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STABILITY OF LINEAR, TIME-VARYING NETWORKS FROM ENERGY FUNCTION ANALYSIS

by

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ABSTRACT

The formalism of classical mechanics is extended to yield an energy function analysis of single degree of freedom, linear time-varying networks. With the aid of the newly found modifications of the classical Lagrangian and Hamiltonian, the behavior of the single loop R(t)-L(t)-C(t) network is discussed through analogy with familiar classical quantities. A general energy analysis is used to investigate the stability of n degree of freedom, time-varying, networks.

The Hamiltonian formulation, through its associated canonical equations, leads to a qualitative phase plane specification of single degree of freedom network behavior. Moreover, the Hamiltonian formulation leads directly to stability criteria which are expressed in terms of the network element values only. Upper and lower bounding functions for the network stored energy are obtained which lead to separate necessary and sufficient conditions for network stability.

The energy function analysis is extended to n degree of freedom, linear, time-varying networks by two different methods. The modified Lagrangian is obtained for some special classes of time-varying networks. In particular, a Lagrangian is obtained for a lossy, time-varying, distributed network, and for an iterated, time-varying network.

An analysis on the basis of general power and energy considerations yields upper and lower bounding functions for the stored energy in n degree of freedom, linear, time-varying networks. These bounding functions are used to investigate the possible stability and instability of such networks. The ground-state behavior of parametric amplifier circuits commonly in use is discussed in examples.

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I. INTRODUCTION

The techniques of analysis of linear, time-varying electrical networks have evolved mainly from two classical standpoints. One method replaces the n^{th} order, variable coefficient differential equation which describes the network with a system function H(s,t), derived by assuming a drive of exp(st). The alternative analytical technique involves describing the network by a single first-order matrix differential equation. Neither of these characterizations has been overly encouraging regarding the ultimate solution of time-varying network problems; they both have their domains of applicability but there is certainly room for yet other techniques.

In classical lumped, stationary network theory, the system function H(s) has become the standard tool of analysis. This function owes its utility to the fact that exp(st) is an eigenfunction[11,16] of the network differential equation for certain values of s, which can be obtained algebraically. It is unfortunate, though, that such a simple drive does not yield the same powerful results for the general time-varying network. To be sure, the H(s,t) characterization has led to many interesting and useful results in the synthesis of special classes of time-varying networks.[6] On the other hand, its tendency to be identified with time-invariant H(s) of old could lead to some serious misconceptions regarding network behavior (e.g., "moving poles" in the left-half s-plane indicating stability).

It appears, from reading surveys of the progress of the theory associated with time-varying networks,[28] that the first-order matrix differential equation characterization is gaining in esteem. The method for explicitly writing out the state equations (A-matrix characterization) of a given time-varying network has been advanced in two different treatments;[4,18]

however, beyond the writing of the equations, their solution entails mathematical rather than engineering research. There is little hope of a major breakthrough in this direction until new mathematical tools are evolved -- while this evolution is a commendable task, it would hardly seem that engineers are the best equipped to undertake it.

The tendency to replace as rapidly as possible the physical network, as though it were something repugnant, with a neutral set of coupled first-order differential equations seems to have been overplayed. A network is not merely a set of equations, but is a collection of physical elements (albeit they may be mathematical models for physical elements) with certain properties which may be used to analytical advantage before the network elements are relegated to equal status in the components of differential equations. To utilize fully the element properties, it will be desirable to seek a mathematical characterization which retains the distinctions among them.

In considering the time-varying network consisting of time-varying resistances, R(t), time-varying inductances, L(t), and time-varying capacitances, C(t), it is natural to retain the second-order differential equation approach:

$$\frac{d}{dt}[L(t)\dot{q}] + R(t)\dot{q} + \frac{1}{C(t)}q = e(t), \qquad (1.1)$$

where e(t) is the driving voltage and $\dot{q}(t)$ is the unknown current in the series RLC branch. It matters not how the elements arise (see Appendix I); any network which can be modeled with linear, time-variable R's, L's, and C's (even a nonreciprocal network) is a valid candidate for considerations. The job of writing the network equations is now as easy as that for time-invariant networks, but it still remains to evolve a new analytical technique.

The network stored energy is a useful measure of performance for any electrical network regardless of whether the network is stationary or time-varying, or even nonlinear. The ability of an electrical network to transfer energy from one form to another is one of its more interesting and useful properties. Thus, a study of the energy behavior of the network can yield much information on the utility of the network. Equations of the form (1.1) can be used to evolve energy functions analogous to those of classical mechanics.[13,19] These energy functions must be tailored to fit the usually dissipative time-varying networks; these modified counterparts of the usual energy functions yield many useful quantitative and qualitative results.

The energy function analysis is not being advanced as a panacea -- it should only be used in conjunction with the other methods mentioned above and within its own domain of applicability. Among its many advantages are, however, suitability to modern computational devices and ability to yield rapidly qualitative information on the response of a given network.

In Section II the formalism of classical mechanics is extended to yield an energy function analysis of single degree of freedom, linear, time-varying networks. With the aid of the newly found modifications of the classical Lagrangian and Hamiltonian, the behavior of the single loop R(t)-L(t)-C(t) network is discussed in the phase plane.

The Hamiltonian formulation, besides providing a set of canonical equations for the system, leads directly to stability criteria which are expressed in terms of the network values only. Upper and lower bounding functions for the network stored energy are obtained in Section III which lead to separate necessary and sufficient conditions for network stability.

The energy function analysis is extended to n degree of freedom, linear, time-varying networks by two different methods in Section IV. The modified Lagrangian is obtained for some special classes of time-varying networks. In particular, a Lagrangian is obtained for a lossy, time-varying, distributed network, and for an iterated time-varying network.

The analysis of Section V on the basis of general power and energy considerations yields upper and lower bounding functions for the stored energy in n degree of freedom, linear, time-varying networks. These bounding functions are used to discuss the possible stability and instability of such networks. The ground state behavior of parametric amplifier circuits commonly in use is discussed in examples.

II. ENERGY FUNCTION ANALYSIS OF LINEAR, TIME-VARYING NETWORKS WITH ONE DEGREE OF FREEDOM

The techniques of analytical mechanics have come to be recognized as powerful devices for the analysis of any physical system. In the classical treatment, however, there have been many assumptions which restrict the utility of these analytical tools. The linear, time-varying electrical network provides a convenient basis for the extension and application of some of the ideas which are embodied in analytical mechanics. A long recognized advantage of the analytical mechanics approach has been the ability to characterize partially a system without seeking a specific solution to the equations of motion; this advantage will be exploited in the following two sections.

II.1 Time-Dependent Energy Functions.

In the Hamiltonian formulation,[13] the absence of explicit time in the Hamiltonian indicates that a first integral of the equations of motion can be obtained. More often than not, the theory is concentrated about just such conservative systems. Electrical networks, as well as most other practical macroscopic systems, usually contain a significant amount of damping (sometimes negative) in the form of resistances. This damping must be evidenced in a Hamiltonian which is explicitly time-dependent, if a Hamiltonian is to be obtainable at all.

In order to obtain the analogs to the classical energy functions, the price paid must be large, but the rewards more than offset it. The words Lagrangian, Hamiltonian, conjugate momentum, etc.[13,19] will be retained, but the mathematical entities which they denote will not be the same. In particular, if the Lagrangian or Hamiltonian is to be thought of as an energy function describing system energy, it only characterizes the immediate system under scrutiny. Thus, a loss or gain in

energy in an isolated part of the system can be accounted for in terms of the time-dependence of its "energy functions." For example, if a resistor is to be considered solely as a network element, the fact that it dissipates electrical energy in the form of heat must be accounted for in an explicit time-dependence of the Lagrangian or Hamiltonian. Similarly, a driven ("pumped") nonlinear energy storage element can be considered to be a time-variable energy storage element to account approximately for energy extracted from or delivered to the pump.

The above described contrivances are necessary to allow the analysis of systems containing energy converting devices where it would be convenient to exclude some of the energy forms (e.g., heat) from the analysis. It is debatable as to whether the issue must be thus skirted because of physical or mathematical dicta. Hamilton's principle[13,19], a very powerful tool when it is a variational principle, although it is couched in terms of physics, seems upon closer inspection to depend upon quirks of mathematics for this property. Consider, for example, the simple stationary network of Fig. 2.1. Its governing equation,

$$\frac{d}{dt}[L\dot{q}] + \frac{1}{C}q = 0, \tag{2.1}$$

is the Euler-Lagrange equation[10],

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0, \tag{2.2}$$

associated with the Lagrangian

$$\mathcal{L}(q,\dot{q}) = \frac{1}{2}L\dot{q}^2 - \frac{1}{2C}q^2. \tag{2.3}$$

But there is an equally valid governing equation (in fact the one more often encountered in electrical engineering) on the current basis,

$$\frac{d}{dt}[Li] + \frac{1}{C}\int idt = 0; \qquad (2.4)$$

it has no associated Lagrangian. Thus, the charge q(t) assumes a more important role than the current i(t) -- strictly from a consideration of the mathematical formulation. Similarly, the flux $\phi(t)$ takes precedence over the voltage $v(t) = \dot{\phi}(t)$ in these analyses.

In application to time-varying networks, the techniques of analytical mechanics are certainly not panaceas. They cannot be expected always to provide closed form solutions where they were previously unobtainable from other more conventional forms of mathematical analysis. But they can provide insight into the qualitative (and even quantitative) behavior of a network without requiring any knowledge of the form of the solution. Thus, as one might expect, for time-invariant networks the energy function analysis does not provide as much information as the exact solution. However, it provides a wealth of information for time-varying networks -- where the exact solution is usually not available. Furthermore, all of the theory of classical mechanics (e. g., the phase plane, state space, and Hamilton-Jacobi theory) can be carried over en masse to provide new insights into the behavior of time-varying networks.

II. 2 Derivation of the Variational Principle.

The single degree of freedom, linear network to be considered is the single loop network of Fig. 2. 2, in which all of the elements are time-dependent. Any linear, single degree of freedom network will obey with obvious modification (e. g., duality) the same set of equations; and, hence; will be governed by similar results.

The time-varying LC network of Fig. 2. 3 enjoys the status of possessing a Lagrangian:

$$\mathcal{L}(q, \dot{q}, t) = \frac{1}{2}L(t)\dot{q}^2 - \frac{1}{2C(t)}q^2 + e(t)q. \qquad (2.5)$$

The Euler-Lagrange equation[10],

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0, \qquad (2.6)$$

associated with this Lagrangian is the governing equation of the network

$$\frac{d}{dt}\left[L(t)\dot{q}\right] + \frac{1}{C(t)}q - e(t) = 0. \qquad (2.7)$$

It is well-known that to formulate the Lagrangian for a holonomic, conservative system, one need only take the difference between kinetic and potential energies,

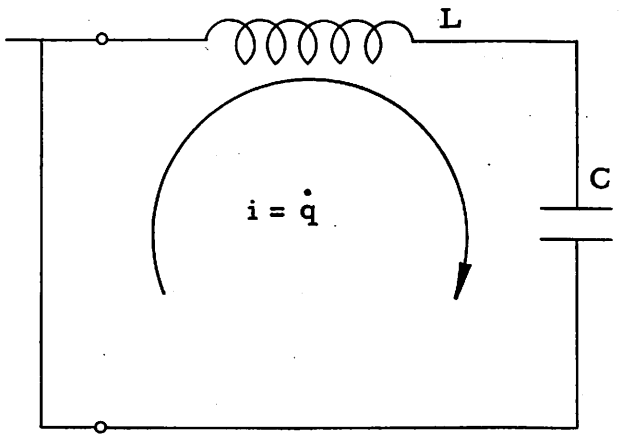

Fig. 2.1. Time-invariant LC Loop.

$$\delta \int_{t_1}^{t_2} \mathcal{L}\, dt = 0. \tag{2.9}$$

The network of Fig. 2.3 has a Lagrangian, and the Euler-Lagrange equation describes its behavior; yet it is surely not conservative -- the time-varying inductance and capacitance indicate an ability to introduce or remove energy from the circuit. The network of Fig. 2.2 is the same network as that of Fig. 2.3, only it includes also a time-dependent resistance. This "dissipative" network is certainly no less conservative in general than the other, yet its governing equation,

$$\frac{d}{dt}\left[L(t)\dot{q}\right] + R(t)\dot{q} + \frac{1}{C(t)}q - e(t) = 0, \tag{2.10}$$

is not the Euler-Lagrange equation associated with any obvious Lagrangian. In usual practice[27] the governing equation (2.10) can be obtained by including the Rayleigh dissipation function,

$$\mathcal{F}(\dot{q}, t) = \frac{1}{2}R(t)\dot{q}^2, \tag{2.11}$$

with the Lagrangian (2.5) in a modification of the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{F}}{\partial \dot{q}} = 0. \tag{2.12}$$

This formulation, although it does lead to the governing equation from so-called energy functions ($\frac{1}{2}R(t)\dot{q}^2$ has the dimensions of power), has the decided disadvantage that it divorces the resistive from the inductive and capacitive parts of the network for any analyses not oriented toward obtaining the governing equation.

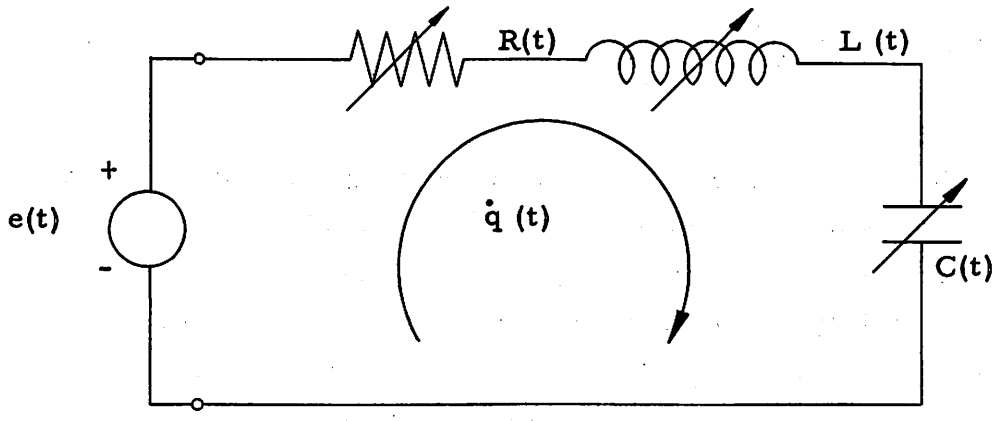

Fig. 2.2. Single-Loop, Linear, Time-variable Network.

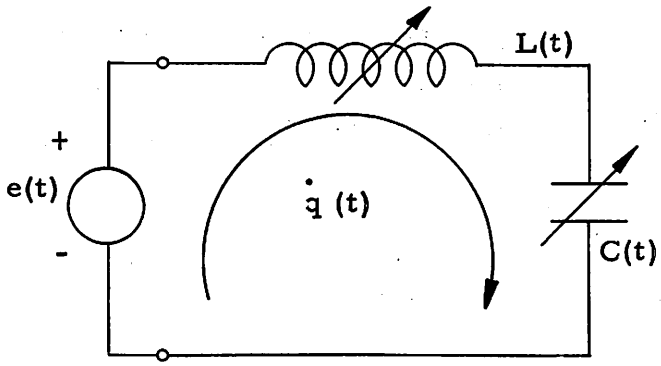

Fig. 2.3. Time-varying LC Loop.

$$\mathcal{L} = T - V \tag{2.8}$$

while the Euler-Lagrange equation governing the system arises from the variational form of Hamilton's principle,[13,19]

It seems that the inability to include the resistance in the compact form of Lagrangian analysis does not stem from conservative or nonconservative aspects of the network element itself, but rather from its peculiar fate of being the coefficient of the term $\dot{q}$ in the governing equation (2.10). This mathematical quirk can be overcome by the formulation of a modified Hamilton's principle for the governing equation multiplied by an arbitrary (for the present), non-zero time function D(t):

$$D(t)\left\{\frac{d}{dt}\left[L(t)\dot{q}\right] + R(t)\dot{q} + \frac{1}{C(t)}q - e(t)\right\} = 0. \tag{2.13}$$

The inverse problem of the calculus of variations -- that of formulating the variational principle from which a given differential equation arises -- has received little attention.[7] Recently, two different approaches to this problem have been evolved[20,24]; the pragmatic approach presented here relies heavily upon the manipulability associated with the variational δ-notation.[16] First one must multiply (2.13) by the function δq and integrate between fixed instants of time t_1 and t_2:

$$\int_{t_1}^{t_2} D(t)\left\{\frac{d}{dt}\left[L(t)\dot{q}\right] + R(t)\dot{q} + \frac{1}{C(t)}q - e(t)\right\}\delta q\,dt = 0, \tag{2.14}$$

where

$$\delta q(t_1) = \delta q(t_2) = 0. \tag{2.15}$$

Upon integration by parts of the first term in the integrand, (2.14) becomes*

*Henceforth, the explicit time-dependence will be omitted except where it is necessary or desirable; thus, C(t) is simply written C, etc.

$$DL\dot{q}\delta q\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\left\{-DL\dot{q}\delta\dot{q} - \dot{D}L\dot{q}\delta q + DR\dot{q}\delta q + \frac{D}{C}q\delta q\right.$$

$$\left. - De\delta q\right\} dt = 0. \tag{2.16}$$

The first term on the left vanishes by virtue of (2.15); and, if,

$$-\dot{D}L + DR = 0, \tag{2.17}$$

(2.16) can be rewritten

$$\delta\int_{t_1}^{t_2} D\left\{-\frac{1}{2}L\dot{q}^2 + \frac{1}{2C}q^2 - eq\right\}dt = 0. \tag{2.18}$$

An acceptable solution to (2.17) is

$$D(t) = \exp\left(\int_{t_o}^{t}\frac{R(\xi)}{L(\xi)}\,d\xi\right), \tag{2.19}$$

although this function multiplied by any constant (positive or negative) will suffice. Multiplication of (2.18) by -1 yields the desired modification of Hamilton's principle (2.9) as

$$\delta\int_{t_1}^{t_2}\exp\left(\int_{t_o}^{t}\frac{R(\xi)}{L(\xi)}\,d\xi\right)\left\{\frac{1}{2}L(t)\dot{q}^2 - \frac{1}{2C(t)}q^2 + e(t)q\right\}dt = 0. \tag{2.20}$$

The integrand in this equation can be identified as the modified Lagrangian,

$$\mathcal{L}(q, \dot{q}, t) = \exp\left(\int_{t_o}^{t}\frac{R(\xi)}{L(\xi)}d\xi\right)\left\{\frac{1}{2}L(t)\dot{q}^2 - \frac{1}{2C(t)}q^2 + e(t)q\right\}, \tag{2.21}$$

which yields through the Euler-Lagrange equation (2.6) the modified governing equation (2.13). For a rigorous demonstra-

tion that the modified Hamilton's principle (2.20) is a variational principle leading to the governing equation (2.13), see Appendix II.

Because it provides the dissipative element R(t) with a place in the energy function formulation, the function D(t) deserves, certainly as much as (2.11), the name dissipation function; it will be designated so in the sequel. Although D(t) could be chosen multiplied by an arbitrary constant, the form (2.19) is desirable because then the condition $R(t) \equiv 0$ leaves the modified Lagrangian (2.21) in the form of the original Lagrangian (2.5).

II. 3 The Hamiltonian Formulation.

The modified form of Hamilton's principle (2.20) makes available to the dissipative network with one degree of freedom a wealth of concepts and techniques from analytical mechanics. The Hamiltonian and Hamilton-Jacobi theory can be evolved from the Lagrangian (2.21) by the usual straightforward manipulative techniques expounded for classical mechanics.[13,19] In the past these concepts have been thoroughly exploited for conservative systems, but little has been done to link their powerful insights with dissipative systems.

The conjugate momentum associated with the coordinate q is

$$p \equiv \frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial \dot{q}}; \tag{2.22}$$

therefore,

$$p = \exp\left(\int_{t_o}^{t} \frac{R(\xi)}{L(\xi)}\, d\xi\right) L(t)\dot{q}, \tag{2.23}$$

or

$$p = D(t)L(t)\dot{q} \ . \tag{2.24}$$

If D(t) were unity (i.e., $R(t) \equiv 0$), the conjugate momentum p would simply be the flux ($\phi \equiv L(t)\dot{q}$ associated with the inductance L(t). Thus, the loss of the simple dual relationship between the charge and the flux is part of the price paid for the inclusion of the resistance in the energy formulation.

The modified Hamiltonian is obtained by performing a Legendre transformation [13] on the Lagrangian:

$$\mathcal{H}(q, p, t) \equiv p\dot{q} - \mathcal{L}(q, \dot{q}, t); \tag{2.25}$$

hence,

$$\mathcal{H}(q, p, t) = \frac{1}{2C(t)L(t)}p^2 + \frac{D(t)}{2C(t)}q^2 - D(t)e(t)q. \tag{2.26}$$

D(t) is no longer the multiplicative factor which it was in the Langrangian (2.21) because of its appearance in p (2.24). Hamilton's canonical equations for this system take the form

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = \frac{1}{D(t)L(t)}p, \tag{2.27a}$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = -\frac{D(t)}{C(t)}q + D(t)e(t). \tag{2.27b}$$

Hence, the modified governing equation (2.13) can be reformulated in terms of a first-order matrix differential equation:

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{D(t)L(t)} \\ -\frac{D(t)}{C(t)} & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} + D(t)\begin{pmatrix} 0 \\ e(t) \end{pmatrix}. \tag{2.28}$$

Naturally, the original network could have been described in terms of any number of sets of two first-order differential

equations [1,4,12,18]; the particular utility of the Hamiltonian approach will become evident as its exploitation develops.

II. 4 Hamilton-Jacobi Theory.

A canonical transformation to obtain the solution q(t) of the governing equation (2.13) can be obtained via Hamilton's principle function S(q,t). [13,19] S(q, t), the generating function of the canonical transformation to the solution coordinates, satisfies the Hamilton-Jacobi partial differential equation

$$\mathcal{H}\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0, \tag{2.29}$$

where

$$\frac{\partial S(q, t)}{\partial q} = p. \tag{2.30}$$

It would suffice to find the solution for the homogeneous case, [5] where the drive e(t) = 0 then, the Hamiltonian (2.26) becomes

$$\mathcal{H}(q, p, t) = \frac{1}{2DL} p^2 + \frac{D}{2C} q^2. \tag{2.31}$$

In this case the Hamilton-Jacobi equation is

$$\frac{1}{2DL}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{D}{2C} q^2 + \frac{\partial S}{\partial t} = 0. \tag{2.32}$$

The solution may be taken to be of the form

$$S = \frac{1}{2} q^2 y(t). \tag{2.33}$$

Inserting (2.33) into (2.32) yields

$$\frac{1}{2} q^2 \left[\frac{1}{DL} y^2 + \frac{D}{C} + \frac{dy}{dt}\right] = 0. \tag{2.34}$$

which leads to the first-order nonlinear differential equation for y(t)

$$\frac{dy}{dt} + \frac{1}{DL}y^2 + \frac{D}{C} = 0. \tag{2.35}$$

This equation is recognized to be the Riccati equation,[8] which can be transformed into a second-order, linear differential equation by the change of variables

$$y(t) = DL\frac{\dot{z}}{z}. \tag{2.36}$$

This substitution leads ultimately to the equation

$$\frac{d}{dt}(DL\dot{z}) + \frac{D}{C}z = 0, \tag{2.37}$$

which is the original governing equation (2.13) for zero drive. Hamilton-Jacobi theory appears in this instance to have led in a circle, but other choices for Hamilton's principal function than the form (2.33) might yield results of greater interest.

It is interesting to note that if the capacitance is omitted from the original network (i.e., $\frac{1}{C(t)} \equiv 0$), the resulting Riccati equation

$$\frac{dy}{dt} + \frac{1}{DL}y^2 = 0, \tag{2.38}$$

has the solution

$$y(t) = -\left[\int_{t_o}^{t} \frac{d\xi}{D(\xi)L(\xi)} + a\right]^{-1} \tag{2.40}$$

From Hamilton-Jacobi theory, q may be solved for in terms of time and constants α and β with the relation

$$\beta = \frac{S(q, t, a)}{\partial a} ; \tag{2.41}$$

therefore,

$$q(t) = \sqrt{2\beta} = \left[\int_{t_o}^{t} \frac{d\xi}{D(\xi)L(\xi)} + a \right] . \tag{2.42}$$

Since in this system only the current is of importance, a is of no consequence, and

$$i(t) = \dot{q}(t) = \frac{\sqrt{2\beta}}{D(t)L(t)} \tag{2.43}$$

or, from (2.19),

$$i(t) = \frac{\sqrt{2\beta}}{L(t)} \exp \left(-\int_{t_o}^{t} \frac{R(\xi)}{L(\xi)} d\xi \right) . \tag{2.44}$$

This result could have been easily obtained, of course, by direct integration of the first-order, linear differential equation governing the network, but derived in the above manner it does demonstrate the utility and unity inherent in the energy approach.

Again, analytical mechanics has shown itself to be readily applicable where the answer is also obtainable by straightforward methods, but the hope for the approach is not in the providing of exact solutions so much as to provide insights into those problems where perhaps exact solutions cannot be obtained.

II.5 Phase Plane Analysis.

The phase plane offers an excellent medium for the investigation of qualitative aspects of the behavior of single degree of freedom networks. In particular, one can obtain restrictions on the nature of the system point trajectory in the phase plane (q-p plane) and upper and lower bounds for

the time of a revolution of the system point (point with instantaneous coordinates q(t) and p(t)) in the phase plane.

For a conservative system, the system point traces a closed trajectory in the phase plane; however, for the general time-varying network, the more exotic behavior pictured in Fig. 2.4 might be expected.

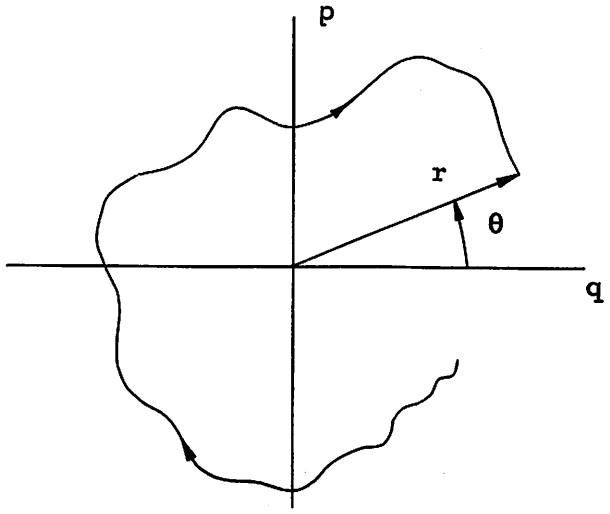

Fig. 2.4. Polar Coordinates and the System Point Trajectory in the Phase Plane.

For unexcited, single loop networks where both L(t) and C(t) are nonnegative (a condition of local passivity, see Appendix I), the system point must always move about the origin in the clockwise direction. This result relies on an analysis of the system point behavior in terms of phase plane polar coordinates:

$$r = \sqrt{q^2 + p^2} \qquad (2.45a)$$

and

$$\theta = \tan^{-1}\left(\frac{p}{q}\right) . \qquad (2.45b)$$

Time differentiation of (2.45b) gives

$$\dot{\theta} = \frac{\dot{p}q - \dot{q}p}{q^2 + p^2} \quad . \tag{2.46}$$

Hamilton's canonical equations (2.27) for the undriven network are

$$\dot{q} = \frac{1}{DL} p \tag{2.47a}$$

and

$$\dot{p} = -\frac{D}{C} q \tag{2.47b}$$

Substitution of these values for $\dot{q}$ and $\dot{p}$ into (2.46) yields

$$\dot{\theta} = \frac{1}{q^2+p^2} \left[-\frac{D}{C} q^2 - \frac{1}{DL} p^2 \right] . \tag{2.48}$$

Substitution of the inverse of the transformation (2.45),

$$q = r \cos \theta \tag{2.49a}$$

and

$$p = r \sin \theta , \tag{2.49b}$$

into (2.48) gives, finally,

$$\dot{\theta} = -\left[\frac{D}{C} \cos^2 \theta + \frac{1}{DL} \sin^2 \theta \right] . \tag{2.50}$$

Thus, the Hamilton's canonical equations for the undriven system are separable when transformed into polar coordinates. But more interesting is the fact that, since D(t) is positive regardless of the sign of R(t), for nonnegative L(t) and C(t), $\dot{\theta}$ is nonpositive and the system point encircles the origin in the clockwise direction.

Another interesting consequence of (2.50) is that it can be used to find bounds on the time required for a revolution of

the system point in the phase plane. Upon the expansion of $\sin^2\theta$ and $\cos^2\theta$, (2.50) becomes

$$\dot{\theta} = -\frac{1}{2}\left[\left(\frac{D}{C} + \frac{1}{DL}\right) + \left(\frac{D}{C} - \frac{1}{DL}\right)\cos 2\theta\right] . \qquad (2.51)$$

This expression indicates the instantaneous bounds on the value of $-\dot{\theta}$:

$$\frac{1}{2}\left[\left(\frac{D}{C} + \frac{1}{DL}\right) - \left|\frac{D}{C} - \frac{1}{DL}\right|\right] \leq -\dot{\theta} \qquad (2.52a)$$

and

$$\frac{1}{2}\left[\left(\frac{D}{C} + \frac{1}{DL}\right) + \left|\frac{D}{C} - \frac{1}{DL}\right|\right] \geq \dot{\theta} . \qquad (2.52b)$$

One can now define the quantity $T_\ell(t_o)$ such that

$$2\pi = \frac{1}{2}\int_{t_o}^{t_o+T_\ell(t_o)}\left[\left(\frac{D}{C} + \frac{1}{DL}\right) + \left|\frac{D}{C} - \frac{1}{DL}\right|\right]dt \geq -\int_{t_o}^{t_o+T_\ell(t_o)}\dot{\theta}\,dt = \theta(t_o) - \theta(t_o+T_\ell(t_o)) \qquad (2.53a)$$

and the quantity $T_u(t_o)$ such that

$$2\pi = \frac{1}{2}\int_{t_o}^{t_o+T_u(t_o)}\left[\left(\frac{D}{C} + \frac{1}{DL}\right) - \left|\frac{D}{C} - \frac{1}{DL}\right|\right]dt \leq -\int_{t_o}^{t_o+T_u(t_o)}\dot{\theta}\,dt = \theta(t_o) - \theta(t_o+T_u(t_o))$$

$$-\theta\,(t_o+T_u\,(t_o)) \qquad (2.53b)$$

It is clear from these relations that the period of a single revolution of the system point in the phase plane beginning at time t_o is bounded by $T_\ell(t_o)$ and $T_u(t_o)$:

$$T_\ell(t_o) \le \tau(t_o) \le T_u(t_o), \tag{2.54}$$

where $\tau(t_o)$ is the actual period. These bounds can be tightened by redefining $T_\ell(t_o)$ and $T_u(t_o)$:

$$2\pi = \frac{1}{2}\int_{t_o}^{t_o+T_\ell(t_o)} \left[\left(\frac{K_1 D}{C} + \frac{1}{K_1 DL}\right) + \left|\frac{K_1 D}{C} - \frac{1}{K_1 DL}\right|\right] dt \tag{2.55a}$$

and

$$2\pi = \int_{t_o}^{t_o+T_u(t_o)} \left[\left(\frac{K_2 D}{C} + \frac{1}{K_2 DL}\right) - \left|\frac{K_2 D}{C} - \frac{1}{K_2 DL}\right|\right] dt \tag{2.55b}$$

The positive constants K_1 and K_2 have been introduced here because D(t) can be taken with any constant multiplier -- recall equations (2.17) and (2.19) and the accompanying discussion; the choice of these denormalized D's affects only the shape of the system point trajectory, not its period of revolution. K_1 and K_2 are to be chosen ultimately so as to minimize the difference between $T_u(t_o)$ and $T_\ell(t_o)$.

It is interesting to note that if

$$\frac{C}{D^2 L} = K^2, \tag{2.56}$$

where K is an arbitrary positive constant, then the time of one revolution from t_o is given exactly by $\tau(t_o)$, where

$$2\pi = \int_{t_o}^{t_o+\tau(t_o)} \frac{dt}{\sqrt{LC}}. \tag{2.57}$$

Condition (2.56) is met in a special class of conservative systems to be discussed in the next section. As an example

of bounding the period, one can consider the stationary series RLC network. There

$$D(t) = e^{\frac{R}{L}(t-t_o)} \tag{2.58}$$

and one can take

$$K_1 = K_2 = \sqrt{\frac{C}{L}} \tag{2.59}$$

(this is not the best choice, but merely a convenient one). Then, from (2.55),

$$T_\ell(t_o) = \frac{L}{R} \log_e (1 + \frac{R}{L} 2\pi\sqrt{LC}) \tag{2.60a}$$

and

$$T_u(t_o) = - \frac{L}{R} \log_e (1 - \frac{R}{L} 2\pi\sqrt{LC}). \tag{2.60b}$$

For this choice of K_2 to be meaningful (2.60b) indicates that

$$1 - \frac{R}{L} 2\pi\sqrt{LC} > 0 \tag{2.61}$$

must hold; otherwise, a better K_2 must be sought or no finite upper bound can be obtained.

II.6 State Space and the Instantaneous Energy Ellipse.

More interesting than the foregoing phase plane analysis is the concept of the instantaneous constant energy surface in the phase plane, which arises in the state space description of the system. In the discussion of the system behavior in state space, ideas drawn from the usual energy function analysis of conservative systems are exploited.

The state space [19] is a three dimensional space (for a single degree of freedom network) consisting of q-, p-, and t-axes; thus, it is the phase plane with an orthogonal time axis appended. For the time-invariant conservative system, the equation

$$\mathcal{H}(q, p) = E_o \tag{2.62}$$

(where E_o is usually the constant stored energy) defines an elliptic cylinder, the surface of which represents the possible area of system point behavior (Fig. 2.5).

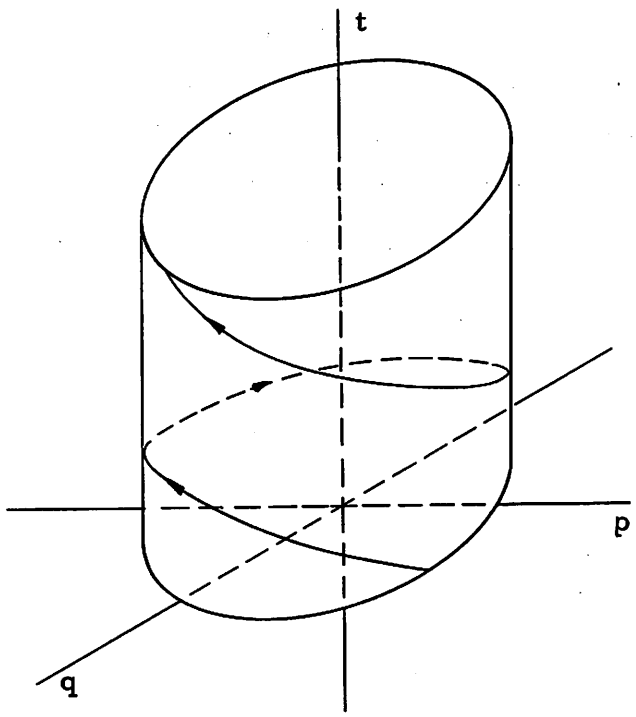

Fig. 2.5. Surface of Possible System Point Behavior in State Space.

For the dissipative, time-varying network, one may define an equivalent cylinder by considering the equation

$$\mathcal{H}(q, p, t) = E_o(t). \tag{2.63}$$

At any time t, the locus of all points (q-p) which satisfy (2.63) will give an ellipse parallel to the q-p plane in state space. As time evolves this variable ellipse will describe a variable cross-section cylinder similar to that in Fig. 2.6; and, as time evolves the system point traces a line of this cylinder. One may think of the surfaces at constant t as orbits of system point behavior (equi-energy) which would occur if explicit time were somehow miraculously stopped and the system were allowed to behave as a conservative system. It is after all only in the evolution of time that energy can be expended; these ellipses could be obtained if time were stopped and the stored energy were allowed to evolve through all of its possible states at that instant.

One criterion for a conservative scleronomous system is

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} = 0. \tag{2.64}$$

For the Hamiltonian of the undriven network,

$$\mathcal{H}(q, p, t) = \frac{1}{2DL}p^2 + \frac{D}{2C}q^2, \tag{2.65}$$

(2.64) yields

$$\frac{1}{2DL}\left(-\frac{\dot{L}}{L} - \frac{R}{L}\right)p^2 + \frac{D}{2C}\left(-\frac{\dot{C}}{C} + \frac{R}{L}\right)q^2 = 0. \tag{2.66}$$

To guarantee this condition would require that

$$-\frac{\dot{L}}{L} - \frac{R}{L} = 0 \tag{2.67a}$$

$$-\frac{\dot{C}}{C} + \frac{R}{L} = 0. \tag{2.67b}$$

This dual requirement is much too stringent, for it is equivalent to requiring that the cylinder of Fig. 2.6 resemble that of Fig. 2.5.

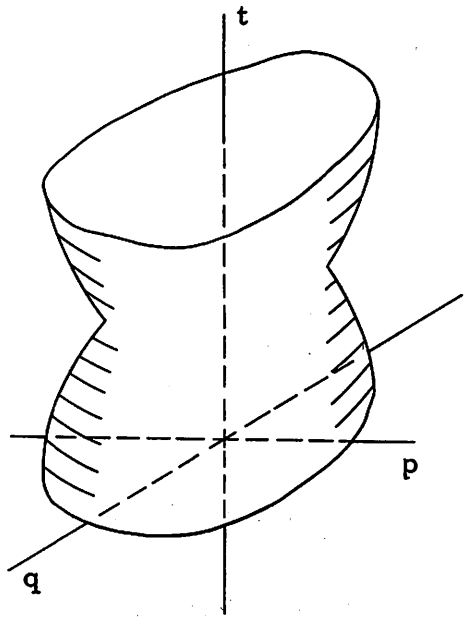

Fig. 2.6. State Space Cylinder for Time-varying Network.

Another condition which is met in the time-invariant conservative system is that the area enclosed by the system point ellipse be the same for any plane t=constant. Equivalently, one can ask that the area enclosed by the instantaneous ellipse defined by

$$\frac{1}{2DL}p^2 + \frac{D}{2C}q^2 = E_o(t) \tag{2.68}$$

be a constant (the ellipse can change orientation). This ellipse has axes

$$a = \sqrt{\frac{2CE_o}{D}} \tag{2.69a}$$

parallel to the q-axis, and

$$b = \sqrt{2DLE_o} \tag{2.69b}$$

parallel to the p-axis; hence, the area of the ellipse is

$$\Gamma(t) = \pi ab = 2\pi\sqrt{LC}\, E_o(t) \tag{2.70}$$

or

$$\Gamma(t) = 2\pi\sqrt{LC}\,\mathcal{H}(q, p, t). \tag{2.71}$$

The time derivative of this area is

$$\dot{\Gamma} = 2\pi\sqrt{LC}\left(\frac{\dot{C}}{C} - \frac{\dot{L}}{L} - \frac{2R}{L}\right)\left(\frac{1}{2DL}p^2 - \frac{D}{2C}q^2\right), \tag{2.72}$$

or, from (2.21)

$$\dot{\Gamma} = 2\pi\sqrt{LC}\left(\frac{\dot{C}}{C} - \frac{\dot{L}}{L} - \frac{2R}{L}\right) \times \mathcal{L}(q, \dot{q}, t). \tag{2.73}$$

Thus, a condition which guarantees that a network is conservative in that its phase plane ellipse is of constant area is

$$\frac{\dot{C}}{C} - \frac{\dot{L}}{L} - \frac{2R}{L} \equiv 0; \tag{2.74}$$

this function will take on much more significance in the next section on stability. Moreover, condition (2.74) is equivalent to condition (2.56), for which the period of revolution of the system point in the phase plane is exactly obtainable.

A network which satisfies condition (2.74) has the general solution

$$q(t) = K_1 \sin\left(\int_{t_o}^{t} \frac{d\xi}{\sqrt{L(\xi)C(\xi)}} + K_2\right), \tag{2.75}$$

a function which is reminiscent of that obtained for the lossless, time-invariant network. In slightly damped systems

this might prove to be a good first-order approximation to the solution. The period of this function agrees with that found in (2.57).

The phase plane and state space analyses have yielded many qualitative aspects of the behavior of the general single degree of freedom, linear, time-varying network. They allow the carry over of much of the intuition associated with conservative systems, but they are far from providing the exact answers obtainable for conservative systems.

II. 7 Analysis of Driven, Time-Varying Networks by the Modified Hamilton's Principle

A common method of analysis for a parametric amplifier circuit containing a single time-varying element is to expand the response into a Fourier series and retain only certain desired terms. [3, 9] The method is justified by assuming, for example, a circuit configuration which by means of band-pass filters allows only the signal frequency ω_s and the idler frequency ω_i ($\omega_s + \omega_i = 2\pi/\gamma$, where γ is the period of variation of the time-varying element) to circulate (Fig. 2.7)

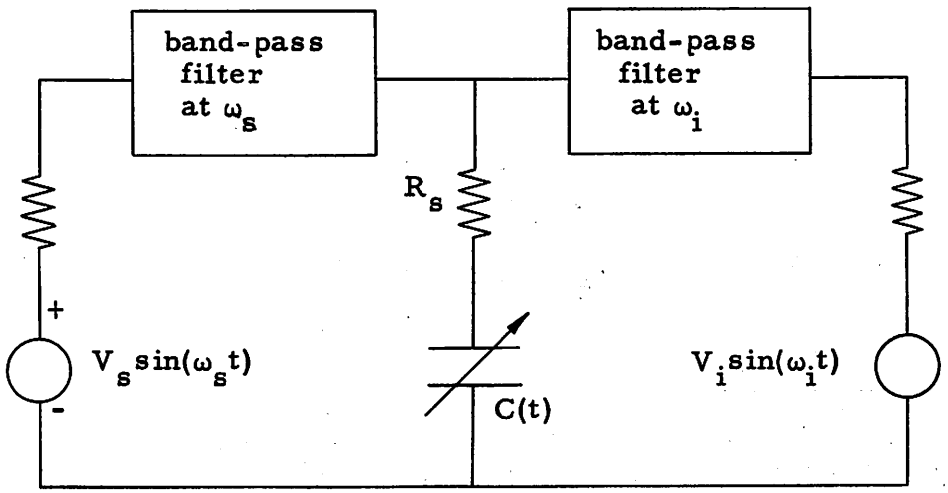

Fig. 2.7. Negative-Resistance-Type Parametric Amplifier.

There can be other assumptions on the model corresponding to other physical assumptions, but this one will suffice for demonstration purposes. The above assumptions are equivalent to the statement that only a flux *

$$\phi(t) = K_1 \sin \omega_s t + K_2 \cos \omega_s t + K_3 \sin \omega_i t + K_4 \cos \omega_i t \quad (2.76)$$

exists across the time-varying capacitor in the equivalent circuit of Fig. 2.8.

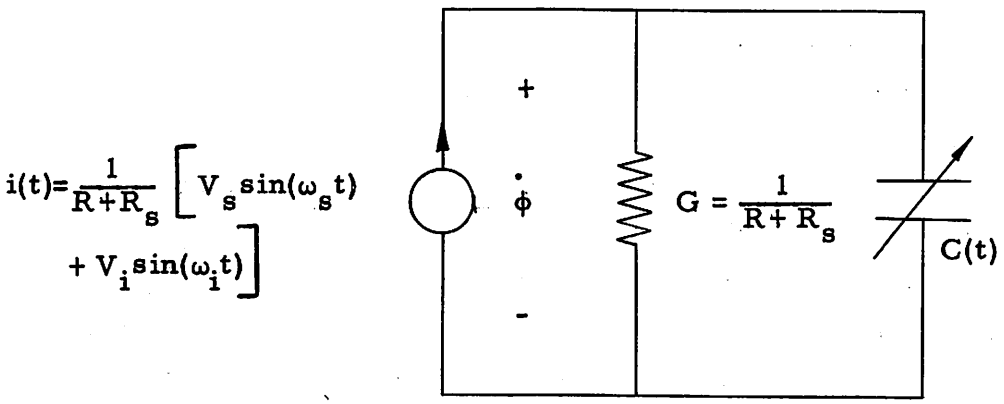

Fig. 2.8. Single Degree of Freedom Equivalent Circuit for Negative-Resistance Parametric Amplifier.

The Lagrangian for this situation is

$$\mathcal{L}(\phi, \dot{\phi}, t) = \exp\left(\int_{t_o}^{t} \frac{G}{C(\xi)}\, d\xi\right) \left\{\frac{1}{2} C(t) \dot{\phi}^2 - i(t)\phi\right\}, \quad (2.77)$$

*On the node basis, dual to that presented previously, flux, which is the time integral of voltage, must serve as the fundamental coordinate.

where

$$G = \frac{1}{R + R_s} \tag{2.78}$$

and

$$i(t) = \frac{1}{R+R_s} \left[V_s \sin \omega_s t + V_i \sin \omega_i t \right] \tag{2.79}$$

Hamilton's principle

$$\delta \int_{t_1}^{t_2} \mathcal{L}(\phi, \dot{\phi}, t) dt = 0. \tag{2.80}$$

may be exploited by the approximation technique of Ritz [16] to obtain approximate values for the steady-state response. To find the steady-state response, one must take the interval

$$t_2 - t_1 = \ell \tau = m \frac{2\pi}{\omega_i} = n \frac{2\pi}{\omega_s} , \tag{2.81}$$

where ℓ, m, and n are all integers. Then the assumption that the natural boundary conditions [16] hold at the end points assures the steady-state. The Ritz method dictates that the variation in (2.80) is equivalent to the four equations

$$\frac{\partial}{\partial K_i} \int_{t_1}^{t_2} \mathcal{L}(\phi, \dot{\phi}, t) dt = 0 \quad (i = 1, \ldots, 4), \tag{2.82}$$

where

$$\mathcal{L}(\phi, \dot{\phi}, t) = \exp\left(\int_{t_o}^{t} \frac{d\xi}{(R+R_s)C(\xi)} \right) \times$$

$$\left\{ \frac{1}{2} C(t) \left[\omega_s K_1 \cos \omega_s t - \omega_i K_2 \sin \omega_i t + \omega_s K_3 \cos \omega_s t - \omega_i K_4 \sin \omega_i t \right]^2 \right.$$

$$\left. - \frac{1}{(R+R_s)} \left[V_s \sin \omega_s t + V_i \sin \omega_i t \right] \right\} \times$$

$$\left[K_1 \sin \omega_s t + K_2 \cos \omega_s t + K_3 \sin \omega_i t + K_4 \cos \omega_i t \right] \Bigg\}. \qquad (2.83)$$

These equations are ideally suited for solution by digital computer to obtain an excellent approximation to the actual solution-- with no approximation as to the nature of C(t). Furthermore, the terms involving C(t) can always be expanded in Fourier series to return to previously given computational methods. The modified Hamilton's principle has shown itself to be a very powerful tool for the quantitative as well as the qualitative analysis of time-varying networks.

II. 8 Summary

In this section a consistent mathematical theory has been developed to place the damped, linear, single degree of freedom, time-varying network in the clothing of classical mechanics. The thorough exploitation of this new approach has led to many new insights into this type of system through analogies with classical conservative systems. Moreover, the stability analysis to be presented in the next section reveals the full utility of the energy function characterization.

III. STABILITY OF LINEAR, TIME-VARYING NETWORKS WITH ONE DEGREE OF FREEDOM

Thus far many analogies between nonconservative and conservative systems have been made by means of the modified Hamiltonian analysis. The avowed goal of this analysis was to include in the energy formulation all forms of energy transfer, including especially that arising from resistances. An added advantage, though, has been obtaining the network differential equations in canonical form. The strange integrating factor D(t) has brought a complete solution no closer, but has yielded a Hamiltonian (Eq. 2.26) capable of rendering a great deal of insight into the qualitative behavior of the network.

The term stability as employed here will mean ground-state stability--where, if an unexcited network is given an arbitrary initial stored energy configuration, there is a net decrease in this energy over a given time interval. Other conditions should certainly enter into stability considerations of time-varying networks; in view of assumptions (A1. 6) in Appendix I, one would also consider the maximum values of the coordinates* and momenta. Upper and lower bounding functions on the stored energy will be obtained which lead directly to sufficient conditions for stability or instability in terms of element value variations only.

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* The coordinates on the loop basis are charges, and on the node basis the coordinates are fluxes.

III. 1 Energy-Bounding Functions

The Hamiltonian (2.26) need not only be considered as the generator of the set of canonical equations of the network (2.27) since it possesses several other properties of great utility. It is the canonical nature of the Hamiltonian and the significance of its explicit time-dependence which set it apart as a tool of network analysis. The total time derivative of the Hamiltonian is also its partial time derivative,[13]

$$\frac{d}{dt}\mathcal{H}(q, p, t) = \frac{\partial}{\partial t}\mathcal{H}(q, p, t), \tag{3.1}$$

and it is this property which renders it so useful in stability analyses.

The Hamiltonian,

$$\mathcal{H}(q, p, t) = \frac{1}{2D(t)L(t)} p^2 + \frac{1}{2C(t)} q^2, \tag{3.2}$$

of the undriven, single loop network of Fig. 3.1 satisfies the relation (3.1).

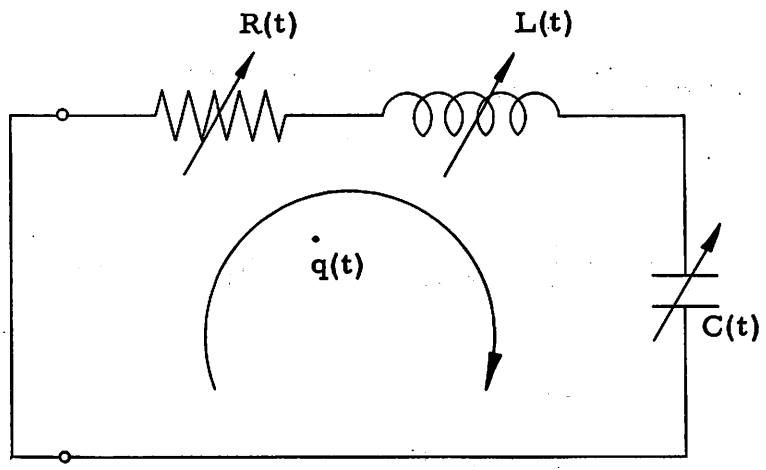

Fig. 3.1. Undriven, Single-Loop, Linear, Time-varying Network.

Hence, one can introduce an arbitrary nonzero time function f(t) so that

$$\frac{d}{dt}\left[f\mathcal{H}\right] = \frac{df}{dt}\mathcal{H} + f\frac{\partial \mathcal{H}}{\partial t}. \tag{3.3}$$

Clearly, since $\mathcal{H}$ is a quadratic form in q and p, $\frac{\partial \mathcal{H}}{\partial t}$ must also be a quadratic form in q and p; this is the bonus of the Hamiltonian formulation which yields the bounding functions. Integration of (3.3) from some arbitrary initial time t_o yields

$$\left[f\mathcal{H}\right]_{t_o}^{t} = \int_{t_o}^{t} \left(\frac{df}{dt}\mathcal{H} + f\frac{\partial \mathcal{H}}{\partial t}\right) dt, \tag{3.4}$$

or

$$f(t)\mathcal{H}(t) = f(t_o)\mathcal{H}(t_o) + \int_{t_o}^{t} \left(\frac{df}{dt}\mathcal{H} + f\frac{\partial \mathcal{H}}{\partial t}\right) dt. \tag{3.5}$$

From (3.2) and (2.17),

$$\frac{\partial \mathcal{H}}{\partial t} = \left(-\frac{\dot{L}}{L} - \frac{R}{L}\right)\frac{1}{2DL}p^2 + \left(-\frac{\dot{C}}{C} + \frac{R}{L}\right)\frac{D}{2C}q^2, \tag{3.6}$$

and

$$\frac{df}{dt}\mathcal{H} + f\frac{\partial \mathcal{H}}{\partial t} = \left[\dot{f} - f\left(\frac{\dot{L}}{L} + \frac{R}{L}\right)\right]\frac{1}{2DL}p^2 + \left[\dot{f} - f\left(\frac{\dot{C}}{C} - \frac{R}{L}\right)\right]\frac{D}{2C}q_o^2 \tag{3.7}$$

Consequently, (3.5) becomes

$$f(t)\mathcal{H}(t) = f(t_o)\mathcal{H}(t_o) + \int_{t_o}^{t} \frac{p^2}{2DL}\left[\dot{f} - f\left(\frac{\dot{L}}{L} + \frac{R}{L}\right)\right] dt + \int_{t_o}^{t} \frac{D}{2C}q^2\left[\dot{f} - f\left(\frac{\dot{C}}{C} - \frac{R}{L}\right)\right] dt. \tag{3.8}$$

For nonnegative L(t) and C(t), it is obvious that

$$\dot{f} - f\left(\frac{\dot{L}}{L} + \frac{R}{L}\right) \leq 0 \tag{3.9a}$$

and

$$\dot{f} - f\left(\frac{\dot{C}}{C} - \frac{R}{L}\right) \leq 0 \tag{3.9b}$$

insure that

$$f(t)\mathcal{H}(t) \leq f(t_o)\mathcal{H}(t_o). \tag{3.10}$$

An f(t) which satisfies (3.9) can be multiplied by an arbitrary constant, so f(t) can be assumed positive; then f(t) can be divided through without altering (3.9) and (3.10):

$$\frac{\dot{f}}{f} - \left(\frac{\dot{L}}{L} + \frac{R}{L}\right) \leq 0; \tag{3.11a}$$

$$\frac{\dot{f}}{f} - \left(\frac{\dot{C}}{C} - \frac{R}{L}\right) \leq 0, \tag{3.11b}$$

and

$$\mathcal{H}(t) \leq f(t_o)\,\mathcal{H}(t_o)\,f^{-1}(t). \tag{3.12}$$

For this upper bound on $\mathcal{H}(t)$, it is desirable to obtain the least possible $f^{-1}(t)$; consequently, the maximum possible f(t) is to be sought. Clearly, the best that can be done is to assume the equality sign in either of the two inequalities (3.11). Suppose, for example, that one had for (3.11a) the actual differential equation

$$\frac{\dot{f}}{f} - \left(\frac{\dot{L}}{L} + \frac{R}{L}\right) = -\epsilon(t), \tag{3.13}$$

where $\epsilon(t)$ is a nonnegative function. The solution of (3.13) is

$$f(t) = KL(t)\exp\left(\int_{t_o}^{t} \frac{R(\xi)}{L(\xi)}d\xi\right) \exp\left(\int_{t_o}^{t} \epsilon(\xi)d\xi\right); \tag{3.14}$$

hence, for maximum f(t) it is necessary that $\epsilon(t) \equiv 0$.

Taking one or the other of the inequalities (3.11) with the equality sign exhausts all possibilities. If (3.11a) is taken with the equality sign and

$$\frac{\dot{f}}{f} = \frac{\dot{L}}{L} + \frac{R}{L}, \tag{3.15}$$

then (3.11b) requires that

$$-\frac{\dot{C}}{C} + \frac{\dot{L}}{L} + \frac{2R}{L} \leq 0. \tag{3.16}$$

Similarly, if (3.11b) is taken with the equality sign and

$$\frac{\dot{f}}{f} = \frac{\dot{C}}{C} - \frac{R}{L}, \tag{3.17}$$

then (3.11a) requires that

$$-\frac{\dot{C}}{C}+\frac{\dot{L}}{L}+\frac{2R}{L}\geq 0. \tag{3.18}$$

The desired solution of (3.15) is

$$f(t) = K_1 L(t) \exp\left(\int_{t_o}^{t} \frac{R(\xi)}{L(\xi)} d\xi\right), \tag{3.19}$$

which gives

$$\mathcal{Y}(t) \leq \mathcal{Y}(t_o) \frac{L(t_o)}{L(t)} \exp\left(-\int_{t_o}^{t} \frac{R(\xi)}{L(\xi)} d\xi\right) . \tag{3.20}$$

But, from equations (2.23) and (2.26),

$$\mathcal{Y}(t) = \exp\left(\int_{t_o}^{t} \frac{R(\xi)}{L(\xi)} d\xi\right)\left[\frac{1}{2}L(t)\dot{q}^2 + \frac{1}{2C(t)}q^2\right] \tag{3.21}$$

$$= \exp\left(\int_{t_o}^{t} \frac{R(\xi)}{L(\xi)} d\xi\right) E(t), \tag{3.22}$$

where E(t) is the stored energy in the network energy storage elements. Thus,

$$E(t) \leq E(t_o) \frac{L(t_o)}{L(t)} \exp\left(-2\int_{t_o}^{t} \frac{R(\xi)}{L(\xi)} d\xi\right) , \tag{3.23a}$$

when

$$-\frac{\dot{C}}{C}+\frac{\dot{L}}{L}+\frac{2R}{L}\leq 0. \tag{3.23b}$$

The desired solution to (3.17) is

$$f(t) = K_2 C(t) \exp\left(-\int_{t_o}^{t} \frac{R(\xi)}{L(\xi)} d\xi\right) . \tag{3.24}$$

Reasoning similar to that just cited leads to

$$E(t) \leq E(t_o) \frac{C(t_o)}{C(t)}, \tag{3.25a}$$

when

$$-\frac{\dot{C}}{C} + \frac{\dot{L}}{L} + \frac{2R}{L} \geq 0. \tag{3.25b}$$

It is remarkable that the function on the left-hand side of (2.74), which in Section II determined a conservative network, has now cropped up in (3.23b) and (3.25b).

Conditions (3.23) and (3.25) lead directly to the construction of an upper bounding function on the stored energy. This function u(t) is to be constructed such that

$$u(t_o) = E(t_o); \tag{3.26}$$

$$u(t) = u(t_k) \frac{L(t_k)}{L(t)} \exp\left(-2 \int_{t_k}^{t} \frac{R(\xi)}{L(\xi)} d\xi \right) \tag{3.27a}$$

on the interval (t_k, t_{k+1}) where

$$-\frac{\dot{C}}{C} + \frac{\dot{L}}{L} + \frac{2R}{L} \leq 0 \tag{3.27b}$$

and

$$u(t) = u(t_k) \frac{C(t_k)}{C(t)} \tag{3.28a}$$

on the interval (t_k, t_{k+1}) where

$$-\frac{\dot{C}}{C} + \frac{\dot{L}}{L} + \frac{2R}{L} \geq 0. \tag{3.28b}$$

Thus, u(t) at the endpoint of one interval provides its initial value for the subsequent interval; this is necessarily so because only the initial value of stored energy can be assumed, from then on only an upper bound is provided. The relationship

$$E(t) \leq u(t) \tag{3.29}$$

and the construction of u(t) are illustrated in Fig. 3.2.

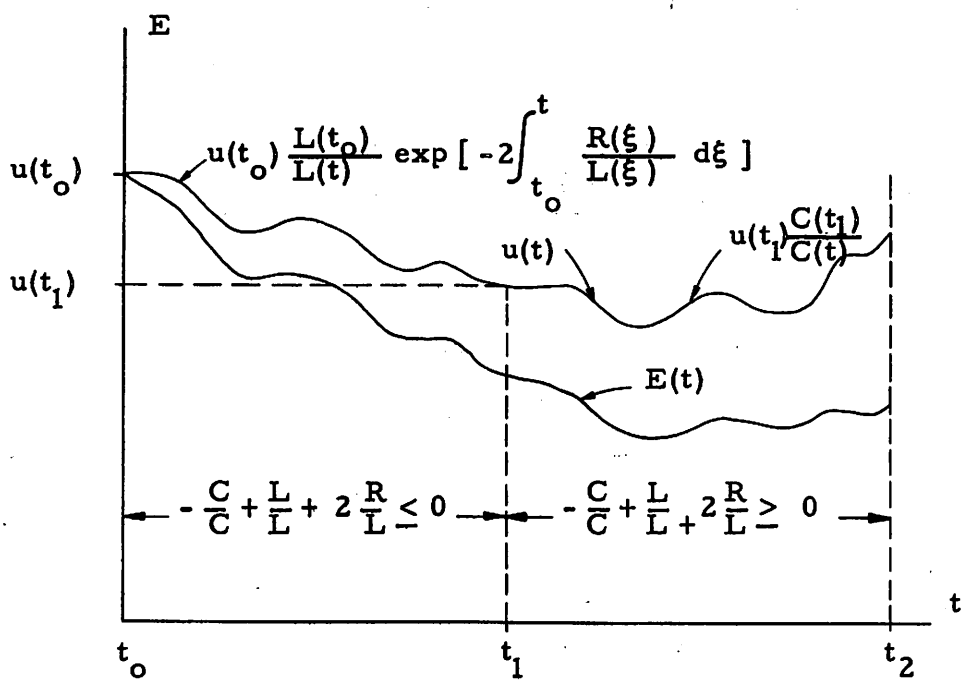

Fig. 3.2. Illustration of upper energy bounding function.

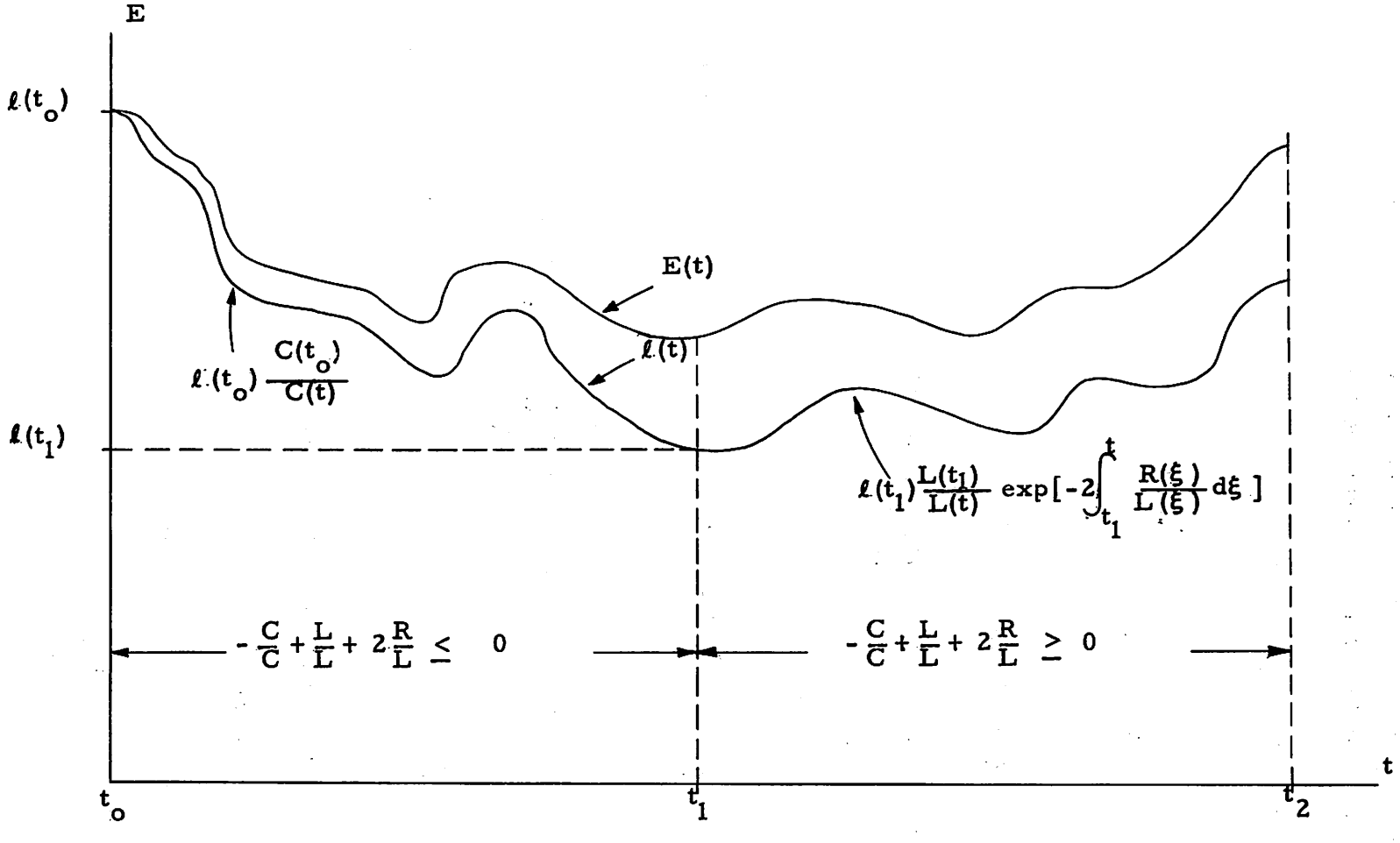

Fig. 3.3. Illustration of lower energy bounding function.

Returning to (3.8), one can evolve a parallel formulation to the above to obtain a lower bounding function. If

$$\dot{f} - f\left(\frac{\dot{L}}{L} + \frac{R}{L}\right) \geq 0 \tag{3.30a}$$

and

$$\dot{f} - f\left(\frac{\dot{C}}{C} - \frac{R}{L}\right) \geq 0, \tag{3.30b}$$

then,

$$f(t)\mathscr{Y}(t) \geq f(t_o)\mathscr{Y}(t_o). \tag{3.31}$$

Following through as above, one obtains

$$E(t) \geq E(t_o)\,\frac{C(t_o)}{C(t)}\,, \tag{3.32a}$$

when

$$-\frac{\dot{C}}{C} + \frac{\dot{L}}{L} + \frac{2R}{L} \leq 0; \tag{3.32b}$$

and

$$E(t) \geq E(t_o)\,\frac{L(t_o)}{L(t)}\,\exp\left(-2\int_{t_o}^{t}\frac{R(\xi)}{L(\xi)}\,d\xi\right) \tag{3.33a}$$

when

$$-\frac{\dot{C}}{C} + \frac{\dot{L}}{L} + \frac{2R}{L} \geq 0. \tag{3.33b}$$

A lower bounding function $\ell(t)$, analogous to u(t), for which

$$E(t) \geq \ell(t), \tag{3.34}$$

can be constructed such that

$$\ell(t_o) = E(t_o); \tag{3.35}$$

$$\ell(t) = \ell(t_k)\frac{C(t_k)}{C(t)} \tag{3.36a}$$

on the interval (t_k, t_{k+1}) where

$$-\frac{\dot{C}}{C} + \frac{\dot{L}}{L} + \frac{2R}{L} \leq 0 \tag{3.36b}$$

and

$$\ell(t) = \ell(t_k)\frac{L(t_k)}{L(t)}\exp\left(-2\int_{t_k}^{t}\frac{R(\xi)}{L(\xi)}d\xi\right) \tag{3.37a}$$

on the interval (t_k, t_{k+1}) where

$$-\frac{\dot{C}}{C} + \frac{\dot{L}}{L} + \frac{2R}{L} \geq 0 \tag{3.37b}$$

The relationship (3.34) and the construction of $\ell(t)$ are illustrated in Fig. 3.3.

Armed with the two energy bounding functions u(t) and $\ell(t)$, one can develop simple stability and instability criteria for periodically variable, single degree of freedom, linear time-varying networks. Suppose that all of the network elements vary in magnitude with the same period T; then,

$$u(t+T) \leq u(t) \tag{3.38}$$

is sufficient for stability,

$$\ell(t+T) < \ell(t) \tag{3.39}$$

is necessary for stability,

$$\ell(t+T) > \ell(t) \tag{3.40}$$

is sufficient for instability, and

$$u(t+T) > u(t) \tag{3.41}$$

is necessary for instability. Although these criteria suffice for most practical networks, there are many situations which they do not cover. Since it is generally the goal to extract power from a network, the resistance level would usually be sufficiently high to overcome any stability difficulties; this point will be more clearly brought out in the examples of SectionIII. 3.

III. 2 Comparison of Stability Criteria with Others obtained by Method of Lyapunov

In treating the same simple example as is shown in Fig. 3.1, Kalman and Bertram [17] arrive via the method of Lyapunov at the following stability criterion:

Assuming

$$\left.\begin{aligned} 0 < \epsilon_1 \leq\ & L(t) \leq \nu_1 < \infty \\ 0 < \epsilon_2 \leq\ & C(t) \leq \nu_2 < \infty \qquad t \geq t_o \\ 0 < \epsilon_3 \leq\ & R(t) \leq \nu_3 < \infty \end{aligned}\right\} \tag{3.42}$$

the system of Fig. 3.1 is asymptotically stable if for all $t \geq t_o$

$$0 < \epsilon_4 \leq 1 + \dot{R}\left(\frac{L}{R^2} - \frac{C}{2}\right) + \frac{\dot{C}L}{RC} - \frac{\dot{L}}{R} \tag{3.43a}$$

and

$$0 < \epsilon_5 \leq 1 + \frac{\dot{R}L}{R^2} \ . \tag{3.43b}$$

Certainly the approach through the Hamiltonian with its restriction to energy functions is no fit challenge to such a powerful tool as Lyapunov's second method. But here, just because of its relation with energy, the stability criterion of SectionIII.1 seems to offer many advantages over other routes

to Lyapunov stability criteria. From (3.25), it is clear that if (3.25b) holds for all $t \geq t_o$, then $C(t)E(t)$ is a Lyapunov function by (3.25a); but for positive $R(t)$, (3.25b) can be divided by $2R/L$ to yield the Lyapunov stability criterion

$$0 < 1 - \frac{1}{2}\left(\frac{\dot{C}L}{RC} - \frac{\dot{L}}{R}\right). \tag{3.44}$$

Here already one sees an advantage in the Hamiltonian formulation in terms of application of (3.44) as compared to applying both of (3.43). Moreover, when $\dot{R} \equiv 0$, (3.44) offers the distinct advantage that its variable term, although of opposite sign, is but half the amplitude of that in (3.43a). Futhermore, if a given network fails the test of (3.44) it is easy to pick out a value of R which brings it into line -- not so with (3.43), where attention must be paid to $\dot{R}$ as well.

Another advantage of the Hamiltonian formulation over taking an arbitrary pair of first order differential equations is that no assumptions need be made regarding the sign of $\dot{R}(t)$; this is particularly useful in connection with investigations of stationary, active networks. Moreover, by a simple modification of the procedure in SectionIII.1, bounding functions can be found even when L(t) and C(t) undergo negative excursions. This is a rare occurrence, it being in violation of the condition of local passivity (see Appendix I), but it is reassuring to know that it can still be handled.

In the Hamiltonian approach the energy and the dissipative effects in the network are brought into prominence. This energy need not decay monotonically, but need only have an average trend toward zero for stability. Furthermore, the growing energy is bounded by a known function so that one may insure that any original assumptions on smallness of q and $\dot{q}$ are not violated.

In their paper,[17] Kalman and Bertram make the statement

> "No procedure ... is available for the determination of the stability of general linear nonstationary systems in an algebraic way. There is little hope that this state of affairs will change soon."

This attitude is certainly borne out upon referring to the two recent Russian articles on the application of Lyapunov stability methods to nonstationary systems by Razumikhin[22] and Roitenberg.[23]

The Hamiltonian approach is not being advocated here to the exclusion of all others, but only as an easier and more convenient starting point.

III. 3 Examples

Example 3.1, A guarantee of stability of Mathieu's equation: Mathieu's equation (normalized),

$$\ddot{q} + (\delta + \epsilon \cos t)q = 0, \tag{3.45}$$

is one for which the bounding functions u(t) and $\ell(t)$ say nothing. It is the governing equation of the network of Fig. 3.1 when

$$L(t) = 1 \tag{3.46a}$$

$$C(t) = \frac{1}{\delta + \epsilon \cos t} \tag{3.46b}$$

and

$$R(t) = 0. \tag{3.46c}$$

The determining function is

$$-\frac{\dot{C}}{C} + \frac{\dot{L}}{L} + \frac{2R}{L} = -\frac{\epsilon \sin t}{\delta + \epsilon \cos t}; \tag{3.47}$$

hence, for the first cycle the bounding functions are

$$u(t) = \begin{cases} E(0), & 0 \leq t \leq \pi \\ E(0)\dfrac{\delta + \epsilon \cos t}{\delta - \epsilon}, & \pi \leq t \leq 2\pi \end{cases} \tag{3.48}$$

and

$$\ell(t) = \begin{cases} E(0)\dfrac{\delta + \epsilon \cos t}{\delta + \epsilon}, & 0 \leq t \leq \pi \\ E(0)\dfrac{\delta - \epsilon}{\delta + \epsilon}, & \pi \leq t \leq 2\pi \end{cases} \tag{3.49}$$

The shaded area in Fig. 3.4 is the possible region of behavior of the stored energy over a cycle. Thus, except in the wholly solvable case where $\epsilon = 0$, nothing definite can be said about the stability of this network by taking the above approach.

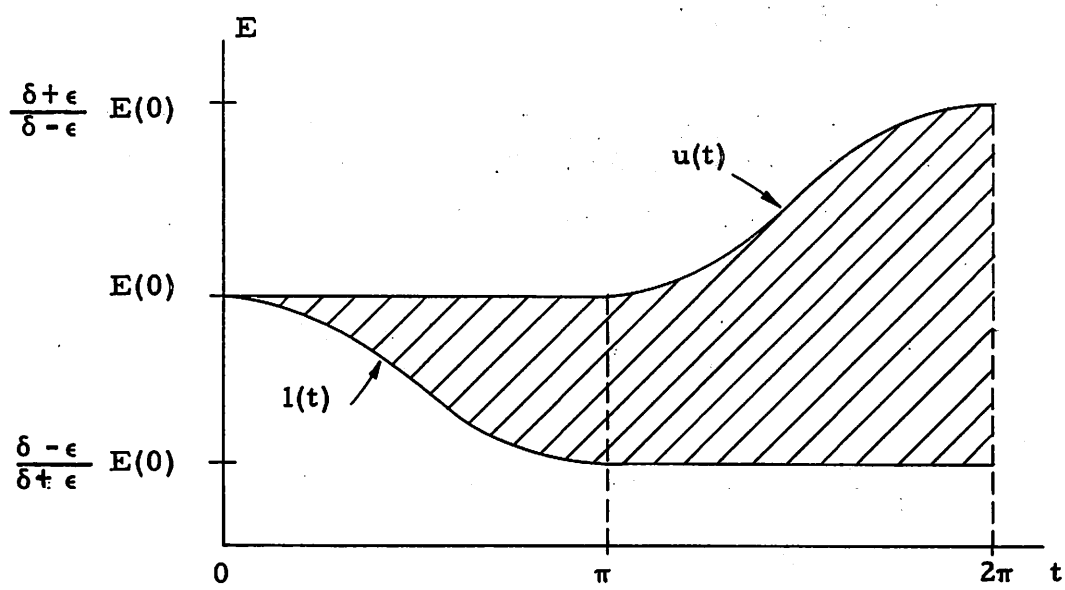

Fig. 3.4. Area of possible energy behavior over one cycle for Mathieu's equation.

There is some fixed resistance R which can be placed in series with the above network to guarantee stability by the above analysis. This R can be found as a function of δ and ϵ by ascertaining the minimum R which renders

$$u(2\pi) \leq u(0). \tag{3.50}$$

The determining function becomes

$$-\frac{\dot{C}}{C} + \frac{\dot{L}}{L} + \frac{2R}{L} = 2R = \frac{\epsilon \sin t}{\delta + \epsilon \cos t}; \tag{3.51}$$

this function is maximum at

$$\cos t = -\epsilon / \delta, \tag{3.52a}$$

$$\sin t = \frac{\sqrt{\delta^2 - \epsilon^2}}{\delta} \tag{3.52b}$$

Consequently, stability is insured if

$$R > \frac{\epsilon}{2\sqrt{\delta^2 - \epsilon^2}} . \tag{3.53}$$

Example 3.2, the G-C Parametric Amplifier : If one chooses the dual of the circuit of Fig. 3.1 (see Fig. 3.5), the upper energy bounding conditions similar to (3.23) and (3.25) become

$$E(t) \leq E(t_o) \frac{C(t_o)}{C(t)} \exp\left(-2\int_{t_o}^{t} \frac{G(\xi)}{C(\xi)} d\xi \right) , \tag{3.54a}$$

when

$$-\frac{\dot{L}}{L} + \frac{\dot{C}}{C} + \frac{2G}{C} \leq 0; \tag{3.54b}$$

and

$$E(t) \leq E(t_o) \frac{L(t_o)}{L(t)} , \tag{3.55a}$$

when

$$-\frac{\dot{L}}{L}+\frac{\dot{C}}{C}+\frac{2G}{C}\geq 0\,. \tag{3.55b}$$

In considering the parallel G-C network of Fig. 3.6, one obtains from (3.55) the stability condition

$$2G - \omega_1 C_1 > 0. \tag{3.56}$$

The exact solution to the homogeneous equation

$$\frac{d}{dt}\left[(C_o + C_1 \sin \omega_1 t)\, v\right] + Gv = 0 \tag{3.57}$$

is

$$v(t) = \frac{K}{C_o+C_1 \sin\omega_1 t} \exp\left(-\int_{t_o}^{t} \frac{G d\xi}{C_o+C_1 \sin\omega_1 \xi} \right), \tag{3.58}$$

which shows the network to be stable so long as C_o is greater than C_1 and G is positive. However, under the condition of optimum power matching (see Appendix III), the matched driven network of Fig. 3.7 dissipates

$$P_{av} \doteq \frac{I_o^2}{4\sqrt{(2G)^2-(\omega_1 C_1)^2}} \tag{3.59}$$

in the load admittance Y_L. Thus, (3.56) provides a practical limit for networks which are to be employed in amplifiers beyond which the possibility of instability arises. Actually, the stability of the network of Fig. 3.6 can be demonstrated even on the energy basis. One need only consider the same network only with an arbitrarily small time-invariant inductance introduced in series with G and C in the single loop (physically, this would usually be the case). Then the network takes the form of Fig. 3.1 and the condition

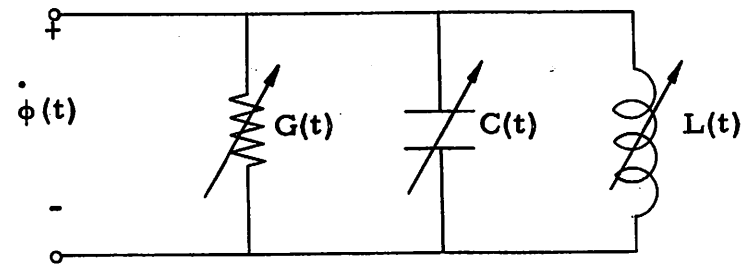

Fig. 3.5. Dual of Circuit of Fig. 3.1.

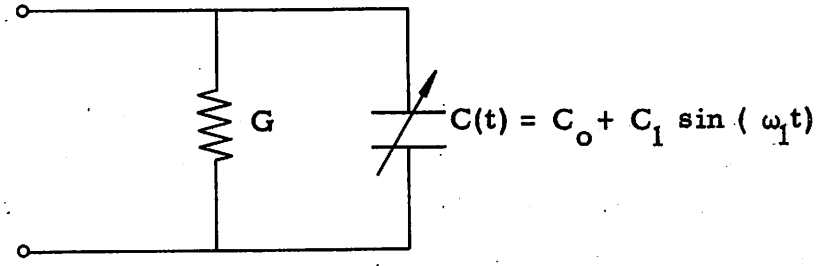

Fig. 3.6. Time-varying G-C Network.

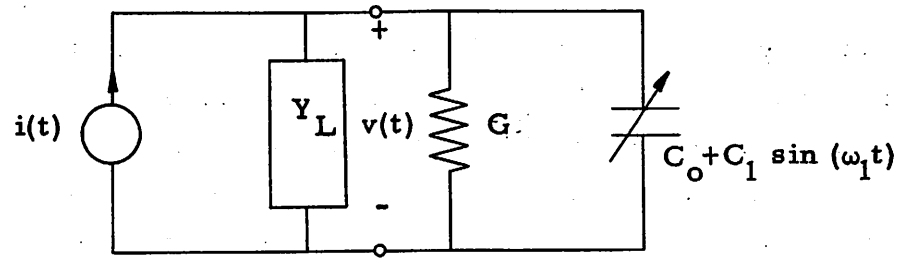

Fig. 3.7. Matched, Driven, Time-varying G-C Network.

$$-\frac{\dot{C}}{C} + \frac{L}{L} + \frac{2}{GL} \geq 0 \tag{3.60}$$

guarantees stability; in this instance L need only be chosen sufficiently small that

$$\frac{2}{GL} - \frac{\omega_1 C_1}{\sqrt{C_o^2 - C_1^2}} > 0 \tag{3.61}$$

to achieve the result dictated by (3.58).

Example 3.3, Two conditions which guarantee the stability of positive RLC networks: If R(t), L(t) and C(t) are positive and periodic in the network of Fig. 3.1, and if either condition (3.27b) or (3.28b) holds always, then the energy decreases on the average. A condition which guarantees (3.27b) for positive R(t) is

$$\frac{d}{dt}\left(\frac{C}{L}\right) > 0, \tag{3.62}$$

and (3.28b) is guaranteed by

$$\frac{d}{dt}\left(\frac{C}{L}\right) - \frac{2RC}{L^2} < 0. \tag{3.63}$$

These two conditions, which are sufficient for stability, were previously obtained by Gadsden.[12] They are, however, but special applications of the stability criteria obtained in Section III.1

IV. ENERGY FUNCTION ANALYSIS OF LINEAR, TIME-VARYING NETWORKS WITH n DEGREES OF FREEDOM

The key to the development of the energy functions for the single degree of freedom network has been in the revision of the governing equation (2.10) so that it ultimately appeared in canonical form. For the n degree of freedom network, the original set of governing equations appears naturally as a matrix (n x n) differential equation of second order. Often, the revision of this equation entails as much work as its actual solution; however, in many interesting and useful cases it does not.

IV. 1 Derivation of the Variational Principle

The development to be presented here closely parallels that of Section II, except that a matrix treatment is employed throughout. One can obtain conditions for a modified Lagrangian and Hamiltonian, and even formulate a state space picture of the system behavior. Most of the networks for which energy functions are simply obtained possess a high degree of symmetry; fortunately, it is just such networks which are of current interest in connection with time-varying devices.

Upon choosing any convenient network tree [14] to define the dependent variables (link currents) $\dot{q}_j$, * one can write the governing loop-basis equations of the network:

*Although an arbitrary tree will suffice, it will become clear later in the development that the best tree, when available, is one with all of the tree branches consisting of capacitances and all of the links containing inductances.

$$\frac{d}{dt}\left\{\sum_{j=1}^{n} L_{ij}(t)\,\dot{q}_j\right\}+\left\{\sum_{j=1}^{n} R_{ij}(t)\,\dot{q}_j\right\}+\left\{\sum_{j=1}^{n} S_{ij}(t)\,q_j\right\} \qquad (4.1)$$

$$= e_i(t);\ i = 1, \ldots, n.$$

This set of equations can be replaced by the single matrix differential equation

$$\frac{d}{dt}\left[\underline{L}(t)\dot{\underline{q}}\right] + \underline{R}(t)\dot{\underline{q}} + \underline{S}(t)\underline{q} = \underline{e}(t); \qquad (4.2)$$

where

$$\underline{L}(t) \equiv \begin{pmatrix} L_{11}(t) & L_{12}(t) & \ldots & L_{1n}(t) \\ L_{21}(t) & L_{22}(t) & \ldots & L_{2n}(t) \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ L_{n1}(t) & L_{n2}(t) & \ldots & L_{nn}(t) \end{pmatrix} \qquad (4.3a)$$

$$\underline{R}(t) \equiv \begin{pmatrix} R_{11}(t) & R_{12}(t) & \ldots & R_{1n}(t) \\ R_{21}(t) & R_{22}(t) & \ldots & R_{2n}(t) \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ R_{n1}(t) & R_{n2}(t) & \ldots & R_{nn}(t) \end{pmatrix} \qquad (4.3b)$$

$$\underline{S}(t) \equiv \begin{pmatrix} S_{11}(t) & S_{12}(t) & \ldots & S_{1n}(t) \\ S_{21}(t) & S_{22}(t) & \ldots & S_{2n}(t) \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ S_{n1}(t) & S_{n2}(t) & \ldots & S_{nn}(t) \end{pmatrix} \qquad (4.3c)$$

$$\underline{q} \equiv \begin{pmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{pmatrix} \tag{4.4a}$$

and

$$\underline{e}(t) \equiv \begin{pmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_n(t) \end{pmatrix} \tag{4.4b}$$

To modify equation (4.2) in a manner similar to that in which (2.10) was revised to the form (2.13), one can premultiply it by an arbitrary (for now)* nonzero n x n matrix D(t):

$$\underline{D}(t)\left\{\frac{d}{dt}\left[\underline{L}(t)\dot{\underline{q}}\right] + \underline{R}(t)\dot{\underline{q}} + \underline{S}(t)\underline{q} - \underline{e}(t)\right\} = 0 \tag{4.5}$$

The variational principle which, via the Euler-Lagrange equations, leads back to the set (4.5) may be obtained by premultiplying (4.5) by the row vector*

$$\delta\tilde{\underline{q}} = (\delta q_1, \delta q_2, \ldots, \delta q_n), \tag{4.6}$$

where

$$\delta\underline{q}(t_1) = \delta\underline{q}(t_2) = 0, \tag{4.7}$$

and integrating between fixed limits t_1 and t_2

$$\int_{t_1}^{t_2} \delta\tilde{\underline{q}}\underline{D}(t)\left\{\frac{d}{dt}\left[\underline{L}(t)\dot{\underline{q}}\right] + \underline{R}(t)\dot{\underline{q}} + \underline{S}(t)\underline{q} - \underline{e}(t)\right\} dt = 0. \tag{4.8}$$

Upon integration of the first term of the integrand by parts, one obtains

---

*$\tilde{\underline{X}}$ indicates the transpose of the matrix $\underline{X}$.

$$\delta \tilde{\underline{q}}\, \underline{D}\,(t)\, \underline{L}\,(t)\, \dot{\underline{q}} \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \Big\{ - \delta \tilde{\dot{\underline{q}}}\, \underline{D}(t) \underline{L}(t) \underline{q} - \delta \tilde{\underline{q}}\, \dot{\underline{D}}\,(t)\, \underline{L}\,(t)\, \dot{\underline{q}}$$

$$+ \delta \tilde{\underline{q}}\, \underline{D}\,(t)\, \underline{R}\,(t)\, \dot{\underline{q}} + \delta \tilde{\underline{q}}\, \underline{D}\,(t)\, \underline{S}\,(t)\, \underline{q} - \delta \tilde{\underline{q}}\, \underline{D}\,(t)\, \underline{e}\,(t) \Big\}\, dt = 0. \quad (4.9)$$

The first term on the left vanishes by virtue of (4.7); and, if

$$- \dot{\underline{D}}\,(t)\, \underline{L}\,(t) + \underline{D}\,(t)\, \underline{R}\,(t) = 0, \quad (4.10)$$

then (4.9) becomes

$$\int_{t_1}^{t_2} \Big\{ - \delta \tilde{\dot{\underline{q}}}\, \underline{D}\,(t)\, \underline{L}(t)\, \dot{\underline{q}} + \delta \tilde{\underline{q}}\, \underline{D}(t)\, \underline{S}(t)\, \underline{q} - \delta \tilde{\underline{q}}\, \underline{D}\,(t)\, e\,(t) \Big\}\, dt = 0 \quad (4.11)$$

Moreover, under the symmetry conditions

$$\underline{D}\,(t)\, \underline{L}\,(t) = \tilde{\underline{L}}\,(t)\, \tilde{\underline{D}}\,(t) \quad (4.12a)$$

and

$$\underline{D}\,(t)\, \underline{S}\,(t) = \tilde{\underline{S}}\,(t)\, \tilde{\underline{D}}\,(t) \quad (4.12b)$$

(4.11) is equivalent to

$$\delta \int_{t_1}^{t_2} \Big\{ - \frac{1}{2} \tilde{\dot{\underline{q}}}\, \underline{D}\,(t) \underline{L}\,(t) \dot{\underline{q}} + \frac{1}{2} \tilde{\underline{q}}\, \underline{D}\,(t)\, \underline{S}\,(t)\, q - \tilde{\underline{q}}\, \underline{D}\,(t)\, \underline{e}\,(t) \Big\}\, dt = 0. \quad (4.13)$$

Equation (4.10) possesses a solution so long as $\underline{L}\,(t)$ is nonsingular, but it is unlikely that conditions (4.12) are met in general. A more general derivation which places fewer restrictions on the matrix $\underline{D}\,(t)$ will be found in Appendix IV. Fortunately, time-varying networks which give rise to matrices $\underline{D}\,(t)$ satisfying (4.12) are attended with a great deal of interest; examples of such networks are presented in the sequel.

If (4.13) is to be identified as the modified form of Hamilton's principle, the modified Lagrangian can be taken to be the negative of the integrand in (4.13):

$$\mathcal{L}(\underline{q}, \dot{\underline{q}}, t) = \frac{1}{2} \tilde{\dot{\underline{q}}}\, \underline{D}(t)\, \underline{L}\,(t)\, \dot{\underline{q}} - \frac{1}{2} \tilde{\underline{q}}\, \underline{D}\,(t) \underline{S}(t) \underline{q} + \tilde{\underline{q}} \underline{D}(t) \underline{e}(t). \tag{4.14}$$

If $\underline{D}(t)$ is taken so that $\underline{D}(t_o) = \underline{I}$ (the identity matrix) and $\underline{D}(t) \equiv \underline{I}$ for $\underline{R}\,(t) \equiv \underline{0}$ (the null matrix), then, as before, the modified Lagrangian reduces to the usual Lagrangian in the absence of resistances. Since it serves to introduce the dissipatances into the energy function formulation, it is reasonable that $\underline{D}\,(t)$ be called the "dissipation matrix." The Euler-Lagrange equations,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \quad i = 1, 2, 3, \ldots, n, \tag{4.15}$$

associated with the modified Lagrangian (4.14) are equivalent to the modified governing equations (4.5) of the network.

Yet another route to the Lagrangian exists, and, what is more, it is never beset with the difficulties encountered above. The method is equivalent to using Lagrange multipliers to account for the network constraints (Kirchoff's current or voltage laws), only now in connection with the inverse problem. To insure the success of the method, one need only assume that every resistive branch also contains some series inductance.* For an arbitrary choice of tree for an m branch network with n independent loop equations, one obtains the (m-n)xn tie-set matrix $\underline{\beta}$. [14] If the current in the j^{th}

*It may be necessary to include a small inductance, say L, in series with each resistance; although aesthetically unpleasing this is generally the case physically.

branch of the network is denoted by $\dot{q}_j$, the tie-set matrix represents the (m-n) constraint relations

$$\underline{\beta}\ \dot{\underline{q}} = 0\ ; \tag{4.16}$$

where $\dot{\underline{q}}$ is an m-vector (column matrix) consisting of all branch currents. Moreover, the tie-set matrix can give the relation between the network branch voltages v_j and the tree branch voltages λ_j (these are the actual Lagrange multipliers),

$$\underline{v} = \tilde{\underline{\beta}}\ \underline{\lambda}\ ; \tag{4.17}$$

where $\underline{v}$ is the m-vector of branch voltages and $\underline{\lambda}$ is the (m-n) - vector of tree-branch voltages. Thus, the set of governing equations for the network can be written as the m x m matrix differential equation

$$\frac{d}{dt}\left[\underline{L}(t)\dot{\underline{q}}\right] + \underline{R}(t)\dot{\underline{q}} + \underline{S}(t)\underline{q} - \underline{e}(t) - \tilde{\underline{\beta}}\underline{\lambda} = 0; \tag{4.18}$$

where $\underline{L}(t)$, $\underline{R}(t)$, and $\underline{S}(t)$ are diagonal matrices with each entry representing an individual branch. The desired matrix $\underline{D}(t)$ to take (4.18) into variational form is the m x m diagonal matrix

$$\underline{D}(t) = \exp\left\{ \operatorname{diag}\left[\int_{t_o}^{t} \left(\frac{R_1(\xi)}{L_1(\xi)}, \ldots, \frac{R_m(\xi)}{L_m(\xi)}\right) d\xi\right]\right\}; \tag{4.19}$$

hence,

$$\underline{D}(t_o) = \underline{I}\,. \tag{4.20}$$

Then, the modified Hamilton's principle becomes

$$\delta\int_{t_1}^{t_2} \left\{ -\frac{1}{2}\tilde{\dot{\underline{q}}}\,\underline{D}(t)\,\underline{L}(t)\,\dot{\underline{q}} + \frac{1}{2}\tilde{\underline{q}}\,\underline{D}(t)\,\underline{S}(t)\,\underline{q} - \tilde{\underline{q}}\,\underline{D}(t)\underline{e}(t) - \tilde{\underline{q}}\,\underline{D}(t)\,\tilde{\underline{\beta}}\,\underline{\lambda}\right\} dt = 0, \tag{4.21a}$$

with the auxiliary constraining relations

$$\underline{\beta}\ \dot{\underline{q}} = 0. \tag{4.21b}$$

The m Euler-Lagrange equations for (4.21a) plus the (m-n) constraining relations (4.21b) provide 2m-n equations for the 2m-n unknowns -- m q_j's and (m-n) tree-branch voltages, λ_j's. Equations (4.21) are a perfectly valid formulation of a modified Hamilton's principle; unfortunately, the presence of the λ_j's renders this formulation useless for stability studies. Nonetheless, one can identify the Lagrangian as the negative of the integrand in (4.21a); consequently,

$$\begin{aligned}\mathcal{L}(\underline{q}, \dot{\underline{q}}, \underline{\lambda}, t) = &\tfrac{1}{2}\tilde{\dot{\underline{q}}}\ \underline{D}(t)\ \underline{L}(t)\ \dot{\underline{q}} - \tfrac{1}{2}\tilde{\underline{q}}\ \underline{D}(t)\ \underline{S}(t)\ \underline{q} \\ &+ \tilde{\underline{q}}\ \underline{D}(t)\ \underline{e}(t) + \tilde{\underline{q}}\ \underline{D}(t)\ \tilde{\underline{\beta}}\ \underline{\lambda}.\end{aligned} \tag{4.22}$$

Hence, the formalism of Hamilton's principle always carries over; however, its utility is sometimes lessened.

IV. 2 The Hamiltonian and the Canonical Equations

Either of the Lagrangians, (4.14) or (4.22), is amenable to the straight-forward manipulation techniques which carry onward toward the Hamiltonian. First one must define the momentum conjugate to the coordinate q_i:

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}. \tag{4.23}$$

The Hamiltonian is then defined by the Legendre transformation [13]

$$\mathcal{H}(\underline{q}, \underline{p}, t) = \sum_i p_i \dot{q}_i - \mathcal{L}(\underline{q}, \dot{\underline{q}}, t). \tag{4.24}$$

The Lagrangian (4.14) yields the relation

$$\underline{p} = \underline{D}(t)\,\underline{L}(t)\,\dot{\underline{q}}\,, \tag{4.25}$$

and, hence, the Hamiltonian,

$$\mathcal{H}(\underline{q},\ \underline{p},\ t) = \frac{1}{2}\,\tilde{\underline{p}}\,\underline{L}^{-1}(t)\,\underline{D}^{-1}(t)\,\underline{p} + \frac{1}{2}\,\tilde{\underline{q}}\,\underline{D}(t)\,\underline{S}(t)\,\underline{q} - \tilde{\underline{q}}\,\underline{D}(t)\,\underline{e}(t); \tag{4.26}$$

this formulation is based on the presumption that $\underline{D}(t)$ and $\underline{L}(t)$ are nonsingular. From (4.10), it is sufficient that $\underline{L}(t)$ be nonsingular; since this condition guarantees the existence of a solution to (4.10), it is again demonstrated that if $\underline{L}(t)$ is originally singular it should be suitably augmented. This Hamiltonian formulation leads to the set of canonical equations

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial q_i}, \quad i = 1, \ldots, n, \tag{4.27a}$$

and

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}, \quad i = 1, \ldots, n; \tag{4.27b}$$

or

$$\dot{\underline{q}} = \underline{L}^{-1}(t)\,\underline{D}^{-1}(t)\,\underline{p} \tag{4.28a}$$

$$\dot{\underline{p}} = -\underline{D}(t)\,\underline{S}(t)\,\underline{q} + \underline{D}(t)\,\underline{e}(t). \tag{4.28b}$$

For networks which lead to dissipation matrices $\underline{D}(t)$ satisfying conditions (4.12), this formulation leads ultimately through all of the concepts presented in Section II. It seems superfluous, however, to pursue them here since a more general stability criterion is soon to be derived from energy considerations.

The Lagrangian (4.22) also gives

$$\underline{p} = \underline{D}(t)\,\underline{L}(t)\,\dot{\underline{q}}\,, \tag{4.29}$$

which leads to the Hamiltonian

$$\mathcal{H}(\underline{q}, \underline{p}, t) = \frac{1}{2} \tilde{\underline{p}} \underline{L}^{-1}(t) \underline{D}^{-1}(t) \underline{p} + \frac{1}{2} \tilde{\underline{q}} \underline{D}(t) \underline{S}(t) \underline{q} - \tilde{\underline{q}} \underline{D}(t) \underline{e}(t) - \tilde{\underline{q}} \underline{D}(t) \tilde{\underline{\beta}} \underline{\lambda}, \tag{4.30a}$$

with

$$\underline{\beta} \underline{L}^{-1}(t) \underline{D}^{-1}(t) \underline{p} = 0. \tag{4.30b}$$

Hamilton's "canonical" equations become

$$\dot{\underline{q}} = \underline{L}^{-1}(t) \underline{D}^{-1}(t) \underline{p}, \tag{4.31a}$$

$$\dot{\underline{p}} = - \underline{D}(t) \underline{S}(t) \underline{q} + \underline{D}(t) \underline{e}(t) + \underline{D}(t) \tilde{\underline{\beta}} \underline{\lambda}, \tag{4.31b}$$

and

$$\underline{\beta} \underline{L}^{-1}(t) \underline{D}^{-1}(t) \underline{p} = 0. \tag{4.31c}$$

Thus, a Hamiltonian and a set of (2m-n) "canonical" equations is always obtainable for an m branch, n degree of freedom time-varying network.

IV. 3 Examples

At this point some examples will serve to indicate the application of the modified Lagrangian. The most obvious networks are those for which

$$\underline{D}(t) = d(t) I; \tag{4.32}$$

this indicates a uniform resistance to inductance ratio in every loop. One must not rule out more complicated matrices $\underline{D}(t)$ than (4.32), but even it, in its simplicity, occurs frequently.

Example 4.1, An Iterated, Time-Varying Network: The network of Fig. 4.1 is described by the modified Lagrangian

$$\mathcal{L} = \exp\left(\int_{t_o}^{t} \frac{R(\xi)}{L(\xi)} d\xi\right)\left\{\frac{1}{2} L(t) \sum_{i=1}^{n} \dot{q}_i^2 - \frac{1}{2C(t)} \sum_{i=1}^{n} [(q_i - q_{i-1})^2 + (q_i - q_{i+1})^2]\right\} \tag{4.33}$$

Clearly, this formulation could be extended to that network where the elements, $R_i(t)$, $L_i(t)$, and $C_i(t)$, all have differing values, and only the function

$$d(t) = \exp\left(\int_{t_o}^{t} \frac{R_i(\xi)}{L_i(\xi)} d\xi\right) \tag{4.34}$$

is invariant from loop to loop:

$$\mathcal{L} = d(t)\left\{\frac{1}{2} \sum_{i=1}^{n} L_i(t) \dot{q}_i^2 - \frac{1}{2} \sum_{i=1}^{n}\left[\frac{(q_i - q_{i-1})^2}{C_i(t)} + \frac{(q_i - q_{i+1})^2}{C_{i+1}(t)}\right]\right\} \tag{4.35}$$

Example 4.2, Nodal Analysis: Throughout the theoretical development of the modified Lagrangian, it was accomplished via loop analysis with the charge q playing the role of fundamental coordinate. Just as an example of the Lagrangian analysis on the (dual) node basis, where the flux ϕ is fundamental, consider the iterated, time-varying network of Fig. 4.2.

For this network, when one has the uniform dissipation function

$$d'(t) = \exp\left(\int_{t_o}^{t} \frac{G_i(\xi)}{C_i(\xi)} d\xi\right) \tag{4.36}$$

at each node, the Lagrangian becomes

$$\mathcal{L} = d'(t)\left\{\frac{1}{2} \sum_{i=1}^{n} C_i(t) \dot{\phi}_i^2 - \frac{1}{2} \sum_{i=1}^{n}\left[\frac{(\phi_i - \phi_{i-1})^2}{L_i(t)} + \frac{(\phi_i - \phi_{i+1})^2}{L_{i+1}(t)}\right]\right\}. \tag{4.37}$$

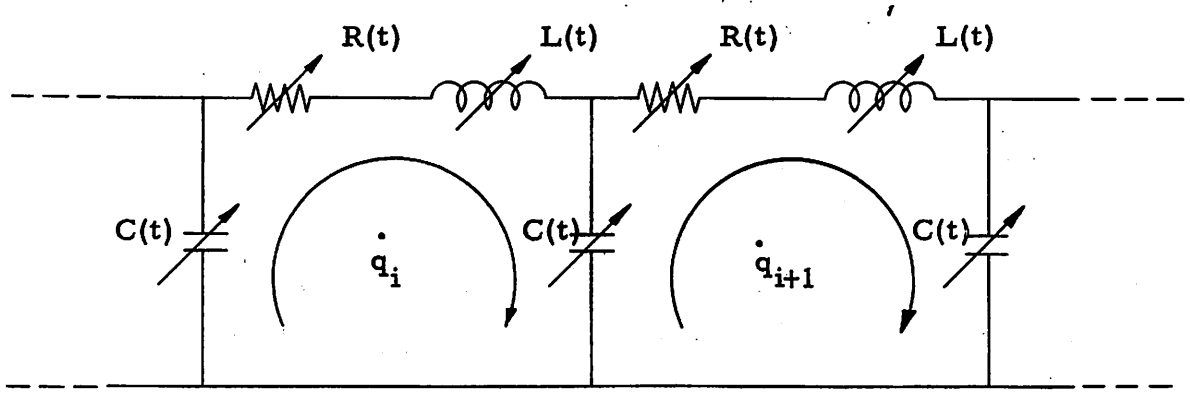

Fig. 4.1. Iterated, time-varying network.

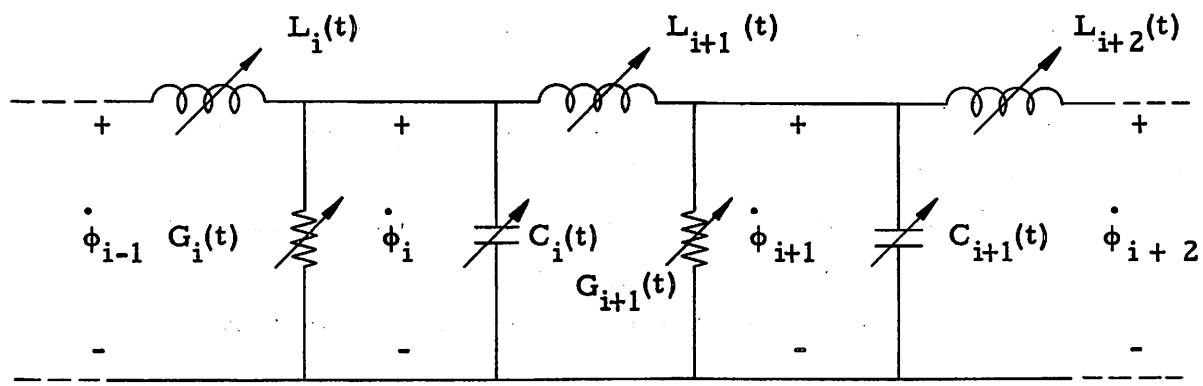

Fig. 4.2. Iterated, time-varying network.

Example 4.3, Continuous Networks: If a variational principle embodies two independent variables, say time t and distance x, then the resulting equation, rather than the Euler-Lagrange, is the Ostrogradski [10]:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial a}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial b}\right) - \frac{\partial \mathcal{L}}{\partial q}, \tag{4.38}$$

where

$$a = \frac{\partial q}{\partial t} \tag{4.39a}$$

and

$$b = \frac{\partial q}{\partial x} \quad . \tag{4.39b}$$

The Lagrangian density [13]

$$\hat{\mathcal{L}} = \frac{1}{2}\ell\ (t,x)\left(\frac{\partial q}{\partial t}\right)^2 - \frac{1}{2}s\ (t,x)\left(\frac{\partial q}{\partial x}\right)^2, \tag{4.40}$$

for example, yields the governing equation

$$\frac{\partial}{\partial t}\ \left[\ell(t,x)\frac{\partial q}{\partial t}\right] - \frac{\partial}{\partial x}\ \left[\ s(t,x)\ \frac{\partial q}{\partial x}\right]\ =\ 0 \tag{4.41}$$

This equation is capable of describing many continuous networks through Lagrangian analysis. One simple example of such a distributed network is the uniform, lossless transmission line, with

$$\ell\ (t,x) = \ell \tag{4.42a}$$

and

$$s\ (t,x) = \frac{1}{C}, \tag{4.42b}$$

which has the wave equation

$$\ell\frac{\partial^2 q}{\partial t^2} - \frac{1}{C}\frac{\partial^2 q}{\partial x^2} = 0. \tag{4.43}$$

Lossy transmission lines and lossy distributed networks are easily described with more esoteric functions, $\ell(t, x)$ and $s(t, x)$.

These examples have served to illustrate the application of energy functions to time-varying networks. The next section will illustrate the utility of the concept of energy in stability studies of such networks.

V. STABILITY OF LINEAR, TIME-VARYING NETWORKS WITH n DEGREES OF FREEDOM

From the physical viewpoint, the behavior of the network stored energy makes a good starting point for network stability investigations. The fact that the network energy can be characterized on two bases -- magnetic stored energy, for instance, can be considered to be analogous to either classical kinetic or potential energy -- leads to results of a more interesting nature than those encountered in the traditional Lyapunov analysis.

V.1 Energy Bounding Functions

Although the energy functions sometimes fail to be describable in a simple manner, the stored energy itself can always be used to investigate stability. Returning to the network equations on the loop basis (4.2), only now for the undriven network,

$$\frac{d}{dt} \left[\underline{L}(t)\, \dot{\underline{q}} \right] + \underline{R}(t)\, \dot{\underline{q}} + \underline{S}(t)\, \underline{q} = 0, \tag{5.1}$$

one can write the network stored energy as

$$E(t) = \frac{1}{2} \tilde{\underline{q}}\, \underline{L}(t)\, \underline{q} + \frac{1}{2} \tilde{\underline{q}}\, \underline{S}(t)\, \underline{q}. \tag{5.2}$$

It is easily verified that the time derivative of this function is given by *

$$\frac{dE(t)}{d(t)} = -\frac{1}{2} \tilde{\dot{\underline{q}}}\, \underline{L}\, \dot{\underline{q}} + \frac{1}{2} \tilde{\underline{q}}\, \underline{S}\, \dot{\underline{q}} - \frac{1}{2} \tilde{\dot{\underline{q}}}\, \underline{R}\, \dot{\underline{q}} - \frac{1}{2} \tilde{\dot{\underline{q}}}\, \tilde{\underline{R}}\, \dot{\underline{q}} . \tag{5.3}$$

---

*The matrices $\underline{L}(t)$ and $\underline{C}(t)$ are assumed symmetrical; thus, any non-reciprocal elements must be introduced via the $\underline{R}(t)$ matrix.

Upon introduction of an arbitrary multiplier f(t), one obtains

$$\frac{d}{dt}\left[f(t)E(t)\right] = \frac{1}{2}\tilde{\underline{q}}\left[\dot{f}\underline{L} - f(\dot{\underline{L}} + \underline{R} + \tilde{\underline{R}})\right]\underline{q} + \frac{1}{2}\tilde{\underline{q}}\left[\dot{f}\underline{S} + f\dot{\underline{S}}\right]\underline{q}. \tag{5.4}$$

Integration of this equation leads to

$$f(t)E(t) = f(t_o)E(t_o) + \frac{1}{2}\int_{t_o}^{t} \tilde{\underline{q}}\left[\dot{f}\underline{L} - f(\dot{\underline{L}} + \underline{R} + \tilde{\underline{R}})\right]\underline{q}\, dt$$

$$+ \frac{1}{2}\int_{t_o}^{t} \tilde{\underline{q}}\left[\dot{f}\underline{S} + f\dot{\underline{S}}\right]\underline{q}\, dt. \tag{5.5}$$

This formulation leads quickly to simple bounding functions for E(t). If

$$\dot{f}L - f(\dot{\underline{L}} + \underline{R} + \tilde{\underline{R}}) \text{ is negative semi-definite} \tag{5.6a}$$

and

$$\dot{f}\underline{S} + f\dot{\underline{S}} \text{ is negative semi-definite,} \tag{5.6b}$$

then

$$f(t)E(t) \leq f(t_o)E(t_o) \tag{5.7}$$

It is a simple matter to find the maximum f(t) which satisfies both conditions (5.6). Under the assumption that $\underline{L}(t)$ is a positive semi-definite matrix,* a similarity transformation represented by $\underline{A}(t)$ can be found such that

$$\tilde{\underline{A}}(t)\underline{L}(t)\underline{A}(t) = \underline{I} \tag{5.8}$$

and

$$\tilde{\underline{A}}(t)\left[\dot{\underline{L}}(t) + \underline{R}(t) + \tilde{\underline{R}}(t)\right]\underline{A}(t) = \underline{\lambda}_1(t), \tag{5.9}$$

where $\underline{\lambda}_1(t)$ is a diagonal matrix which has the n roots of

*If $\underline{L}(t)$ is not a positive semi-definite matrix, it can be augmented by additional elements L_{ϵ_i} (later to be reduced to zero) to make it so.

$$\det. \left[\lambda_1(t)\underline{L}(t) - \dot{\underline{L}}(t) - \underline{R}(t) - \tilde{\underline{R}}(t) \right] = 0 \qquad (5.10)$$

as entries.[13,16] Thus, (5.5a) holds if

$$\dot{f}\underline{I} - f\,\underline{\lambda}_1(t) \text{ is negative semi-definite.} \qquad (5.11)$$

Similarly, if the pertinent part of $\underline{S}(t)$ is positive semi-definite,* a similarity transformation represented by $\underline{B}(t)$ can be found such that

$$\tilde{\underline{B}}(t)\underline{S}(t)\underline{B}(t) = \underline{I} \qquad (5.12)$$

and

$$\tilde{\underline{B}}(t)\dot{\underline{S}}(t)\underline{B}(t) = \underline{\lambda}_2(t), \qquad (5.13)$$

where $\underline{\lambda}_2(t)$ is the diagonal matrix which has the n roots of

$$\det \left[\lambda_2(t)\,\underline{S}(t) + \dot{\underline{S}}(t) \right] = 0 \qquad (5.14)$$

as entries. Hence, (5.5b) holds if

$$\dot{f}\underline{I} - f\,\underline{\lambda}_2(t) \text{ is negative semi-definite.} \qquad (5.15)$$

For positive f(t), the obvious choice for maximum f(t) which just satisfies (5.11) and (5.15) is given instantaneously by

$$\dot{f}/f = \lambda_{ui}(t) = \min_t \left\{ \lambda_{1i}(t),\ \lambda_{2i}(t) \right\}, \qquad (5.16)$$

where $\lambda_{1i}(t)$ and $\lambda_{2i}(t)$ are the roots of (5.10) and (5.14), respectively. Then,

$$f(t) = \exp\left(\int_{t_o}^{t} \lambda_u(\xi)\,d\xi \right), \qquad (5.17)$$

and from (5.7),

---

*Obviously, it is inconsequential to consider entire rows and corresponding columns of $\underline{S}(t)$ which have all zero entries.

$$E(t) \leq E(t_o) \exp\left(-\int_{t_o}^{t} \lambda_{\mu}(\xi)\, d\xi\right). \tag{5.18}$$

Returning to (5.5), one sees that if

$$\dot{f}\underline{L} - f(\dot{\underline{L}} + \underline{R} + \tilde{\underline{R}}) \text{ is positive semi-definite} \tag{5.19a}$$

and

$$\dot{f}\underline{S} + \dot{\underline{S}} \text{ is positive semi-definite,}$$

then

$$f(t)E(t) \geq f(t_o)E(t_o). \tag{5.20}$$

Following reasoning parallel to that used above, one sees that conditions (5.19) just hold for positive f(t) when

$$\frac{\dot{f}}{f} = \lambda_{\ell}(t) = \max_{t} \left\{\lambda_{1j}(t), \lambda_{2i}(t)\right\}, \tag{5.21}$$

where $\lambda_{1i}(t)$ and $\lambda_{2i}(t)$ are the roots of (5.10) and (5.14) respectively. Then,

$$f(t) \geq \exp\left(\int_{t_o}^{t} \lambda_{\ell}(\xi)\, d\xi\right), \tag{5.22}$$

and, from (5.20),

$$E(t) \geq E(t_o) \exp\left(-\int_{t_o}^{t} \lambda_{\ell}\right)(\xi)d\xi \quad . \tag{5.23}$$

The energy of the linear, time-varying network with n degrees of freedom has been bounded in a manner similar to that found in Section III. Some interesting special cases are immediately recognizable from the above:

(1) For a time-invariant network, if all of the roots of

$$\det\left[\lambda\underline{L} - \underline{R} - \tilde{\underline{R}}\right] = 0 \tag{5.24}$$

are positive, the network is stable;

(2) Similarly, for a time-invariant network, if all of the roots of

$$\det\left[\lambda \underline{L} - \underline{R} - \underset{\sim}{\underline{R}}\right] = 0 \tag{5.25}$$

are negative, the network is unstable;

(3) If only $\underline{L}(t)$ and $\underline{R}(t)$ are time-varying, then the roots of

$$\det\left[\lambda(t)\underline{L}(t) - \dot{\underline{L}}(t) - \underline{R}(t) - \underset{\sim}{\underline{R}}(t)\right] = 0. \tag{5.26}$$

indicate stability if they are all positive, and instability if they are all negative.

Case (3) does not exhaust all possibilities, and the more general method given above must be used when it does not hold.

Still, one has but half of the picture, for the entire formulation can be done on a nodal basis as well. Then, instead of (5.1), one begins with a set of network equations

$$\frac{d}{dt}\left[\underline{C}(t)\dot{\underline{\phi}}\right] + \underline{\Gamma}(t)\,\dot{\underline{\phi}} + \underline{\Gamma}(t)\,\underline{\phi} = 0, \tag{5.27}$$

where $\underline{\phi}(t)$ is the n-vector* of tree-branch voltages. Now the stored energy can be written as

$$E(t) = \frac{1}{2}\,\underset{\sim}{\underline{\phi}}\,\underline{C}(t)\dot{\underline{\phi}} + \frac{1}{2}\,\underset{\sim}{\underline{\phi}}\,\underline{\Gamma}(t)\,\underline{\phi}; \tag{5.28}$$

its time derivative is given by

---

*The dimension of the matrix equations on the node basis need not equal that obtained on the loop basis -- it is, as in time-invariant networks, strictly a topological consideration.[14] Neither the loop basis nor the node basis dimensionality need equal the number of dynamic degrees of freedom, which has been given by Bryant.[4]

$$\frac{dE(t)}{dt} = -\frac{1}{2}\dot{\tilde{\phi}}\,\underline{\dot{C}}\,\dot{\phi} + \frac{1}{2}\tilde{\phi}\,\underline{\dot{\Gamma}}\,\phi - \frac{1}{2}\dot{\tilde{\phi}}\,\underline{G}\,\phi - \frac{1}{2}\dot{\tilde{\phi}}\,\underline{\tilde{G}}\,\phi. \tag{5.29}$$

Proceeding as before, one obtains

$$f(t)E(t) = f(t_o)E(t_o) + \frac{1}{2}\int_{t_o}^{t} \dot{\tilde{\phi}}\left[\dot{f}\underline{C} - f(\dot{\underline{C}} + \underline{G} + \tilde{\underline{G}})\right]\dot{\phi}\ dt$$

$$+ \frac{1}{2}\int_{t_o}^{t} \tilde{\phi}\left[\dot{f}\,\underline{\Gamma} + f\,\underline{\dot{\Gamma}}\right]\phi\ dt. \tag{5.30}$$

Consequently, if

$$\dot{f}\underline{C} - f(\dot{\underline{C}} + \underline{G} + \tilde{\underline{G}}) \text{ is negative semi-definite} \tag{5.31a}$$

and

$$\dot{f}\underline{\Gamma} + f\,\underline{\dot{\Gamma}} \text{ is negative semi-definite} \tag{5.31b}$$

then

$$f(t)E(t) \leq f(t_o)E(t_o). \tag{5.32}$$

Similarly, if

$$\dot{f}\underline{C} - f(\dot{\underline{C}} + \underline{G} + \tilde{\underline{G}}) \text{ is positive semi-definite} \tag{5.33a}$$

and

$$\dot{f}\underline{\Gamma} + f\,\underline{\dot{\Gamma}} \text{ is positive semi-definite} \tag{5.33b}$$

then

$$f(t)E(t) \geq f(t_o)E(t_o). \tag{5.34}$$

The pertinent quantities here are the roots $\lambda_{3i}(t)$ of

$$\det\left[\lambda_3(t)\,\underline{C}(t) - \dot{\underline{C}}(t) - \underline{G}(t) - \tilde{\underline{G}}(t)\right] = 0 \tag{5.35}$$

and the roots $\lambda_{4i}(t)$ of

$$\det\left[\lambda_4(t)\ \underline{\Gamma}(t) + \dot{\underline{\Gamma}}(t)\right] = 0. \tag{5.36}$$

For positive $f(t)$,

$$\frac{\dot{f}}{f} \equiv \lambda_u(t) = \min_t \left\{\lambda_{3i}(t),\ \lambda_{4i}(t)\right\} \tag{5.37}$$

gives

$$E(t) \leq E(t_o)\ \exp\left(-\int_{t_o}^{t} \lambda_u(\xi)\, d\xi \right) \tag{5.38}$$

and

$$\frac{\dot{f}}{f} = \lambda_\ell(t) = \max_t \left\{\lambda_{3i}(t),\ \lambda_{4i}(t)\right\} \tag{5.39}$$

gives

$$E(t) \geq E(t_o)\ \exp\left(-\int_{t_o}^{t} \lambda_\ell(\xi)\, d\xi \right). \tag{5.40}$$

Thus, (5.16) and (5.37) give two choices for the energy upper bounding function at any instant; it is desirable to have the minimum upper bounding function (i.e., the maximum $\lambda_u(t)$). The best obtainable upper bounding function for this formulation is given by

$$\bigwedge_u(t) = \max_t \left[\min_t\left(\lambda_{1i}(t),\ \lambda_{2i}(t)\right),\ \min_t\left(\lambda_{3i}(t),\ \lambda_{4i}(t)\right)\right] \tag{5.41}$$

and

$$E(t) \leq E(t_o)\ \exp\left(-\int_{t_o}^{t} \bigwedge_u(\xi)\, d\xi \right). \tag{5.42}$$

Similarly, the best obtainable lower bounding function for this formulation is given by

$$\bigwedge_\ell(t) = \min_t \left[\max_t\left(\lambda_{1i}(t),\ \lambda_{2i}(t)\right),\ \max_t\left(\lambda_{3i}(t),\ \lambda_{4i}(t)\right)\right], \tag{5.43}$$

and

$$E(t) \geq E(t_o) \exp\left(-\int_{t_o}^{t} \bigwedge_{\ell}(\xi)\,d\xi\right). \qquad (5.44)$$

It should be recalled that the functions $\lambda_{1i}(t)$, $\lambda_{2i}(t)$, $\lambda_{3i}(t)$, and $\lambda_{4i}(t)$ are the respective roots of

$$\det\left[\lambda_1(t)\underline{L}(t) - \dot{\underline{L}}(t) - \underline{R}(t) - \tilde{\underline{R}}(t)\right] = 0, \qquad (5.45a)$$

$$\det\left[\lambda_2(t)\underline{S}(t) + \dot{\underline{S}}(t)\right] = 0, \qquad (5.45b)$$

$$\det\left[\lambda_3(t)\underline{C}(t) - \dot{\underline{C}}(t) - \underline{G}(t) - \tilde{\underline{G}}(t)\right] = 0, \qquad (5.45c)$$

and

$$\det\left[\lambda_4(t)\underline{\Gamma}(t) + \dot{\underline{\Gamma}}(t)\right] = 0. \qquad (5.45d)$$

For networks where all of the elements vary periodically with the same period T, equations (5.42) and (5.44) may be used to formulate simple stability criteria:

$$\int_{t_o}^{t_o+T} \bigwedge_{u}(\xi)\,d\xi > 0 \text{ is sufficient for stability,} \qquad (5.46a)$$

and

$$\int_{t_o}^{t_o+T} \bigwedge_{\ell}(\xi)\,d\xi \geq 0 \text{ is necessary for stability;} \qquad (5.46b)$$

similarly,

$$\int_{t_o}^{t_o+T} \bigwedge_{\ell}(\xi)\,d\xi < 0 \text{ is sufficient for instability,} \qquad (5.47a)$$

and

hence,

$$\lambda_1 = \frac{\dot{L}}{L} + \frac{2R}{L} \tag{5.50a}$$

and

$$\lambda_2 = \frac{\dot{C}}{C} \tag{5.50b}$$

On the node basis the network equations become

$$\frac{d}{dt}\left[\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}\right] + \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L} & -\frac{1}{L} \\ -\frac{1}{L} & \frac{1}{L} \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 \tag{5.51}$$

Hence, the equations for λ_3 and λ_4 are *

$$\det \begin{pmatrix} \lambda_3 C_\epsilon - \frac{2}{R} & 0 \\ 0 & \lambda_3 C - \dot{C} \end{pmatrix} = 0 \tag{5.52a}$$

and

$$\det \begin{pmatrix} \lambda_4 \Gamma_\epsilon + \lambda_4 \frac{1}{L} - \frac{\dot{L}}{L^2} & -\lambda_4 \frac{1}{L} + \frac{\dot{L}}{L^2} \\ -\lambda_4 \frac{1}{L} + \frac{\dot{L}}{L^2} & \lambda_4 \frac{1}{L} - \frac{\dot{L}}{L^2} \end{pmatrix} = 0 \tag{5.52b}$$

Solution of these two equations yields

$$\lambda_{31} = \frac{\dot{C}}{C}\,, \tag{5.53a}$$

$$\lambda_{32} = \frac{2}{RC_\epsilon} \longrightarrow +\infty\,, \tag{5.53b}$$

$$\lambda_{41} = \frac{\dot{L}}{L}\,, \tag{5.53c}$$

---

*Here $\underline{C}$ and $\underline{\Gamma}$ have been temporarily augmented by C_ϵ and Γ_ϵ to insure that they are positive definite; these quantities will later be reduced to zero.

$$\int_{t_o}^{t_o + T} \bigwedge_u(\xi)\, d\xi < 0 \text{ is necessary for instability.} \qquad (5.47b)$$

V. 2 Examples

Some simple examples will serve to illustrate the theory and to point up its usefulness in investigations of stability.

(1) Single Loop RLC Time-Varying Network

The network of Fig. 5.1 can be described on the loop basis by the single equation

$$\frac{d}{dt}\left[L\dot{q}\right] + R\dot{q} + \frac{1}{C}q = 0. \qquad (5.48)$$

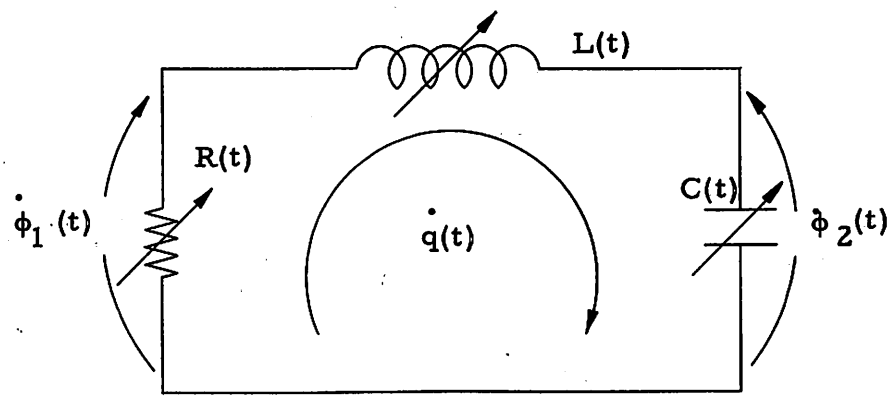

Fig. 5.1. Single Loop RLC Time-Varying Network.

The equations for λ_1 and λ_2 are, therefore,

$$\lambda_1 L - \dot{L} - 2R = 0 \qquad (5.49a)$$

and

$$\lambda_2 \frac{1}{C} - \frac{\dot{C}}{C^2} = 0; \qquad (5.49b)$$

and

$$\lambda_{42} = 0. \tag{5.53d}$$

Consequently, for the single loop network, the upper bounding function is obtained as in (5.41) from

$$\bigwedge_u(t) = \max_t \left\{ \min_t \left[\left(\frac{\dot{L}}{L} + \frac{2R}{L}\right), \left(\frac{\dot{C}}{C}\right) \right], \min_t \left[\left(\frac{\dot{C}}{C}\right), (+\infty), \left(\frac{\dot{L}}{L}\right), (0) \right] \right\}. \tag{5.54}$$

Similarly, the lower bounding function is obtained as in (5.43) from

$$\bigwedge_\ell(t) = \min_t \left\{ \max_t \left[\left(\frac{\dot{L}}{L}\right) + \frac{2R}{L}\right), \left(\frac{\dot{C}}{C}\right) \right], \max_t \left[\left(\frac{\dot{C}}{C}\right), (+\infty), \left(\frac{\dot{L}}{L}\right), (0) \right] \right\} \tag{5.55a}$$

$$= \max_t \left[\left(\frac{\dot{L}}{L} + \frac{2R}{L}\right), \left(\frac{\dot{C}}{C}\right) \right], \tag{5.55b}$$

since infinity dominates the second square bracket. This example has shown clearly the advantage of the n degree of freedom formulation with its two bases (q and ϕ) over the single degree of freedom formulation of Section III.

(2) Stability of Iterated Time-Varying Network

The iterated time-varying network in Fig. 5.2 is a lossy version of those previously described in the literature.[2, 3] If this network is to be described by a set of matrix differential equations of the form

$$\frac{d}{dt}\left[\underline{C}(t)\dot{\underline{\phi}} \right] + \underline{G}(t)\dot{\underline{\phi}} + \underline{\Gamma}(t)\underline{\phi} = 0, \tag{5.56}$$

the pertinent matrices become

$$\underline{C}(t) = \begin{pmatrix} (\underline{M}_1)_m & C(t) & 0 & \\ & (\underline{M}_1)_m & & \\ 0 & & C(t) & \ddots \\ & & & \ddots \end{pmatrix} \tag{5.57a}$$

$$\underline{G}(t) = \begin{pmatrix} (\underline{0})_m & G & 0 & \\ & (\underline{0})_m & & \\ 0 & & G & \ddots \\ & & & \ddots \end{pmatrix} \tag{5.57b}$$

and

$$\underline{\Gamma}(t) = \begin{pmatrix} (\underline{M}_2)_m & 0 & 0 & \\ & (\underline{M}_2)_m & & \\ 0 & & 0 & \ddots \\ & & & \ddots \end{pmatrix} \tag{5.5 c}$$

$\underline{M}_1$ and $\underline{M}_2$ are m x m, time- invariant matrices describing the arbitrary m degree of freedom lossless networks intervening between each G-C(t) set. The equation

$$\det \left[\lambda_3 \underline{C} - \dot{\underline{C}} - 2 \underline{G} \right] = 0 \tag{5.58}$$

has the repeated roots

$$\lambda_3 = 0 \tag{5.59a}$$

and

$$\lambda_3 = \frac{\dot{C}}{C} + \frac{2G}{C} . \tag{5.59b}$$

the equation

$$\det \left[\lambda_4 \underline{\Gamma} + \dot{\underline{\Gamma}} \right] = 0 \tag{5.60}$$

has the repeated roots

$$\lambda_4 = 0. \tag{5.61}$$

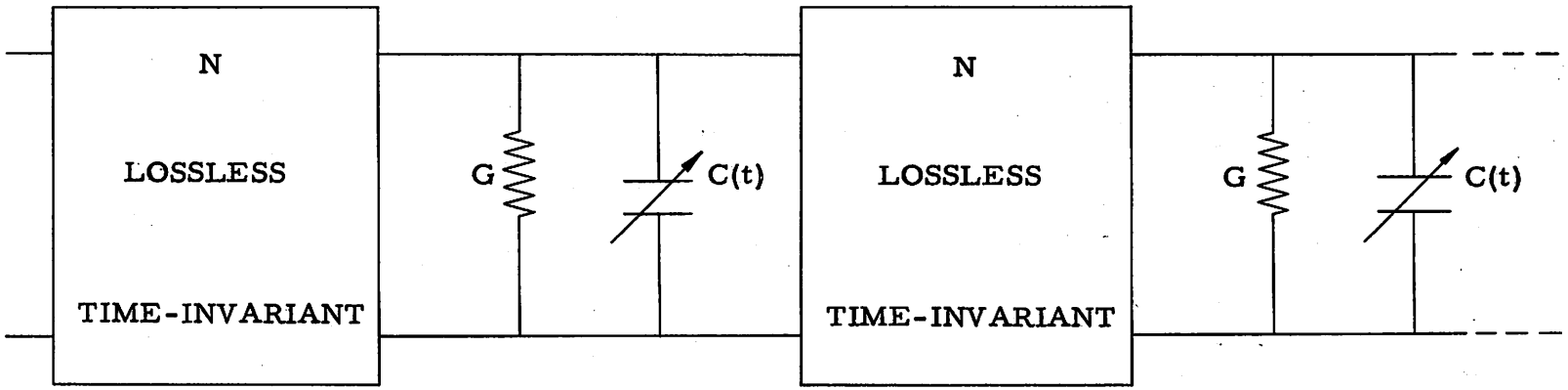

Fig. 5.2. Iterated time-varying network.

Thus, the stability of this iterated time-varying network is guaranteed, regardless of the nature of the intervening time-invariant, lossless networks, if

$$\dot{C} + 2G > 0. \tag{5.62}$$

This criterion is useful in a preliminary investigation of such networks; however, once the lossless networks have been specified the problem can be reworked to obtain less stringent requirements.

Example 5.3, Shunt Version of Negative-Resistance Parametric Amplifier: The network of Fig. 5.3 has been employed in the analysis of negative-resistance type parametric amplifiers.

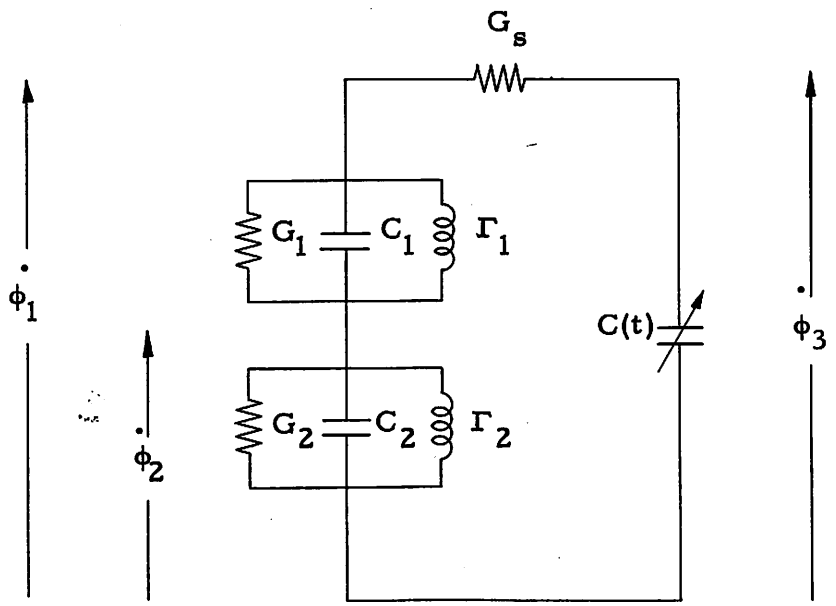

Fig. 5.3. Shunt version, negative-resistance parametric amplifier.

For this network, one has the matrices

$$\underline{\underline{C}}(t) = \begin{pmatrix} C_1 & -C_1 & 0 \\ -C_1 & C_1+C_2 & 0 \\ 0 & 0 & C_2(t) \end{pmatrix} \tag{5.63a}$$

$$\underline{G} = \begin{pmatrix} G_1+G_s & -G_1 & -G_s \\ -G_1 & G_1+G_2 & 0 \\ -G_s & 0 & G_s \end{pmatrix} \tag{5.63b}$$

and

$$\underline{\Gamma} = \begin{pmatrix} \Gamma_1 & -\Gamma_1 & 0 \\ -\Gamma_1 & \Gamma_{1+}\Gamma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.63c}$$

The equation

$$\det\left[\lambda_3 \underline{C} - \dot{\underline{C}} - \underline{G} - \underline{\tilde{G}}\right] = 0 \tag{5.64}$$

becomes

$$\lambda_3^3(C_1C_2C)-\lambda_3^2(C_1C_2\dot{C}+2G_1C_2C+2C_1G_2C+2C_1G_sC+2C_2G_sC+2C_1C_2G_s)$$

$$+\lambda_3^2(C_1G_2\dot{C}+C_1G_s\dot{C}+C_2G_s\dot{C}+G_1G_sC+G_2G_sC+2G_1G_2C+2G_1C_2G_s+2C_1G_2G_s)$$

$$-4(G_1G_s\dot{C}+G_2G_s\dot{C}+G_1G_2\dot{C}+2G_1G_2G_s) = 0. \tag{5.65}$$

To insure stability, one must ascertain that all of the roots $\lambda_{3i}(t)$ are positive for all time; this can be done by a time-dependent Hurwitz criteria, Thus, for the polynomial

$$\lambda^3 - a(t)\lambda^2 + b(t)\lambda - c(t) = 0, \tag{5.66}$$

the roots $\lambda(t)$ are always positive if

$$a > 0, \tag{5.67a}$$

$$c > 0, \tag{5.67b}$$

and

$$ab - c > 0. \tag{5.67c}$$

Condition (5.67b) yields, for example, from (5.65), that

$$\dot{C} + \frac{2G_1G_2G_3}{G_1G_2+G_1G_s+G_2G_s} > 0 \tag{5.68}$$

for stability. Hence, it is the series combination of three resistances shunting C(t) which forms one single degree of freedom type of stability condition. The other two conditions must also be satisfied, but they are a bit difficult to study in the absence of specific element values.*

V. 3 Conclusions

The use of functions bounding the network stored energy on both the loop and the node basis has allowed a quite general formulation of stability criteria. This two-basis approach possesses the obvious advantage that one may switch from one to the other as the need arises to obtain an even tighter bound on the network stored energy than one alone could give. In many instances where norms other than the stored energy are to be employed, this approach for more than one characterization of the same norm might also prove expedient. Specifically, the many basis characterization should be an excellent alternative to the Lyapunov search for a singly specified decreasing norm.[15]

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* For a table of typical values for C(t), $\dot{C}$(t) and G_s see Uenohara.[26]

VI. CONCLUSIONS AND FURTHER PROBLEMS

The characterization of single degree of freedom networks by modified Lagrangian and Hamiltonian energy functions, and the usefulness of such an approach as presented in Sections II and III, indicates that the extension to n degree of freedom networks as presented in Section IV should be put on a firmer foundation. The modified forms of Hamilton's principle -- (4.13), when it is available, or the always obtainable (4.21a) -- are both valid starting points for approximate analysis schemes as discussed in II. 7. It would be worthwhile from this standpoint alone to attempt to find the simplest Lagrangian characterization for a given time-varying network.

Although it was not mentioned, the Lagrangian formulation is readily extendable to cover networks for which only one kind of energy storage element is nonlinear. An energy function analysis capable of far-reaching consequences would emerge if all kinds of nonlinear elements -- particularly resistances -- could be included. The phase plane analyses of Section II in conjunction with the true Lagrangian and Hamiltonian would seem to be the ideal starting point for the study of autonomous oscillations and energy converting properties of nonlinear networks.

In the stability analysis the stored energy is but one of the infinite number of functions which may be used to investigate the network. Because of its characterization on two bases (loop-charge and node-flux), the energy approach has yielded some manipulative advantages over the Lyapunov search for a method of single basis characterization. Certainly, the search should be continued for other multi-basis descriptions of network quantities as an alternative to the viewpoint often propounded. In terms of Lyapunov stability, canonic transformations [13, 19, 27] on the network Hamiltonian

might present a good route to Lyapunov functions.

The results presented here only represent a beginning. Energy analysis of networks is an alternative to many excellent methods -- sometimes better, sometimes worse. But if the Lagrangian analysis is strengthened and better understood, vast bodies of theory from analytical mechanics can be brought to bear on electrical networks. The employment of 300 years of theory could be of theoretical value to the understanding as well as of practical value to the utilization of electrical networks.

APPENDIX I. ORIGINS OF TIME-VARYING NETWORKS

Time-varying network elements can, of course, be obtained by mechanically varying the element values in time. Examples of this type of time-varying element might be a driven rheostat [R(t)], a parallel plate capacitor with a moving slab of dielectric [C(t)], or an inductor with a moving iron core [L(t)]. Although these are plausible devices, they could hardly be expected to be practically operated at high frequencies

The time-varying network elements usually considered are but small signal approximations to electrically driven nonlinear elements. Consider, for example, the single degree of freedom, nonlinear, time-invariant network of Fig. A1.1.

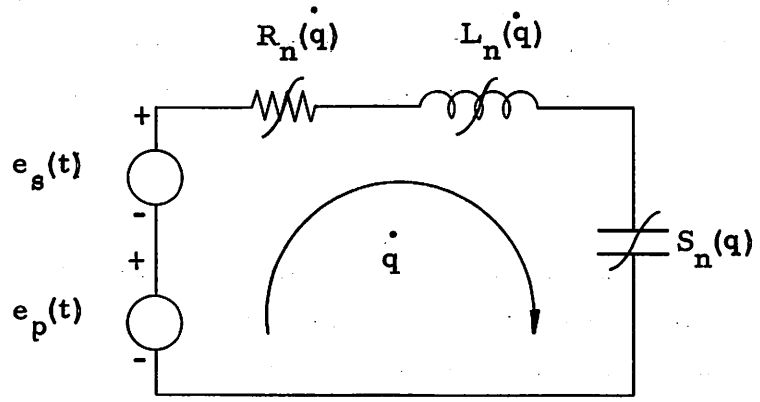

Fig. A1.1. Driven, Nonlinear, Single Degree of Freedom Network.

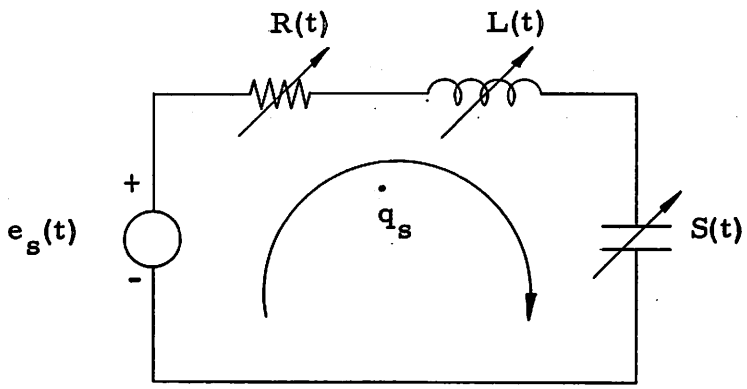

Fig. A1.2. Driven, Linear, Time-varying, Single Degree of Freedom Network.

The differential equation governing the network behavior is

$$\frac{d}{dt}\left[L_n(\dot{q})\,\dot{q}\right] + R_n(\dot{q})\,\dot{q} + S_n(q)q = e_s(t) + e_p(t). \qquad \text{(A1.1)}$$

Suppose that the signal source is idle, i.e.,

$$e_s(t) = 0, \qquad \text{(A1.2)}$$

then equation (A1.1) becomes

$$\frac{d}{dt}\left[L_n(\dot{q}_p)\,\dot{q}_p\right] + R_n(\dot{q}_p)\dot{q}_p + S_n(q_p)\,q_p = e_p(t). \qquad \text{(A1.3)}$$

The solution to this equation, whether it is obtained experimentally or analytically, can be considered to be the known function q_p; then the solution to (A1.1) can be denoted by

$$q = q_p + q_s. \qquad \text{(A1.4)}$$

Under the assumption that the elements are well-behaved functions of $\dot{q}$ and q, equation (A1.1) can be expanded about the fundamental solution q_p to obtain the equation

$$\frac{d}{dt}\Big\{\left[L_n(\dot{q}_p)+L_n'(\dot{q}_p)\dot{q}_s+\ldots\right](\dot{q}_p+\dot{q}_s) + \left[R_n(\dot{q}_p)+R_n'(\dot{q}_p)\dot{q}_s+\ldots\right]$$

$$(\dot{q}_p+\dot{q}_s) + \left[S_n(q_p) + S_n'(q_p) +\ldots\right](q_p+q_s)\Big\} = e_p(t)+ e_s(t) \qquad \text{(A1.5)}$$

(where prime (') denotes the derivative of a function with respect to its argument.) Under the assumptions

$$\left|q_s\right|_{\max} \ll \left|q_p\right|_{\max} \qquad \text{(A1.6a)}$$

and

$$\left|\dot{q}_s\right|_{\max} \ll \left|\dot{q}_p\right|_{\max} \qquad \text{(A1.6b)}$$

one can subtract equation (A1. 3) from (A1. 1) and retain only first-order terms to obtain

$$\frac{d}{dt}\left\{ [L_n(\dot{q}_p) + L_n'(\dot{q}_p)\dot{q}_p]\dot{q}_s \right\} + [R_n(\dot{q}_p) + R_n'(\dot{q}_p)\dot{q}_p]\dot{q}_s$$

$$+ [S_n(q_p) + S_n'(q_p)q_p]\, q_s = e_s(t). \tag{A1. 7}$$

Since q_p is a known function of time, one can make the identifications

$$L(t) = L_n(\dot{q}_p) + L_n'(\dot{q}_p)\dot{q}_p, \tag{A1. 8a}$$

$$R(t) = R_n(\dot{q}_p) + R_n'(\dot{q}_p)\dot{q}_p, \tag{A1. 8b}$$

and

$$S(t) = S_n(q_p) + S_n'(q_p)q_p \tag{A1. 8c}$$

to obtain the linear, second-order, time-varying differential equation which approximately describes the relation between the marginal response $q_s(t)$ and the signal $e_s(t)$:

$$\frac{d}{dt}[L(t)\dot{q}_s] + R(t)\dot{q}_s + S(t)q_s = e_s(t). \tag{A1. 9}$$

Thus, if there were some way of filtering out the ground-state (i. e., $e_s(t) \equiv 0$) response $q_p(t)$, equation (A1. 9) could be considered to be the governing equation for the linear, time-varying network of Fig A1. 2. It is required at many points in the development of the bounding functions that L(t) and C(t) be positive-definite time-functions (or in the more general case that $\underline{L}(t)$ and $\underline{C}(t)$ be positive-definite matrices), this condition is called local passivity by Duinker.[9] From equations (A1. 8), it is clear that local passivity of L(t) and

C(t) is guaranteed if $[\dot{q}L(\dot{q})]$ and $qS(q)]$ are both monotonically increasing functions of their arguments (see Fig. A1. 3), since (A1. 8a) yields

$$\frac{\partial}{\partial \dot{q}} [\dot{q}L_n(\dot{q})] > 0 \qquad \text{(A1. 10a)}$$

and (A1. 8c) yields

$$\frac{\partial}{\partial q} [qS_n(q)] > 0. \qquad \text{(A1. 10b)}$$

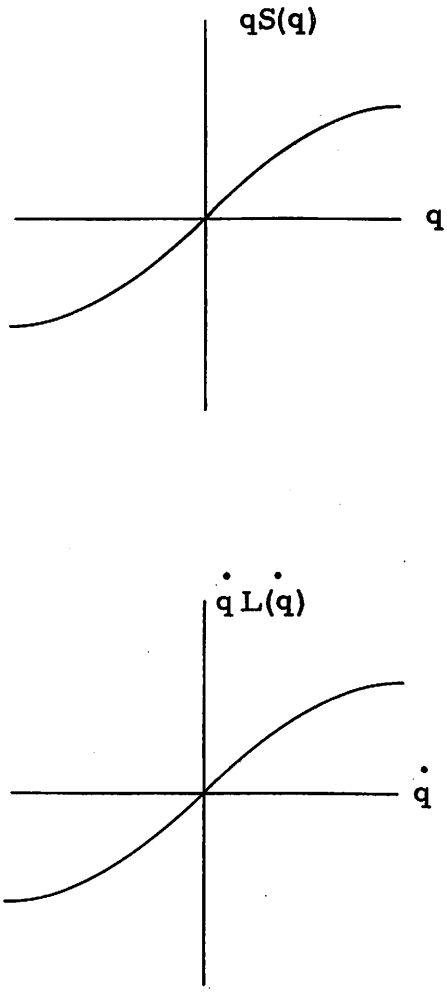

Fig. A1. 3. Dependences which produce locally passive S(t) and L(t).

APPENDIX II. DEMONSTRATION THAT THE MODIFIED HAMILTON'S PRINCIPLE IS A VARIATIONAL PRINCIPLE

It is the purpose here to demonstrate rigorously that the modified Hamilton's principle derived in Section II is a variational principle which leads back to the original equation of motion. The modified Hamilton's principle is

$$\delta \int_{t_1}^{t_2} D(t)\left[\frac{1}{2} L(t)\dot{q}^2 - \frac{1}{2C(t)} q^2 + e(t)q\right] dt = 0; \qquad (A2.1)$$

to take this variation one can consider the integral

$$J_{\bar{c}} = \int_{t_1}^{t_2} D(t)\left[\frac{1}{2} L(t)(\dot{q}+\epsilon\dot{\eta})^2 - \frac{1}{2C(t)}(q+\epsilon\eta)^2 + e(t)(q+\epsilon\eta)\right]dt \qquad (A2.2)$$

where $\eta(t)$ is an arbitrary function which vanishes at t_1 and t_2. It must be shown that

$$\delta J_{\bar{c}} = \epsilon \left(\frac{\partial J_{\bar{c}}}{\partial \epsilon}\right)_{\epsilon = 0} \qquad (A2.3)$$

leads to the equation

$$D(t)\left\{\frac{d}{dt}\left[L(t)q\right] + R(t)\dot{q} + S(t)q - e(t)\right\} = 0. \qquad (A2.4)$$

$$\frac{\partial J_{\bar{c}}}{\partial \epsilon} = \int_{t_1}^{t_2} D(t)\left[L(t)(\dot{q}+\epsilon\dot{\eta})\dot{\eta} - \frac{1}{C(t)}(q+\epsilon\eta)\eta + e(t)\eta\right]dt; \qquad (A2.5)$$

$$\left(\frac{\partial J}{\partial \epsilon}\right)_{\epsilon=0} = \int_{t_1}^{t_2} D(t)\left[L(t)\ddot{q}\eta - \frac{1}{C(t)}q\eta + e(t)\eta\right] dt. \qquad (A2.6)$$

Integrating the first term by parts and setting the right hand side equal to zero, from (A2.3), yields

$$[D(t)L(t)\dot{q}\eta]\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\left\{-\frac{d}{dt}[D(t)L(t)\dot{q}] - \frac{D(t)}{C(t)}q + D(t)e(t)\right\}\eta\, dt = 0. \tag{A2.7}$$

The bracketed term vanishes because $\eta(t)$ is zero at the end points, and by virtue of the arbitrariness of η (the fundamental theorem of the calculus of variations) the term in braces in the integrand must be zero; hence,

$$-\frac{d}{dt}[D(t)L(t)\dot{q}] - \frac{D(t)}{C(t)}q + D(t)e(t) = 0, \tag{A2.8}$$

or

$$D(t)\frac{d}{dt}[L(t)\dot{q}] + \dot{D}(t)L(t)\dot{q} + \frac{D(t)}{C(t)} - D(t)e(t) = 0. \tag{A2.9}$$

But, since

$$\dot{D}(t)L(t) = D(t)R(t), \tag{A2.10}$$

equation (A2.9) becomes

$$D(t)\left\{\frac{d}{dt}[L(t)\dot{q}] + R(t)\dot{q} + \frac{1}{C(t)}q - e(t)\right\} = 0, \quad \text{Q.E.D.} \tag{A2.11}$$

APPENDIX III. OPTIMUM POWER FROM A TIME-VARYING PARALLEL G-C COMBINATION

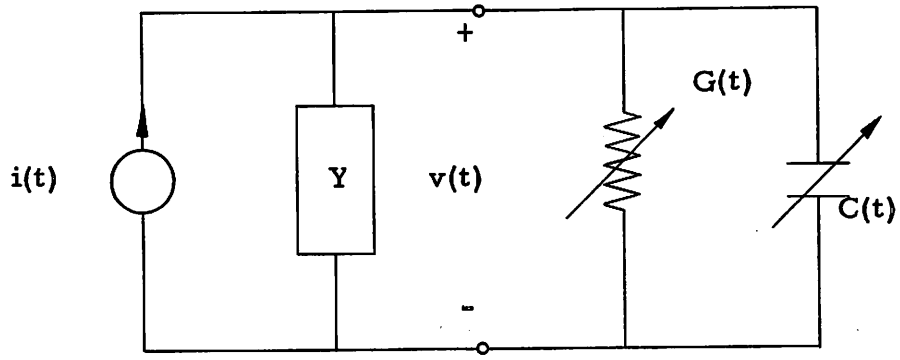

Fig. A3.1. Time-varying G-C Matching Network.

Given the network of Fig. A3.1; its impulse response $h(t,\tau)$ is to be determined so that maximum power is dissipated in the unknown network Y. The equation describing the network behavior is

$$\int_{-\infty}^{t} h(t,\tau)v(\tau)d\tau + G(t)v(t) + \frac{d}{dt}\left[C(t)v(t)\right] = i(t). \qquad (A3.1)$$

The instantaneous power into the unknown network load is

$$v(t)\int_{-\infty}^{\infty} h(t,\tau)v(\tau)\,d\tau = v(t)i(t) - v(t)\frac{d}{dt}\left[C(t)v(t)\right] - G(t)v^2(t); \qquad (A3.2)$$

hence, the average power into the load is

$$P_{av} = \frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left[v_i - v\frac{d}{dt}(Cv) - Gv^2\right]dt \qquad (A3.3)$$

(where the arguments of v(t) and i(t) have been omitted for convenience). To find a condition for maximum power, one may employ the variational formulation[10,16]

$$\delta P_{av} = 0 \qquad \text{(A3.4a)}$$

and

$$\delta^2 P_{av} < 0. \qquad \text{(A3.4b)}$$

But

$$P_{av} = \delta \left(\frac{1}{t_2-t_1}\right)\int_{t_1}^{t_2} [vi - v\frac{d}{dt}(Cv) - Gv^2]\, dt$$

$$= \delta \left(\frac{1}{t_2-t_1}\right)\int_{t_1}^{t_2} [vi - \frac{d}{dt}(\frac{1}{2}Cv^2) - \frac{1}{2}\dot{C}v^2 - Gv^2]\, dt;$$

thus, conditions (A3.4) become

$$\delta \int_{t_1}^{t_2} [vi - \frac{d}{dt}(\frac{1}{2}Cv^2) - \frac{1}{2}\dot{C}v^2 - Gv^2]\, dt = 0 \qquad \text{(A3.5a)}$$

and

$$\delta^2 \int_{t_1}^{t_2} [vi - \frac{d}{dt}(\frac{1}{2}Cv^2) - \frac{1}{2}\dot{C}v^2 - Gv^2]\, dt < 0. \qquad \text{(A3.5b)}$$

The perfect differential $\frac{d}{dt}(\frac{1}{2}Cv^2)$ is inconsequential (the stored energy does not participate in the exchange of power), and (A3.5a) indicates that the Euler-Lagrange equation

$$i(t) - \dot{C}(t)v(t) - 2G(t)v(t) = 0 \qquad \text{(A3.6)}$$

must be satisfied. Moreover, from (A3.5b), the sufficient condition for a maximum becomes

$$-\dot{C} - 2G < 0\ ; \qquad \text{(A3.7)}$$

but his is the condition which also guarantees stability (see Example 3.42).

Combination of (A3.6) with (A3.1) yields

$$\int_{-\infty}^{t} h(t,\tau)\, v(\tau)\, d\tau = G(t)v(t) - C(t)\dot{v}(t); \qquad \text{(A3.8)}$$

this is the condition of adjoint match, where the equation of the loading network is the adjoint of that of the loaded network. Combination of (A3.6) with (A3.2) yields the expression for the instantaneous power

$$P_i(t) = \frac{i^2}{2G+\dot{C}} - \frac{d}{dt}\left[\frac{1}{2}C\left(\frac{i}{2G+\dot{C}}\right)^2\right] - \frac{1}{2}\dot{C}\left(\frac{i}{2G+\dot{C}}\right)^2 - G\left(\frac{i}{2G+\dot{C}}\right)^2. \qquad \text{(A3.9)}$$

Again, in computing the average power, the second term on the right can be dropped as it does not contribute; therefore,

$$P_{av} = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} \frac{i^2(t)}{2[2G(t)+\dot{C}(t)]}\, dt. \qquad \text{(A3.10)}$$

Now, if, for example,

$$i(t) = I_o \cos \omega_o t, \qquad \text{(A3.11a)}$$

$$G(t) = G_o, \qquad \text{(A3.11b)}$$

$$C(t) = C_o + C_1 \cos \omega_1 t, \qquad \text{(A3.11c)}$$

and

$$\omega_o \ll \omega_1, \qquad \text{(A3.11d)}$$

then an approximate expression for average power extracted under optimum loading is given by

$$P_{av} \doteq \frac{1}{4} I_o^2 \frac{1}{\sqrt{(2G_o)^2-(\omega_1 C_1)^2}}. \qquad \text{(A3.12)}$$

If condition (A3. 7) is not met, the integral in (A3.10) diverges. It has been shown here that the stability condition found in Section III and optimum power matching are intimately related.

APPENDIX IV. GENERALIZED ENERGY FUNCTIONS FOR NETWORKS WITH n DEGREES OF FREEDOM

If a given network is described on the loop basis by an n-dimensional matrix differential equation

$$\frac{d}{dt}[L(t)\dot{q}] + \underline{R}(t)\dot{q} + \underline{S}(t)\underline{q} = \underline{e}(t), \tag{A4.1}$$

it is desirable to seek a variational principle for which this equation represents the set of associated Euler-Lagrange equations. The original matrix can, of course, be modified by premultiplication by an arbitrary nonsingular matrix $\underline{D}(t)$:

$$\underline{D}(t)\left\{\frac{d}{dt}[L(t)\dot{q}] + \underline{R}(t)\dot{\underline{q}} + \underline{S}(t)\underline{q} - \underline{e}(t)\right\} = 0 \tag{A4.2}$$

The most general Lagrangian which could be employed has the form *

$$\mathcal{L}(\underline{q}, \dot{\underline{q}}, t) = \tilde{\underline{q}}\underline{M}_1\dot{\underline{q}} + \tilde{\underline{q}}\underline{M}_2\underline{q} + \tilde{\underline{q}}\underline{M}_3\dot{\underline{q}} + \tilde{\underline{q}}\underline{M}_4\underline{q} + \tilde{\underline{q}}\underline{M}_5\underline{e}. \tag{A4.3}$$

Employment of the variational principle

$$\delta\int_{t_1}^{t_2}\mathcal{L}(\underline{q}, \dot{\underline{q}}, t)\,dt = 0 \tag{A4.4}$$

yields the vector Euler-Lagrange equation

$$-\frac{d}{dt}[(\underline{M}_1 + \tilde{\underline{M}}_1)\dot{\underline{q}}] + [-\underline{M}_2 + \tilde{\underline{M}}_2 + \underline{M}_3 - \tilde{\underline{M}}_3]\underline{q} + [-\dot{\underline{M}}_2 - \dot{\tilde{\underline{M}}}_3 + \underline{M}_4 + \tilde{\underline{M}}_4]q + \underline{M}_5\underline{e} = 0. \tag{A4.5}$$

---

*$\tilde{\underline{M}}$ denotes the transpose of $\underline{M}$; and the arguments of $\underline{M}_1(t), \ldots, \underline{M}_5(t)$ are omitted for convenience, although they are all time-dependent.

Comparison of this equation with (A4. 2) yields immediately the following relations:

$$\underline{M}_1 + \tilde{\underline{M}}_1 = \underline{D}\,\underline{L} \tag{A4. 6a}$$

and

$$\underline{M}_5 = -\,\underline{D}. \tag{A4. 6b}$$

One must still somehow solve the set of equations

$$-\underline{M}_2 + \tilde{\underline{M}}_2 + \underline{M}_3 - \tilde{\underline{M}}_3 = \underline{D}\,\underline{R} - \dot{\underline{D}}\,\underline{L} \tag{A4. 7a}$$

and

$$-\dot{\underline{M}}_2 - \tilde{\underline{M}}_3 + \underline{M}_4 + \tilde{\underline{M}}_4 = \underline{D}\,\underline{S}. \tag{A4. 7b}$$

Combining these two equations (and recalling that $\underline{S}$ is symmetrical), one obtains the second-order differential equation for $\underline{D}(t)$

$$\frac{d}{dt}(\underline{D}\,\underline{R} - \dot{\underline{D}}\,\underline{L}) = \underline{D}\,\underline{S} - \underline{S}\,\tilde{\underline{D}}. \tag{A4. 8}$$

A solution to (A4. 8) such that $(\underline{D}\,\underline{L})$ is symmetrical will yield all of the desired matrices and the Lagrangian. Qualitatively, (A4. 7a) indicates that $(\underline{D}\,\underline{R} - \dot{\underline{D}}\,\underline{L})$ must have zeros on the diagonal. Beyond this, however, little else can be said about these matrices in general. Except in special cases, the solution to (A4. 8) is per haps more difficult than that of the original equation (A4. 1). Special cases for which an acceptable $\underline{D}(t)$ is easily found abound; some of these are discussed in the body of the manuscript.

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