \dot{x}_o) = \frac{1}{2\pi} e^{-\frac{1}{2}(x_o^2 + \dot{x}_o^2)}$ and integrate over $-\infty < x_o < 0$ and $-\infty < \dot{x}_o < \infty$. This results in

$$\frac{1}{4} - \frac{1}{2\pi} \sin^{-1} \rho(t) =$$

$$\frac{1}{2\pi} \int_0^t \int_0^{\infty} \dot{x} \, e^{-\frac{\dot{x}^2}{2}} \varphi(0^+, \dot{x}, t-\tau) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{r(\tau)\dot{x}}{\sqrt{\sigma(\tau)}}} e^{-\frac{1}{2}z^2} \, dz \right) d\dot{x} \, d\tau \tag{13}$$

where

$$r(t) = \frac{1}{\beta} e^{-\frac{at}{2}} \sin \beta t \left(\beta = \sqrt{1 - \frac{a^2}{4}} \right) \tag{14}$$

$$\sigma(\;\;) - 1 - \rho^2(t) - r^2(t) \tag{15}$$

and $\rho(t)$ is given by (2).

Using the expression (12) for $\varphi(0^+, \dot{x}, t)$ in (13) and expanding both sides in powers of t, we find after some tedious but routine integration,

$$\frac{1}{2} - \frac{1}{2\pi}\left(\frac{t}{2!} + \frac{a}{3}\frac{t^3}{3!}\right) + 0(t^3) \tag{16}$$

$$= \frac{1}{2} - \frac{1}{2\pi}\left\{\frac{t}{2!} + \left[\frac{a}{6} + \frac{1}{2a^2} - \frac{1}{4\left(a + \sqrt{\frac{3}{2a}}\right)^2}\right]\frac{t^2}{3!}\right\} + 0(t^3).$$

Thus, a must satisfy the equation:

$$\frac{a}{3} = \frac{1}{a2} - \frac{1}{2\left(a + \sqrt{\frac{3}{2a}}\right)^2} \tag{17}$$

The only positive real root of (17) is

$$a = \frac{1}{\sqrt{a}}\left[\frac{3}{2\sqrt{2}} + \frac{\sqrt{3}}{2}(3)^{\frac{1}{4}} - \frac{1}{2}\sqrt{\frac{3}{2}}\right] \tag{18}$$

Using (12) in (7), we find that for small t P(t) is given by

$$P(t) = \frac{1}{2} - \frac{1}{2\pi}t + \frac{1}{4\pi}\frac{1}{a^2}\frac{t^2}{2!} + 0(t^3) \tag{19}$$

Hence, from (9) and (18), we find

$$q(0^+) = \frac{1}{2a^2} \tag{20}$$

$$= \frac{a}{\left[\frac{3}{\sqrt{2}} + (3)^{\frac{3}{4}} - \sqrt{\frac{3}{2}}\right]^2}$$

$$\cong .198a---$$

the bounds given by Lonquèt-Higgins being

$$.1911\ a < q(0^+) < .203\ a \tag{21}$$

Since powers of t in p(t) higher than the third do not enter into the calculation for $q(0^+)$, the result applies not only to the two-dimensional Markoff case but applies in general to any Gaussian process with covariance function of the form of (1).

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