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THIRD ORDER, TUNNEL-DIODE OSCILLATORS

by

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Supported in part by the Air Force Office of Scientific Research of the Office of Aerospace Research; the Department of the Army, Army Research Office; and the Department of the Navy, Office of Naval Research, under Grant AF-AFOSR-139-63.

July 23, 1963

ABSTRACT

A nonlinear analysis procedure and a design procedure are given for the tunnel-diode oscillator with a parallel RC load (TDRC). For most tunnel diodes the maximum frequency of oscillation of the tunnel-diode oscillator with a series RL load (TDRL) is obtained for a nonharmonic mode of oscillation. The TDRC achieves its maximum frequency of oscillation for a nearly harmonic mode. In addition, the maximum oscillation frequency of the TDRC is usually higher than that of the TDRL. Thus, if maximum frequency is important, the TDRC usually has an advantage over the TDRL. The conditions are derived to achieve maximum frequency of harmonic oscillation for the TDRC. This frequency is only slightly less than the maximum oscillation frequency of the TDRC.

The TDRC is described by a third order, nonlinear, differential equation. The analysis technique used for the third order equation is a perturbation method. The results can be used to obtain the amplitude, frequency, dc component, and an indication of the amplitude of the second harmonic for many oscillators described by a third order equation.

I. INTRODUCTION

The tunnel-diode oscillator with a series RL load (TDRL) has been studied extensively.[1-3] Pepper[1] has found the minimum period of oscillation of the TDRL and has shown that the minimum period is obtained for a nonharmonic mode of oscillation. Sterzer and Nelsen[2] and Pepper[1] indicate that for many tunnel diodes a shorter period of oscillation can be obtained with a parallel RC load (TDRC). This is indeed the case, and in addition the TDRC achieves its minimum period for a nearly harmonic mode of oscillation. Thus, for many tunnel diodes, the TDRC has a shorter period and an output which is more nearly a pure sinusoid than can be obtained with the TDRL.

The TDRC is described by a third order, nonlinear, differential equation. Nonlinear analysis techniques are available for many second order oscillator equations.[1, 4-6] Third and higher order equations are often solved by numerical methods. For analytic results, one usually has to be content with the results of an approximate linear analysis. The purpose of this paper is to perform a nonlinear analysis of the TDRC equation and to obtain a design procedure from the results. The design procedure realizes specified values of period and amplitude. In addition, it is shown that a close approximation to the maximum frequency of oscillation for the tunnel-diode can be obtained with the TDRC. (Throughout the remainder of this paper the term frequency will be used in place of reciprocal period. For nonsinusoidal waveforms, the frequency referred to is the frequency of the fundamental Fourier component).

The analysis assumes that the oscillator waveform is nearly harmonic. However, the experimental work and the numerical analysis which are presented indicate that the analysis is valid for rather nonharmonic oscillations.

II. THE OSCILLATOR EQUATION

The tunnel-diode can be represented by the circuit model shown in Fig. 1.[1-3] All dc bias sources are included in $F_1(v)$. The characteristic equation of the TDRC shown in Fig. 1 is

$$\left[s^3 + \left(\frac{G_L}{C_L} + \frac{R_s}{L} \right) s^2 + \frac{1}{L} \left(\frac{1}{C_D} + \frac{1 + R_s G_L}{C_L} \right) s + \frac{G_L}{L C_L C_D} \right] v$$

$$+ \left[\frac{1}{C_D} s^2 + \left(\frac{G_L}{C_L} + \frac{R_s}{L} \right) s + \frac{1 + R_s G_L}{L C_L C_D} \right] F_1(v) = 0 \tag{1}$$

where

$$s = \frac{d}{dt}$$

$$R_s = R_1 + R_2$$

$$L = L_1 + L_2 \tag{2}$$

It is convenient at this stage to approximate the tunnel diode nonlinearity with a polynomial. The method used is given in the Appendix. As indicated, for purposes of analysis the following can be used

$$F_1(v) = -\alpha' v + \beta_{13} v^3 \tag{3}$$

The TDRC equation becomes

$$\left\{ s^3 + \left(\frac{G_L}{C_L} + \frac{R_s}{L} - \frac{\alpha'}{C_D} \right) s^2 + \left[\frac{1}{L} \left(\frac{1}{C_D} + \frac{1+R_s G_L}{C_L} \right) - \frac{\alpha'}{C_D} \left(\frac{G_L}{C_L} + \frac{R_s}{L} \right) \right] s \right.$$

$$\left. + \frac{G_L - \alpha' (1+R_s G_L)}{L C_L C_D} \right\} v + \frac{\beta_{13}}{C_D} \left[s^2 + \left(\frac{G_L}{C_1} + \frac{R_s}{L} \right) s + \frac{1+R_s G_L}{L C_L C_D} \right] v^3 = 0 \tag{4}$$

For convenience, we write (4) as

$$M_1 v + M_3 v^3 = 0 \tag{5}$$

where

$$M_1 = s^3 + a_2 s^2 + a_1 s + a_0$$

$$M_3 = b_{32} s^2 + b_{31} s + b_{30} \tag{6}$$

III. STARTING CONDITION

When the oscillator equation is written in the form of (5) all linear terms are included in $M_1 v$. That is, the "small signal" equation, linearized about the operating point $v = 0$ (the variational equation) is

$$M_1 v = (s^3 + a_2 s^2 + a_1 s + a_0)\, v = 0 \tag{7}$$

If the oscillator described by (5) is to be self starting, (7) must have an unstable solution, i. e., the operating point must be a point of unstable equilibrium.. This in turn requires that the polynomial in s, (7), be non-Hurwitz:

$$\frac{a_0}{a_2} - a_1 \geq 0 \tag{8a}$$

or

$$a_2 < 0 \tag{8b}$$

In addition, (7) has an unstable solution if

$$a_0 < 0 \tag{9a}$$

or

$$a_1 < 0 \tag{9b}$$

If the oscillation is to be nearly harmonic, $M_1(s)$ must have a pair of complex zeros. With (9a), the real zero is in the right half plane (RHP). This leads to bistability or very nonsinusoidal oscillations.[7] If (9b) is satisfied, the real zero has a large negative value and (7) can be reduced to an approximate second order equation. From an inspection of this second order equation, it is clear for the third order equation that the conditions of (9) are not satisfied; i. e.,

$$a_0 ; a_1 > 0 \tag{10}$$

With (10), (8a) and (8b) can be combined and the complete starting condition becomes

$$a_0 - a_1 a_2 \geqslant 0 \tag{11}$$

From (4) and (11), the TDRC starting condition is

$$\frac{\alpha'}{C_D}\left(\frac{G_L}{C_L} + \frac{R_s}{L}\right)^2 - \left(\frac{G_L}{C_L} + \frac{R_s}{L}\right)\left(\alpha'^2 + \frac{1+R_s G_L}{LC_L}\right) + \frac{1}{LC_D}\left(\frac{\alpha'}{C_D} - \frac{R_s}{L}\right) \geqslant 0 \tag{12}$$

IV. THE MAXIMUM FREQUENCY OF HARMONIC OSCILLATION OF THE TDRC.

The tunnel diode can be a harmonic oscillator only when $F_1(v)$ is a linear function of v, i. e., if $F_1(v)$ can be replaced with its small signal value linearized about the operating point $v = 0$.

$$F_1(v) \rightarrow \alpha' v \tag{13}$$

For the assumed, linear, tunnel-diode circuit the maximum frequency of harmonic oscillation of the TDRL is given by either of the following, whichever is less.[1,2]

$$\omega_R^2 = \frac{a'}{C_D^2}\left(\frac{1}{R_s} - a'\right) \tag{14}$$

$$\omega_L^2 = \frac{1}{LC_D} - \frac{a'^2}{C_D^2} \tag{15}$$

The quantity ω_R is found by setting $\text{Re}Z_D(j\omega) = 0$; ω_L is found by setting $\text{Im}Z_D(j\omega) = 0$, where Z_D is defined in Fig. 1 for a linearized $F_1(v)$. A tunnel-diode oscillator can never oscillate harmonically at a frequency greater than ω_R. If

$$\omega_L > \omega_R$$

series inductance can be added until the circuit has natural frequencies at $\pm j\omega_R$ and the circuit will oscillate harmonically with frequency ω_R. If

$$\omega_R > \omega_L$$

which is usually the case, series capacitance can be added until the circuit has a pair of natural frequencies at $\pm j\omega_R$. However, this circuit has a third natural frequency which lies on the positive real axis. This introduces a growing exponential term in the solution which must be limited by the nonlinearity of the circuit, and the circuit cannot oscillate harmonically. If a parallel RC load is used, the third natural frequency can be kept in the LHP. It is shown below that the linearized TDRC can oscillate harmonically at a frequency ω_h where

$$\omega_L < \omega_h < \omega_R \tag{16}$$

provided

$$\omega_R^2 > \omega_L^2 \tag{17a}$$

$$\omega_R^2 > 0 \tag{17b}$$

In the following it is assumed that the conditions of (17) are satisfied since the TDRC then has the previously mentioned frequency advantage over the TDRL. (if (17a) is not satisfied a simple inductive load can be used to realize the maximum frequency of harmonic oscillation. If (17b) is not satisfied the tunnel diode will not oscillate harmonically).

The problem can be stated as follows: for a given tunnel-diode (R_1, L_1, α', and C_0 are fixed), find the values of R_L and C_L that produce the maximum frequency of harmonic oscillation[8] (R_2, L_2 = 0). Alternately, we must find the values of R_L and C_L that provide a pair of $j\omega$ axis natural frequencies which lie at a maximum distance from the origin and still keep the real natural frequency in the LHP.

The Laplace transform of the characteristic equation of the linearized TDRC shown in Fig. 1, with $F_1(v)$ given by (13) is

$$p^3 + \left(\frac{G_L}{C_L} + \frac{R_s}{L} - \frac{\alpha'}{C_D} \right) p^2 + \left[\frac{1}{L} \left(\frac{1}{C_D} + \frac{1+R_sG_L}{C_L} \right) - \frac{\alpha'}{C_D} \left(\frac{G_L}{C_L} + \frac{R_s}{L} \right) \right] p$$

$$+ \frac{G_L - \alpha' (1+R_sG_L)}{LC_LC_D} = 0 \tag{18}$$

The solution of the maximum frequency of oscillation problem is obtained using root locus arguments with (18). This equation can be rearranged as

$$TD(p) = 1 + \frac{G_L}{C_L} \frac{Q(p)}{D(p)} = 0 \tag{19}$$

where

$$Q(p) = p^2 + \left(\frac{R_s + R_L}{L} - \frac{a'}{C_D} \right) p + \frac{1 - a'(R_s + R_L)}{LC_D} \tag{20}$$

$$D(p) = p\left[p^2 + \left(\frac{R_s}{L} - \frac{a'}{C_D} \right) p + \frac{1 - a'R_s}{LC_D} \right] \tag{21}$$

The zeros of Q(p) are

$$z_{1,2} = \frac{1}{2} \left(\frac{a'}{C_D} - \frac{R_s + R_L}{L} \pm \sqrt{\left(\frac{a'}{C_D} + \frac{R_s + R_L}{L} \right)^2 - \frac{4}{LC_D}} \right) \tag{22}$$

The zeros of D(p) are

$$p_1 = 0$$

$$p_{2,3} = \frac{1}{2} \left(\frac{a'}{C_D} - \frac{R_s}{L} \pm \sqrt{\left(\frac{a'}{C_D} + \frac{R_s}{L} \right)^2 - \frac{4}{LC_D}} \right) \tag{23}$$

We will examine the root locus of (19) as C_L is varied. First it will be shown that $p_{2,3}$ must be complex for harmonic oscillation. Assume $p_{2,3}$ are real. From (22) it is seen that ($z_{1,2}$ are also real. Fig. 2 shows the locations of $p_{2,3}$ and $z_{1,2}$ for $p_{2,3}$ real ($z_{1,2}$ can lie anywhere on their respective dotted segments, depending on the value of R_L).

Now from (14), (15), and (17a)

$$\frac{a'}{C_D} > \frac{R_s}{L} \tag{24}$$

Therefore, p_2 must be in the RHP. Then for $\frac{1}{C_L} = 0$ the two farthest to the right on the real axis are p_1 and z_1, and the segment of the real axis between p_1 and z_1 is part of the root locus.

Therefore, there is always a real natural frequency in the RHP when $p_{2,3}$ are real and the circuit can not oscillate harmonically.

For the problem at hand, $p_{2,3}$ must be complex, and because of (24) $p_{2,3}$ must be in the RHP. The root locus of (19) as $\frac{1}{C_L}$ is varied is shown in Fig. 3. (where for clarity the construction in detail for only the upper half plane is shown). The zeros of Q(p), $z_{1,2}$ lie somewhere on the dotted lines depending on the value of R_L. The $j\omega$ axis crossing of the locus is denoted by ω_a.

Next it will be shown that the maximum value of ω_a is obtained with the maximum permissible value of R_L (a larger value of R_L moves the real zero into the RHP). It can be seen from Fig. 3 that

$$\text{Arg} \quad \frac{Q(j\omega_a)}{D(j\omega_a)} = -90^\circ - (90^\circ + \gamma_1) - (90^\circ + \gamma_2) + \beta_1 + \beta_2 = -180^\circ$$

$$\beta_1 + \beta_2 = 90^\circ + \gamma_1 + \gamma_2 \tag{25}$$

If $\beta_1 + \beta_2$ is decreased by changing R_L, ω_a must increase since this in turn increases $\beta_1 + \beta_2$ and decreases $\gamma_1 + \gamma_2$. The next step is to show that $\beta_1 + \beta_2$ is a monotonically decreasing function of R_L. Let $Q(j\omega_a) = D + jN$. From (20) and Fig. 3,

(26)

$$\frac{d(\beta_1 + \beta_2)}{dR_L} = \frac{\left[\frac{1 - a'(R_s + R_L)}{LC_D} - \omega_a^2 \right] \frac{\omega_a}{L} + \left(\frac{R_s + R_L}{L} - \frac{a'}{C_L} \right) \frac{a'\omega_a}{LC_D}}{\left[1 + \left(\frac{N}{D}\right)^2 \right] D^2}$$

(27)

The denominator of (27) is always positive. The numerator is

$$\frac{\omega_a}{L} \left(\omega_L^{\,2} - \omega_a^{\,2} \right) \tag{28}$$

Define ω_1 as Im p_2.

From (23) and (24)

$$\omega_1^{\,2} = \frac{1}{LC_D} - \frac{1}{4} \left(\frac{a'}{C_D} + \frac{R_s}{L} \right)^2 > \frac{1}{LC_D} - \left(\frac{a'}{C_D} \right)^2 = \omega_L^{\,2} \tag{29}$$

Next we show that $\omega_a^{\,2} > \omega_1^{\,2}$.

$$\text{Arg} \; \frac{Q(j\omega)}{D(j\omega)} = \beta_1 + \beta_2 - (\gamma_1 + \gamma_2) - 270^\circ \tag{30}$$

Arg $\frac{Q}{D}$ is a monotonically increasing function of ω since $(\beta_1 + \beta_2)$ is monotonically increasing and $(\gamma_1 + \gamma_2)$ is monotonically decreasing.

$$\text{Arg} \; \frac{Q(0)}{D(0)} = -450^\circ \tag{31}$$

$$\text{Arg} \; \frac{Q(j\omega_a)}{D(j\omega_a)} = -180^\circ \tag{32}$$

$$\text{Arg} \; \frac{Q(j\omega_1)}{D(j\omega_1)} = \beta_1 + \beta_2 - \gamma_2 - 360^\circ \tag{33}$$

$$0 < \beta_1 + \beta_2 < 180^\circ \tag{34}$$

$$-90 < -\gamma_2 < 0 \tag{35}$$

Combining (33), (34), and (35),

$$-450 < \text{Arg}\ \frac{Q(j\omega_1)}{D(j\omega_1)} < -180$$

Therefore

$$\omega_a^{\ 2} > \omega_1^{\ 2} > \omega_L^{\ 2} \tag{36}$$

and from (27), (28), and (36),

$$\frac{d(\beta_1 + \beta_2)}{dR_2} < 0 \ \text{ for all } R_L$$

The maximum allowable value of R_L therefore provides the maximum frequency of harmonic oscillation.

The values of R_L and C_L required to achieve the maximum frequency of harmonic oscillation, ω_h, can now be determined along with the value of ω_h. With $j\omega$ axis zeros at $\mp j\omega_a$, (18) becomes

$$\beta^3 + \sigma_1 p^2 + \omega_a^{\ 2} p + \sigma_1 \omega_a^{\ 2} = 0 \tag{37}$$

if the real zero, $-\sigma_1$, is in the LHP

$$\sigma_1 \geqslant 0 \tag{38}$$

$$\sigma_1 \omega_a^{\ 2} \geqslant 0 \tag{39}$$

or, for the TDRC

$$\sigma_1 = \frac{G_L}{C_L} + \frac{R_s}{L} - \frac{a'}{C_D} \geqslant 0 \tag{40}$$

$$\sigma_1 \omega_a^{\ 2} = \frac{G_L - a'(1 + R_s G_L)}{L C_L C_D} \geqslant 0 \tag{41}$$

Then, from (41)

$$R_L \leq \frac{1}{\alpha'} - R_s \tag{42}$$

The maximum allowable value of R_L, R_{Lh} is

$$R_{Lh} = \frac{1}{\alpha'} - R_s \tag{43}$$

Therefore, $\sigma_1 = 0$. From (40) and (43), the required value of C_L is

$$C_{Lh} = \frac{\alpha'}{\left(\frac{\alpha'}{C_D} - \frac{R_s}{L}\right)(1 - \alpha' R_s)} \tag{44}$$

Note that (14) and (17b) give

$$1 - \alpha' R_s > 0 \tag{45}$$

When (17) are satisfied, (24) and (45) are also, and

$$R_{Lh}, \; C_{Lh} > 0$$

The frequency of oscillation is

$$\omega_a^2 = \frac{1}{L}\left(\frac{1}{C_D} + \frac{1 + R_s G_L}{C_L}\right) - \frac{\alpha'}{C_D}\left(\frac{R_s}{L} + \frac{G_L}{C_L}\right) \tag{46}$$

The maximum frequency of harmonic oscillation is found by substituting (43) and (44) into (46).

$$\omega_h^2 = \frac{2}{LC_D} - \frac{\alpha'^2}{C_D^2} - \frac{R_s}{\alpha' L^2} \tag{47}$$

This can be rewritten as $\omega_h^2 = \omega_L^2 + \frac{1}{\alpha' L}\left(\frac{\alpha'}{C_D} - \frac{R_s}{L}\right)$

Then with (24) and (45) $\omega_L^2 < \omega_h^2 < \omega_R^2$

Note that (47) has a maximum when $L = \frac{R_s C_D}{a'}$ (48)

With (48), $\omega_h = \omega_R$ and $C_L = \infty$, i.e., the TDRC has become a TDRL. A value of L given by (48), of course, violates assumption (17a) (see (24)). For $L > \frac{R_s C_D}{a'}$ ω_h^2 is a monotonically decreasing function of L.

It has been determined from numerical solutions and experimental results that ω_h is only slightly less than the maximum frequency of oscillation of the TDRC. Therefore, the maximum frequency of oscillation of the TDRC is a nearly harmonic oscillation, and the maximum frequency is approximately ω_h.

V. PERTURBATION METHOD

If the second order equation

$$s^2 x + x + \mu f(x, sx) = 0 \tag{49}$$

has a periodic solution and if μ is a small parameter, the solution can be obtained by a perturbation method known as the Lindstedt method.[6] For a given second order oscillator equation, the problem is to put the equation into the form of (49). Pepper has shown how to do this for a class of second order equations and has obtained the perturbation solution for the TDRL.[1,9] Similarly, if the third order equation

$$(s^3 + \sigma_1 s^2 + \omega_a^2 s + \sigma_1 \omega_a^2)\ x + \mu f\ (x, sx, s^2 x, s^3 x) = 0 \tag{50}$$

has a periodic solution, it can be solved by the Lindstedt method. Thus, the Lindstedt method can be used for the TDRC if (4) can be put into the form of (50). This can be done provided the stable, steady-state solution of (4) is a pure sinusoid with zero amplitude when the starting condition (12) is satisfied with the equal sign (the reason for this will be seen below).

For the sake of generality, we will find the perturbation solution of

$$M_1 x + M_2 x^2 + M_3 x^3 = 0 \qquad (51)$$

where

$$M_1 = s^3 + a_2 s^2 + a_1 s + a_0 \qquad (52)$$

$$Mj = b_{j3} s^3 + b_{j2} s^2 + b_{j1} s + b_{j0}; \; j = 2,3 \qquad (53)$$

The starting condition for (51) is (11). Equation (51) can be put into the form of (50) as follows. First consider the linear part of (51).

$$M_1 x = (s^3 + a_2 s^2 + a_1 s + a_0) x \qquad (54)$$

The zeros of (54) are labeled:

$$-\sigma_2$$

$$-\sigma_3 \pm j\omega_3 \qquad (55)$$

where

$$a_0 = \sigma_2(\sigma_3^2 + \omega_3^2)$$

$$a_1 = \sigma_3^2 + \omega_3^2 + 2\sigma_2\sigma_3$$

$$a_2 = \sigma_2 + 2\sigma_3 \qquad (56)$$

If the oscillator is to be nearly harmonic

$$\left|\frac{\sigma_3}{\omega_3}\right| < < 1 \tag{57}$$

With (57)

$$\omega_3^2 \simeq a_1$$

$$\sigma_2 \simeq \frac{a_0}{a_1}$$

Then

$$(s^3 + \frac{a_0}{a_1} \; s^2 + a_1 s + a_0) \; x$$

has approximately the same zeros as (54) but the complex pair are on the $j\omega$ axis. Let

$$x = hu \tag{58}$$

Then (51) becomes

$$(s^3 + \frac{a_0}{a_1} \; s^2 + a_1 s + a_0) \; u - \mu^2 s^2 u + hM_2 u^2 + h^2 M_3 u^3 = 0 \tag{59}$$

where

$$\mu^2 = \frac{a_0}{a_1} - a_2 \tag{60}$$

Now let

$$h^2 = \mu^2 \tag{61}$$

(59) becomes

$$(s^3 + \frac{a_0}{a_1} s^2 + a_1 s + a_0) u + \mu \ (-\mu s u + M_2 u^2 + \mu M_3 u^3) = 0 \tag{62}$$

Equation (62) is of the form (50). The Lindstedt method can now be used for (62) (e.g., see stoker[6]). The results are

$$x = \mu(A_0 + \mu A_1) \cos \omega t + \mu^2(-2D_1 \sin \omega t + C_1 \cos 2\omega t + D_1 \sin 2\omega t + E_1) \tag{63}$$

where

$$A_0^2 = \frac{a_1}{H} \tag{64}$$

$$\omega = \sqrt{a_1}\left\{1 + \frac{\mu^2}{2H}\left[\frac{3}{4}(b_{31} - b_{33} a_1) - \frac{k_s}{\sqrt{a_1}}\right]\right\} \tag{65}$$

$$H = \frac{3}{4}\ (a_1 b_{32} + \frac{a_0}{a_1} b_{31} - b_{30} - a_0 b_{33}) - k_c - \frac{a_0}{a_1\sqrt{a_1}} k_s \tag{66}$$

$$C_1 = \frac{A_0^2 (8a_1^2 b_{21} + 3a_0 b_{20} - 24a_1^3 b_{23} - 12a_0 a_1 b_{22})}{2\ (16 a_1^3 + 9a_0^2)} \tag{67}$$

$$D_1 = \frac{A_0^2 \sqrt{a_1}(2a_1 b_{20} + 9a_0 a_1 b_{23} - 8a_1^2 b_{22} - 3a_0 b_{21})}{16a_1^3 + 9a_0^2} \tag{68}$$

$$E_1 = \frac{b_{20} A_0^2}{2a_0} \tag{69}$$

$$k_c = \frac{1}{A_0^2}\left[(C_1 + 2E_1)\ (b_{20} - a_1 b_{22}) + \sqrt{a_1} D_1\ (b_{21} - a_1 b_{23})\right] \tag{70}$$

$$k_s = \frac{1}{A_0^2}\left[\sqrt{a_1}\,(C_1 + 2E_1)\ (a_1 b_{23} - b_{21}) + D_1\ (b_{20} - a_1 b_{22})\right] \tag{71}$$

The next term in the solution must be found to obtain A_1.

Equation (61) imposes an important restriction on the perturbation analysis. If $x_1(t)$ is the stable steady state solution of (51), (61) requires that

$$x_1(t) \to 0 \quad \text{when} \quad \mu \to 0 \tag{72}$$

When $\mu = 0$, the starting condition (11) is satisfied with the equal sign, and a solution of (51) is a sinusoid with zero amplitude. This is a stable solution when (72) is satisfied; it is an unstable solution when (72) is not satisfied.

We will define a potentially hard oscillator as an oscillator that has a non-zero amplitude when the starting condition (11) is satisfied with the equal sign. Thus, the perturbation method is not valid for a potentially hard oscillator. For very small amplitudes, a potentially hard oscillator becomes "more unstable" as the amplitude increases, i. e., the net average ac power generated by the circuit increases as the amplitude increases. A potentially hard oscillator becomes a hard oscillator when the starting condition is not quite satisfied. It can be shown by average power arguments[10] that if the following inequalities hold, the solution of (51) is not that of a potentially hard oscillator.

$$\frac{3}{4}(a_1 b_{32} + a_2 b_{31} - b_{30} - a_0 b_{33}) - \frac{b_{20}}{a_0}(a_1 b_{22} + a_2 b_{21} - b_{20} - a_0 b_{23}) > 0 \tag{73 a}$$

or

$$\frac{3}{4}(a_1 b_{32} + \frac{a_0}{a_1} b_{31} - b_{30} - a_0 b_{33}) - \frac{b_{20}}{a_0}(a_1 b_{22} + \frac{a_0}{a_1} b_{21} - b_{20} - a_0 b_{22}) > 0 \tag{73b}$$

I. e., (73) must be satisfied for the perturbation method to be valid.

If (73b) is satisfied and $C_1 = D_1 = 0$, $\mu^2 A_0^2$ is always positive. The validity condition (73) assures a positive value for the amplitude squared except for the possibility of the second harmonic terms, C_1 and D_1, altering the situation.

The results of the perturbation analysis can now be specialized to the TDRC case. From (73a) and (4) the TDRC is not potentially hard if

$$\frac{\beta_{13}}{C_D}\left[\left(\frac{G_L}{C_L}+\frac{R_s}{L}\right)^2-\frac{2\alpha'}{C_D}\left(\frac{G_L}{C_L}+\frac{R_s}{L}\right)+\frac{1}{LC_D}\right]>0 \tag{74}$$

The discriminant of (74) considered as a quadratic in $(G_L/C_L + R_s/L)$ is

$$4\left(\frac{\beta_{13}}{C_D}\right)^2\left(\frac{(\alpha')^2}{C_D^2}-\frac{1}{LC_D}\right)$$

For most tunnel diodes $\alpha'^2/C_D < 1/L$, (74) is satisfied, and the perturbation method can be used. The results are obtained by comparing (8) and (63) to (66) (note that b_{33}, $b_{2j} = 0$)

$$v = \mu A_0 \cos \omega t \tag{75}$$

$$\omega = \sqrt{a_1}\left[1+\frac{b_{31}\left(\frac{a_0}{a_1}-a_2\right)}{2\left(a_1 b_{32}+\frac{a_0}{a_1}b_{31}-b_{30}\right)}\right] \tag{76}$$

$$\mu^2 A_0^2 = \frac{4\,(a_0 - a_1 a_2)}{3\left(a_1 b_{32}+\frac{a_0}{a_1}b_{31}-b_{30}\right)} \tag{77}$$

The ai, b_{ij} are obtained from (4).

VI. DESIGN PROCEDURE FOR THE TDRC

It will be assumed that (17) (i.e., (24) and (45)) are satisfied so that the TDRC has a higher frequency of harmonic oscillation than the TDRL. If (47) is used as a basis for determining the design frequency, the design procedure can be used to obtain a frequency near the maximum possible. If a lower frequency is desired, series L_2 or series R_2 can be added. Design requirements may dictate whether L_2, R_2 or both are to be added. For example, it can be seen from (43) that R_2 can be specified within limits by choosing the appropriate value of $R_s = R_1 + R_2$. Then $L = L_1 + L_2$ is determined from the frequency requirement. The value of L or R_s is determined from (51)

$$\frac{1}{L} = \frac{\alpha'}{R_s C_D} \left(1 - \sqrt{1 - \alpha' R_s - \frac{R_s C_D^2 \omega^2}{\alpha'}}\right) \tag{78}$$

or

$$R_s = \alpha' L \left(\frac{2}{C_D} - \frac{\alpha'^2 L}{C_D^2} - \omega^2 L \right) \tag{79}$$

where ω is the required frequency of oscillation. The negative square root is taken in (78) in order that $\alpha'/C_D > R_s/L$. The value of R_L is determined from (47)

$$R_L = R_{Lh} = \frac{1}{\alpha'} - R_s \tag{80}$$

From (4) and (80) $a_0 = 0$. Since $a_1 > 0$, the starting condition is $a_2 < 0$.

Now C_L can be obtained from (4) and (77) noting that $\omega \approx \sqrt{a_1}$

$$C_L = \frac{a'L\,(4\omega^2 C_D{}^2 - 3a'\beta_{13}\mu^2 A_0{}^2)\,C_{Lh}}{4a'\omega^2 LC_d{}^2 - 3\beta_{13} C_{Lh}\mu^2 A_0{}^2 (1 - a'R_s)^2} \tag{81}$$

where μA_0 is the amplitude across the nonlinearity. The starting condition becomes $C_L > C_{Lh}$.

VII. EXPERIMENTAL RESULTS

Tunnel-diode oscillators have been built using a Hoffman type IN 2928 silicon tunnel diode. The static v-i characteristics are shown in Fig. 4 along with the cubic approximations determined by the method in the Appendix. The operating point is taken as the point of minimum slope and is 0.125 volts. For this operating point $a' = 2.2 \times 10^{-3}$ mho, $C_D = 27$ pf, $R_1 \approx 1$ ohm, $L_1 \approx 5 \times 10^{-9}$ henry. In order to have R_s determined accurately an external series resistance of 24 ohms is added so that $R_s = 25$ ohms.

Two tunnel-diode oscillators were designed. The design or predicted results are given in Table 1 along with the experimental values of amplitude and frequency. The values of L, R_2, and C_L are determined from (78), (80), and (81). The design procedure calls for a load resistance of 429 ohms. A value of 400 ohms was used to make certain that the linearized real natural frequency was in the LHP. The circuit used is shown in Fig. 5. Since $R_s >> R_1$, $L >> L_1$ the voltage across the nonlinearity is very nearly equal to v_1. Table 1 shows excellent frequency agreement. The measured amplitude is too low in both cases, being 35% below the design value for the worst case. The accuracy of the amplitude is discussed in the following paragraph.

In order to check the accuracy of the TDRC analysis the amplitude and frequency were measured for several values of C_L. This procedure was repeated for several values of R_L. In Fig. 6 the measurements are compared with values obtained from the

Frequency (mc)		Amplitude (mv)				L	C_L	R_L
		v_1		v_L		(μh)	pf	Ω
Design	Meas	Design	Meas	Design	Meas			
20	20	100	65	71.5	45	3.2	37	400
20	19.8	150	100	79	63	3.2	50	400

TABLE. 1

perturbation method for $R_L = 400$. The results are similar for other values of R_L. It can be seen that the perturbation solution agrees quite well with the experimental results for amplitudes up to about 0.18 volts (0.36 volts peak to peak). This is relatively large signal operation and the voltage across the tunnel-diode becomes quite nonharmonic. An examination of Fig. 6 shows that the predicted amplitude is too high for measured amplitudes below about 0.15 volts; while the opposite is true for higher amplitudes. This can be explained by examining Fig. 4. In most of the negative resistance region the slope of the cubic approximation is less (more negative) than that of the actual characteristic. Thus for voltages in this region an element with a cubic characteristic would supply more energy to the passive elements than the actual element does, so a larger amplitude is predicted. In most of the positive resistance region the cubic has a greater slope. In this region the cubic would dissipate more energy than the actual nonlinearity. Thus for large amplitudes, use of the cubic approximation should result in smaller predicted amplitudes. This is borne out by Fig. 6.

For reference, designate the TDRC equation with the nonlinearity approximated by the cubic as the approximate equation. This approximate equation was solved numerically with the aid of a computer by a method similar to the phase plane analysis of Kuh[10]. The results are also shown in Fig. 6. The amplitude and frequency of the perturbation method are in excellent agreement with that of the numerical solution. The perturbation solution is an excellent approximation to the solution of the approximate equation. This indicates that the limiting factor is the approximation of the nonlinearity rather than the method of solution.

VIII. GENERALIZATION

The perturbation results given in Section V., can be applied to a large class of oscillators described by a third order equation. Specifically, the results can be used for any oscillator which

a) can be described by three dynamically independent variables,

b) have frequency independent nonlinearities that are all single-valued functions of one and the same variable, and

c) are not potentially hard.

Due to b) the oscillator can be described by a single third order equation. As shown previously, property c) is required if the perturbation method is to be valid.

The TDRC equation (1) can be written as $M_0 v + M_1 F_1(v) = 0$ where M_0 and M_1 are linear differential operators. Generalizing, we can write the characteristic equation of any oscillator satisfying conditions a) and b) above as

$$M_0 x + \sum_{i=1}^{n} M_i F_i(x) = 0 \tag{82}$$

where M_0 and the M_i are linear operators of order three or less, i. e.,

$$M_i = a_{i3} s^3 + a_{i2} s^2 + a_{i1} s + a_{i0};\ i = 0, 1, \dots, n \tag{83}$$

As is done for the TDRC, the use of a transformation of variables is assumed in (82) so that the dc operating point is $x = 0$ and

$$\sum_{i=1}^{n} a_{i0} F_i(0) = 0 \tag{84}$$

Due to (84) the constant terms of the $F_i(x)$ do not affect (82). For convenience the constant terms will be omitted. If the nonlinearities in (82) are approximated by third degree polynomials

$$F_i(x) = \alpha_i x + \beta_{i2}\, x^2 + \beta_{i3}\, x^3 \tag{85}$$

With (85), (82) becomes

$$M_1 x + M_2 x^2 + M_3 x^3 = 0 \tag{86}$$

where

$$M_1 = s^3 + a_2 s^2 + a_1 s + a_0$$

$$M_j = b_{j3} s^3 + b_{j2} s^2 + b_{j1} s + b_{j0}; \quad j = 2,3$$

$$a_3 = a_0 + \sum_{j=1}^{n} a_i a_{i3}$$

$$a_k = \frac{1}{a_3} \left(a_{0k} + \sum_{i=1}^{n} a_0 a_{ik}\right)$$

$$b_{jk} = \frac{1}{a_3}\, \beta_{ij}\, a_{ik}$$

(87)

Equation (86) is the equation analyzed in Section V. Thus, the results of Section V can be applied directly to (86) provided the oscillator described by (88) is not potentially hard, (i. e., provided (73) is satisfied). Some oscillators described by (82) (therefore, by (86) approximately) are the Colpitts, Hartley and some phase shift oscillators provided the active device can be considered frequency independent.

APPENDIX: APPROXIMATION OF THE TUNNEL-DIODE CHARACTERISTIC

For the tunnel-diode nonlinearity $F_1(v)$. the value of $-\alpha'$ is chosen as the slope of the actual characteristic at the operating point. This is necessary if the small-signal operation of the oscillator (e. g., the starting condition) is to be accurately described.

A third degree polynomial has odd symmetry about its point of minimum slope, but the tunnel-diode characteristic does not, as can be seen from Fig. 1. A better fit to the tunnel-diode characteristic can be obtained if, instead of a conventional third degree polynomial, the following approximation is used

$$
\begin{aligned}
F_1(v) &= -\alpha' v + \beta_{13}^{+} v^3 , \quad v > 0 \\
&= -\alpha' v + \beta_{13}^{-} v^3 , \quad v < 0
\end{aligned} \tag{88}
$$

β_{13}^{+} and β_{13}^{-} are chosen so that the maximum and minimum values of (88) coincide with those of the actual tunnel-diode characteristic. Then

$$
\begin{aligned}
\beta_{13}^{+} &= \frac{\alpha'}{3v_2^2} \\
\beta_{13}^{-} &= \frac{\alpha'}{3v_1^2}
\end{aligned} \tag{89}
$$

where v_1 and v_2 are defined in Fig. 1.

To the accuracy obtained in Section V., the perturbation solution can be obtained when the nonlinearities are approximated by

$$F_i(x) = \alpha_i x + \beta_{i3}^{+} x^3 , \quad x > 0$$

$$= \alpha_i x + \beta_{i3}^{-} x^3 , \quad x < 0 \qquad (90)$$

The results are (63), (64), (65) and (66) where the β_{i3} are given by

$$\beta_{i3} = \frac{1}{2} \quad (\beta_{i3}^{+} + \beta_{i3}^{-})$$

and the $\beta_{i2} = 0$ (hence, the $\beta_{2i} = 0$).

This can be shown as follows. Equation (62) can be written as an integral equation. The first perturbation term can be obtained by following the method used by Stoker to establish existance of the perturbation series for a second order oscillator equation.[6] This involves definite integrals in which x is replaced by its first approximation, $x_0 = A_0 \cos \omega t$. These integrals can be evaluated when the $F_i(x)$ are given by (90). The amplitude and frequency can then be determined; the results are given above.

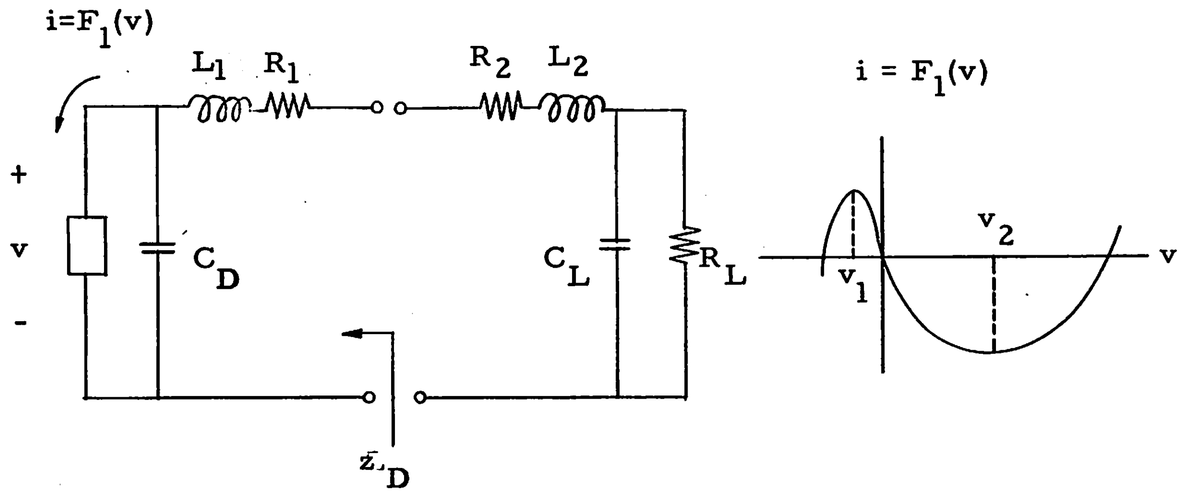

Figure 1. Tunnel Diode Circuit Model

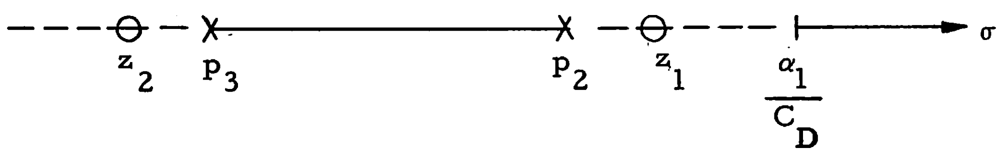

Figure 2. Complex Frequency Plane Locations of $P_{2,3}$ and $z_{1,2}$ for $P_{2,3}$ Real

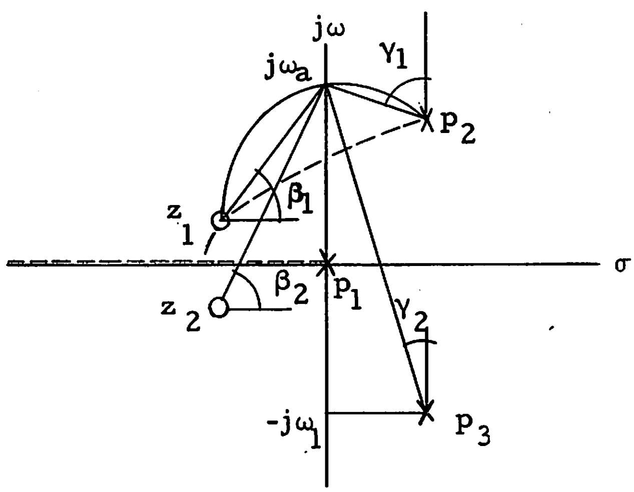

Figure 3. Root Locus of (19) as $1/C_L$ is Varied.

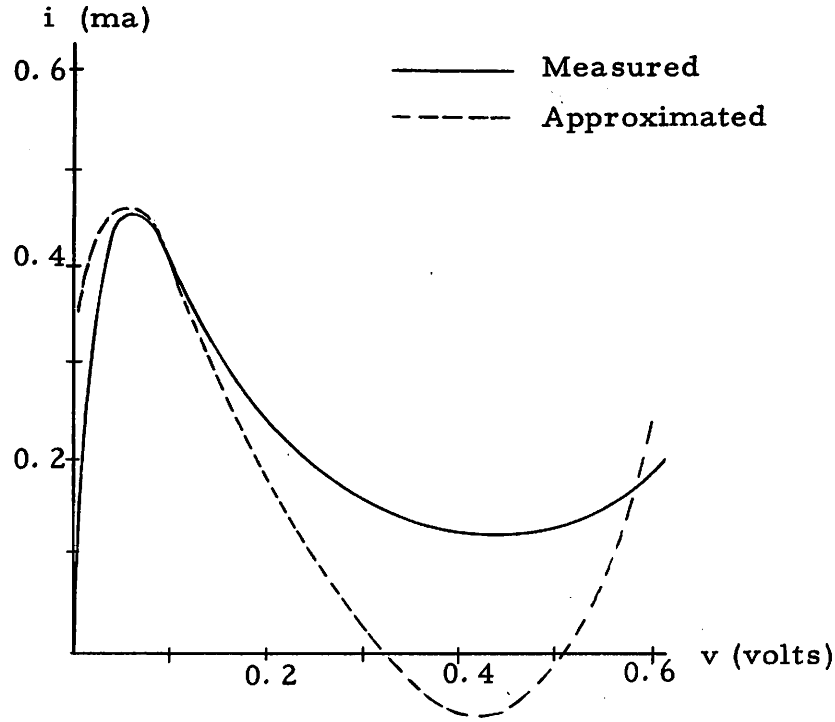

Figure 4. Tunnel Diode Static v-i Characteristics together with Cubic Approximation

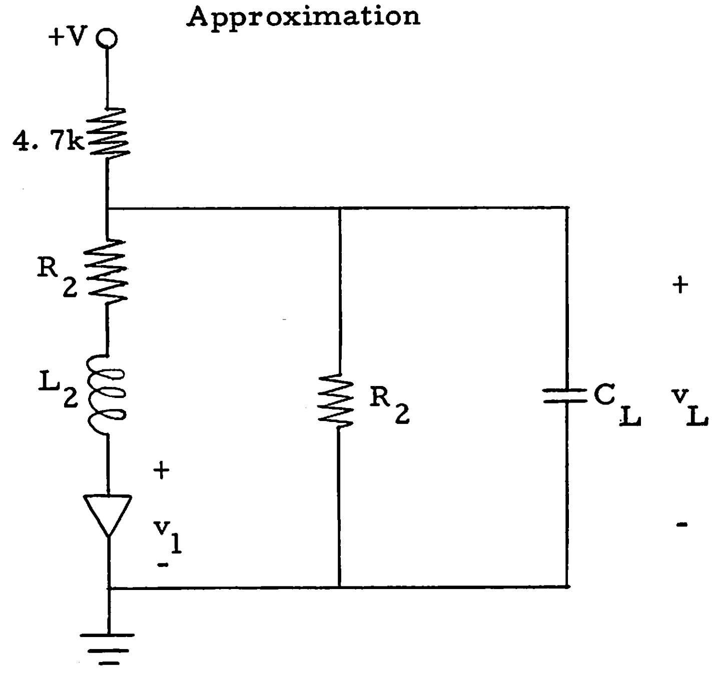

Figure 5. Tunnel-Diode Oscillator Circuit

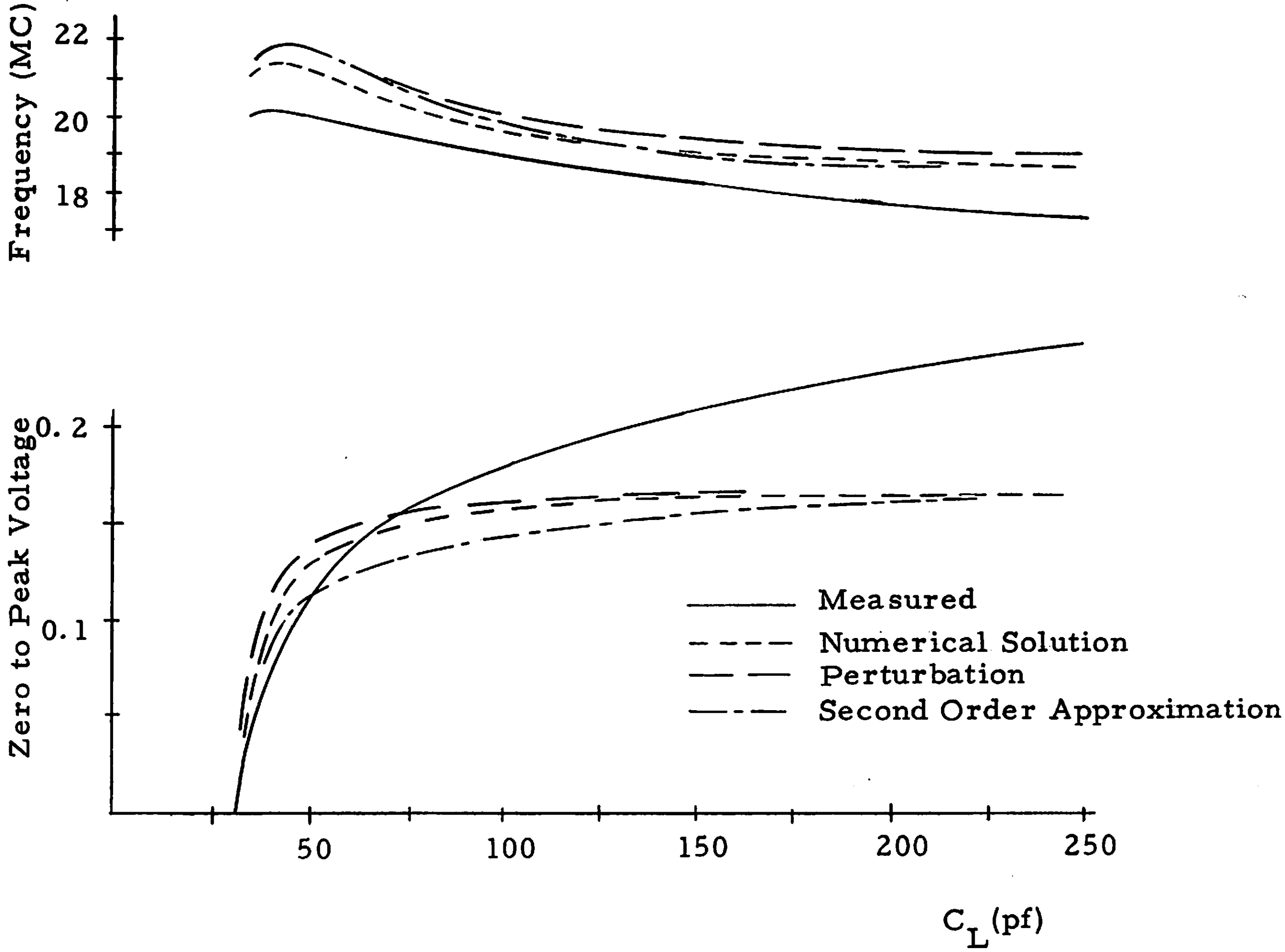

Figure 6. Comparison of Experimental and Calculated Values for Amplitude and Frequency of the Tunnel-Diode Oscillator

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7. E.g., When a_o is a very small negative number for the TDRC, the dc load line intersects the $F_1(v)$ curve at three points and the circuit oscillates about either the upper or lower intersection depending on initial conditions. As a_o is made slightly more negative, the TDRC becomes bistable. This behavior was found with a numerical analysis and confirmed experimentally.

8. Sterzer and Nelson[2] give $(\hat{\omega})^2 \leq \frac{1 - \alpha'(R_s + G_2)}{LC_D + R_L C_L (R_s C_D - \alpha' L)}$ as

   an approximation to the frequency of oscillation of the TDRC. This approximation gives $\omega = 0$ when R_L and C_L are adjusted for the maximum frequency of harmonic oscillation.

9. D. K. Lynn, R. S. Pepper, D. O. Pederson, "The Minimum Period of Oscillation of Simple Tunnel Diode Oscillators," Institute of Engineering Research Series 60, Issue No. 388, University of California, Berkeley; Aug. 2, 1961.

10. Y. H. Kuh, "A method for Solving Third and Higher Order Nonlinear Differential Equations," J. Franklin Inst., 256, pp. 229-244; Sept. 1953.