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A NOTE ON ZERO-STATE STABILITY OF LINEAR SYSTEMS

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We consider a linear, time-varying, possibly anticipative system whose zero-state response y to an input x is given by

$$y(t) = \int_{-\infty}^{\infty} w(t, \tau)\, x(\tau)\, d\tau \qquad -\infty < t < \infty \qquad (1)$$

For simplicity we assume that all functions are real-valued. We will consider only bounded inputs and we require that the response always be defined and finite for any such input. Since we take the response to be defined by (1), we consider exclusively the zero-state response of the system.† Now, for any fixed t_o, we can define a bounded input by

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†The zero-state response of a linear system is the response of the system when it starts from the zero state. For further information on this subject, consult Zadeh, L.A. and C.A. Desoer: Theory of Linear Systems, The state-space approach, McGraw-Hill Book Co; New York, 1963.

$$x_{t_o}(\tau) = \text{sgn}\, w(t_o, \tau) \qquad -\infty < \tau < \infty \tag{2}$$

and for this input the response at t_o is

$$\int_{-\infty}^{\infty} |w(t_o, \tau)| d\tau \tag{3}$$

Therefore our requirement implies that the integral (3) is finite for each t_o. Thus we are led to our basic assumption concerning w, which characterizes the system:

Assumption A: The linear time-varying system defined by (1) satisfies the condition that for every fixed t_o, $-\infty < t_o < \infty$, $w(t_o, \tau)$ is a measurable function of τ and the integral (3) is finite.

Now let $\mathcal{Y}$ be the set of all functions on $(-\infty, \infty)$ and if a function y happens to be bounded, define its norm by

$$\| y \| = \sup | y(t) | \qquad -\infty < t < \infty \tag{4}$$

Let $\mathcal{X}$ be the set of bounded measurable functions on $(-\infty, \infty)$. It is well known that the space $\mathcal{X}$ with the norm (4) is complete, therefore is a Banach space. All of our inputs x will be long to $\mathcal{X}$, and so have finite norm ($\| x \| < \infty$) while, because of Assumption A, all of our responses y are well defined (the integral in (1) is taken as a Lebesque integral), finite ($y(t)$ is finite for each t, $-\infty < t < \infty$), and belong to $\mathcal{Y}$. Thus under

Assumption A, formula (1) defines a linear operator T from $\mathcal{X}$ to $\mathcal{Y}$, so that $y = Tx$.

Now probably the most natural definition of "zero-state stability" in the sense of, "any bounded input produces a bounded output," is given by Property 1 below, while Property 2 is an apparently stronger requirement. Property 3 is a common hypothesis for a linear system, We list now these three properties of interest to us, which a linear system satisfying Assumption A might have:

Property 1. For all x belonging to $\mathcal{X}$, the output $y = Tx$ is bounded, i.e. x belongs to $\mathcal{X}$ (so $\| x \| < \infty$) implies $\| Tx \| < \infty$.

Property 2. Property 1 holds and furthermore there is a finite constant C such that $\| Tx \| \leq C \| x \|$ for all x belonging to $\mathcal{X}$.

Property 3. There is a finite constant M such that

$$\int_{-\infty}^{\infty} |w(t, \tau)| \, d\tau < M \text{ for all } t, \ -\infty < t < \infty.$$

Thus Property 1 requires that $\| Tx \| / \| x \|$ be finite for every $x \neq 0$ belonging to $\mathcal{X}$, while Property 2 requires that all of these ratios be bounded by a constant C which is independent of x. The interesting point is that all these properties are equivalent.

Theorem. A linear system satisfying Assumption A either has all three of the above properties or none of them.

Proof. We will show $3 \Longrightarrow 2 \Longrightarrow 1 \Longrightarrow 3$. $2 \Longrightarrow 1$ is immediate.

Proof that $3 \Longrightarrow 2$. For any $x \in \mathcal{X}$ we have†

$$\| Tx \| = \sup_t \left| \int (w(t,\tau)\, x(\tau)\, d\tau \right| \leq \sup_t \int |W(t,\tau)| \; |x(\tau)| d\tau$$

$$\leq \sup_t (\| x \| \int |(w(t,\tau)| \, d\tau) \leq M \| x \|$$

Proof that $1 \Longrightarrow 3$. The key to the proof is the "Principle of Uniform Boundedness," which asserts: * if for each t in some index set R, T_t is a bounded linear operator mapping a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$, and if, for every $x \in \mathcal{X}$, the set of real numbers $\{\| T_t(x) \| : t \in R\}$ is bounded, then the set of numbers $\{\| T_t \| : t \in R\}$ is bounded.

Now, for each $t \in R = (-\infty, \infty)$, (1) defines a linear operator from the Banach space $\mathcal{X}$ into the Banach space $\mathcal{Z} = (-\infty, \infty)$ (with absolute value norm) by $T_t(x) = y(t)$. Clearly $\| T_t \| = \sup_{\|x\|=1} \{\| T_t(x) \|\} \leq \int |w(t,\tau)| d\tau$ while for the particular x given by (2) we have $\| T_{t_o} \| \geq \int | w(t_o, \tau) | d\tau$ so $\| T_t \| = \int |w(t,\tau)| \, d\tau < \infty$ for all t. Property 1 states that for all $x \in \mathcal{X}$ $\sup_{t \in R} \| T_t(x) \| = \sup_{t \in R} \| y(t) \| = \| T(x) \| < \infty$. Hence by the conclusion of the Principle of Uniform Boundedness, Property 3 follows.

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†All suprema and all integrals are taken over $(-\infty, \infty)$.

*See for example, p. 168 of C. Goffman's article in Buck, R. C., "Studies in Modern Analysis," published by the Mathematical Association of America, Distributed by Prentice Hall, 1962. See also Dunford, N. and Schwartz, J. T. Linear Operators, Part I. Interscience Publishers, 1958, Theorem II, p. 52.