Electronics Research Laboratory
University of California
Berkeley, California

SOLUTIONS TO SPHERICAL ANISOTROPIC ANTENNAS

by

K. K. Mei
M. Meyer

This research was supported by the U. S. Army Signal Research and Development Laboratory under Contract No. DA 36-039 SC-88924, and was presented at the PTGAP Symposium, National Bureau of Standards, Boulder, Colorado, July 9, 1963.

November 13, 1963

SOLUTIONS TO SPHERICAL ANISOTROPIC ANTENNAS

by

K. K. Mei and M. Meyer

Department of Electrical Engineering
University of California
Berkeley, California

ABSTRACT

An exact solution is found for the fields of a spherical antenna with an anisotropic surface. The surface, for which a rather simple solution is found, conducts perfectly only along spiral lines which go from pole to pole, and is otherwise non-conducting. The antenna is either excited by fields at a gap around the equator or between adjacent spirals. The method of solution is described, and results of several representative cases are presented.

This research was supported by the U. S. Army Signal Research and Development Laboratory under Contract No. DA 36-039 SC-88924, and was presented at the PTGAP Symposium, National Bureau of Standards, Boulder, Colorado, July 9, 1963.

SOLUTIONS TO SPHERICAL ANISOTROPIC ANTENNAS

by

K. K. Mei* and M. Meyer†

1. Introduction

The antenna as an anisotropic-boundary-value problem was first considered by Cheo, Rumsey and Welch[1] in their solution to the frequency-independent-antenna problem. A solution of Maxwell's equation is obtained for a boundary consisting of an infinite number of equally spaced wires in the form of coplanar equiangular spirals. Despite the seemingly academic nature of the boundary conditions, the radiation patterns so obtained agree closely with experimental results. Therefore, the directionally conductive surface as a mathematical model of an electromagnetic radiator is a potentially powerful approach to antenna analysis. A theoretical investigation of the solutions of a class of spherical-spiral antennas, utilizing the same model as considered by Cheo, et al., has been made. Because of the geometry of such antennas, the solutions will facilitate the design of numerous antennas, which possibly may be used in satellites.

2. Description of the Antennas

The antennas discussed consist of spherical-directionally conductive surfaces. The direction of conducting curves can be described in terms of the functional relation of φ and θ as

---

*Department of Electrical Engineering, University of California, Berkeley 4, California.

†Formerly of Department of Electrical Engineering, University of California, now at Hughes Aircraft Co., E.M. Staff Dept. Fullerton, Calif.

1. Cheo, B., Rumsey, V.H. and Welch, W.J., "A Solution to the Frequency-Independent-Antenna Problem," IRE Trans., Vol. AP-9; pp. 535-545, Nov. 1961.

$$\underset{\sim}{T} = \hat{\theta} + \sin\theta \frac{d\varphi}{d\theta} \hat{\varphi}. \tag{1}$$

We shall confine our discussion to two particular functions of $\varphi(\theta)$, namely

$$\varphi(\theta) = \frac{a}{2} \log \frac{1 - \cos\theta}{1 + \cos\theta}, \tag{2}$$

and

$$\varphi(\theta) = b \log \sin\theta \;, \tag{3}$$

where a and b are arbitrary constants. These functions are illustrated in Figs. 1a and 1b. It is noticed that the main difference between these two curves is that (3) represents a spiral with a mirror image about the $\theta = \pi/2$ plane, whereas (2) represents a single-sensed spiral. The method of solution described can be applied to all $\varphi(\theta)$ with derivatives

$$d\varphi/d\theta = a_{nm} \cos^n\theta \sin^{2m+1}\theta \tag{4}$$

for all integers m and n. Thus (2) and (3) are special cases of $m = -1$ and $n = 0, 1$, respectively.

The exciting field for this antenna is established across a narrow slot encircling the antenna at $\theta = \theta_o$, and has a variation of $e^{jm\varphi}$.

3. Basis of Analysis

By applying the well-known method of the Hertz potential,[2] the general solutions of the Maxwell's equations can be represented in terms of two scalar functions, Ψ_1 and Ψ_2, as

2. Harrington, R. F., Time-Harmonic Electromagnetic Fields, McGraw-Hill Book Company, New York, 1961; Chap. 6.

$$\underset{\sim}{E} = -\frac{jZ_o}{k} \nabla \times \nabla \times \Psi_1 \underset{\sim}{r} - \nabla \times \Psi_2 \underset{\sim}{r} ; \tag{5}$$

$$\underset{\sim}{H} = -\frac{j}{kZ_o} \nabla \times \nabla \times \Psi_2 \underset{\sim}{r} + \nabla \times \Psi_1 \underset{\sim}{r} , \tag{6}$$

where Ψ_1 and Ψ_2 are solutions of the scalar wave equations

$$(\nabla^2 + k^2) f = 0 . \tag{7}$$

Because we have assumed the excitation to be of the form $e^{jm\varphi}$, as the antenna structure is uniform in azimuth, we shall consider the φ-variation of the resulting fields to be everywhere $e^{jm\varphi}$, where m is any integer. Thus, Ψ_1 and Ψ_2 may be represented by their expansions in spherical harmonics

$$\Psi_1^{o^i} = \sum_{n=m}^{\infty} B_n^{o^i} z_n (kr) P_n^m(\cos \theta)\, e^{jm\varphi} ; \tag{8}$$

$$\Psi_2^{o^i} = \sum_{n=m}^{\infty} B_n^{o^i} z_n(kr) P_n^m(\cos \theta)\, e^{jm\varphi} , \tag{9}$$

where the superscripts i and o denote the potentials inside and outside the sphere, respectively. The symbol $z_n(kr)$ represents the spherical Hankel function $h_n^{(2)}(kr)$ outside and the spherical Bessel function $j_n(kr)$ inside the sphere. In terms of (8) and (9), the components of the electric fields (5) are

$$E_r^{o^i} = -jZ_o e^{jm\varphi} \sum_{n=m}^{\infty} A_n^{o^i} n(n+1) \frac{z_n(kr)}{kr} P_n^m(\cos \theta); \tag{10}$$

$$E_\theta^{o^i} = jZ_o e^{jm\varphi} \sum_{n=m}^{\infty} A_n^{o^i} \left(\frac{z_n(kr)}{kr} + z_n'(kr) \right) P_n^{m'}(\cos \theta) \sin \theta -$$

$$- jme^{jm\varphi} \sum_{n=m}^{\infty} B_n^{o^i} a_n (kr) \frac{P_n^m(\cos\theta)}{\sin \theta} ; \tag{11}$$

$$E_\varphi{}^{o^i} = mZ_o e^{jm\varphi} \sum_{n=m}^{\infty} A_n{}^{o^i} \left(\frac{z_n(kr)}{kr} + z_n'(kr) \right) \frac{P_n{}^m(\cos\theta)}{\sin\theta}$$

$$- e^{jm\varphi} \sum_{n=m}^{\infty} B_n{}^{o^i} z_n(kr) P_n{}^{m'}(\cos\theta) \sin\theta \,, \tag{12}$$

where the coefficients $A_n{}^{o^i}$ and $B_n{}^{o^i}$ are to be determined from the boundary conditions.

The anisotropic boundary condition of the antenna is characterized by the conducting direction of the spherical surface; that is, the electric fields in the direction $\underset{\sim}{T}$ vanish everywhere on the surface except across the gap. Hence,

$$\underset{\approx}{E} \cdot \underset{\approx}{T} = E_\theta + \sin\theta \frac{d\varphi(\theta)}{d\theta} E_\varphi = C\,\delta(\theta - \theta_o)\, e^{jm\varphi} \,, \tag{13}$$

where C is a constant denoting the magnitude of the exciting field.

For the antenna of Fig. 1a, the boundary condition (13) requires

$$E_\theta + aE_\varphi = C\delta(\theta - \theta_o)\, e^{jm\varphi}, \quad \text{at } r = r_o \,, \tag{14}$$

where r_o is the radius of the antenna. When (11) and (12) are substituted into (14) and multiplied by $\sin\theta$, we have

$$jZ_o \sum_{n=m}^{\infty} A_n{}^{o^i} x_n{}^{o^i} P_n{}^{m'}(\cos\theta) \sin^2\theta - jm \sum_{n=m}^{\infty} B_n{}^{o^i} W_n{}^{o^i} P_n{}^m(\cos\theta)$$

$$+ amZ_o \sum_{n=m}^{\infty} A_n{}^{o^i} x_n{}^{o^i} P_n{}^m(\cos\theta) - a \sum_{n=m}^{\infty} B_n{}^{o^i} W_n{}^{o^i} P_n{}^{m'}(\cos\theta)\sin^2\theta$$

$$= C\,\delta(\theta - \theta_o) \sin\theta, \tag{15}$$

where

$$x_n{}^{o^i} = \frac{z_n(kr)}{kr_o} + z_n'(kr_o)\,, \quad \text{and} \quad W_n{}^{o^i} = z_n(kr_o).$$

The left-hand side of (15) may be rewritten in a series of $P_n^m(\cos\theta)$ by applying the recurrence relation

$$P_n^{m'}(\cos\theta)\sin^2\theta = \frac{(n+m)(n+1)}{2n+1} P_{n-1}^m - \frac{n(n-m+1)}{2n+1} P_{n+1}^m . \tag{16}$$

After expanding the right-hand side of (15) into the associated Legendre polynomials, and equating term by term in (15), we have

$$\left[jZ_o x_{n+1}^{o^i} \frac{(n+m+1)(n+2)}{2n+3}\right] A_{n+1}^{o^i} + amZ_o x_n^{o^i} A_n^{o^i}$$

$$- \left[jZ_o x_{n-1}^{o^i} \frac{(n-1)(n-m)}{2n-1}\right] A_{n-1}^{o^i} - \left[aW_{n+1}^{o^i} \frac{(n+m+1)(n+2)}{2n+3}\right] B_{n+1}^{o^i}$$

$$- jmW_n^{o^i} B_n^{o^i} + \left[aW_{n-1}^{o^i} \frac{(n-1)(n-m)}{2n-1}\right] B_{n-1}^{o^i}$$

$$= \frac{CP_n^m(\cos\theta_o)\sin^2\theta_o(2n+1)(n-m)!}{2(n+m)!}, \quad (n = m,\ m+1,\ m+2, \dots) \tag{17}$$

These equations and the conventional boundary conditions of the antenna uniquely determine the radiated fields everywhere.

For the antenna of Fig. 1b, the characteristic boundary condition is

$$E_\theta + b\cos\theta\, E_\varphi = C\,\delta(\theta - \theta_o) . \tag{18}$$

If we make use of the recurrence relations (16) and

$$P_n^m(\cos\theta)\cos\theta = \frac{1}{2n+1}\left[(n-m+1)P_{n+1}^m + (n+m)P_{n-1}^m\right] \tag{19}$$

we obtain a set of equations

$$\left[jZ_o \frac{(n+m+1)(n+2)}{2n+3} + bm\, Z_o \frac{(n+m+1)}{2n+3}\right] x_{n+1}^{o^i} A_{n+1}^{o^i}$$

$$- \left[jZ_o \frac{(n-1)(n-m)}{2n+1} + bmZ_o \frac{(n-m)}{2n-1} \right] x^{o^i}_{n+1} A^{o^i}_{n-1}$$

$$- b \left[\frac{(n+m+2)(n+3)(n+m+1)}{(2n+5)(2n+3)} \right] W^{o^i}_{n+2} B^{o^i}_{n+2}$$

$$- \left[b \frac{3n^2 - n(4m^2-1) - 3m^2}{(2n+1)(2n-1)(2n+3)} - jm \right] W_n^{o^i} B_n^{o^i}$$

$$+ \left[b \frac{(n-2)(n-m-1)(n-m)}{(2n-3)\ (2n-1)} \right] W^{o^i}_{n-2} B^{o^i}_{n-2}$$

$$= \frac{CP_n^m(\cos\theta_o)\sin^2\theta_o(2n+1)(n-m)!}{2(n+m)!} \; ; (n = m,\ m+1,\ m+2, \dots) \qquad (20)$$

4. Additional Boundary Conditions

When the antenna is constructed of an anisotropic sheet of vanishing thickness, it is transparent to waves polarized perpendicular to the conducting direction. The remaining boundary conditions for such an antenna are that:

(a) The tangential electric fields be continuous across the boundary, i. e.,

$$E_\theta^{\ o}(r_o) = E_\theta^{\ i}(r_o)\,, \text{ and } E_\varphi^{\ o}(r_o) = E_\varphi^{\ i}(r_o)\,. \qquad (21)$$

(b) The magnetic fields parallel to the current be continuous across the boundary, i. e.,

$$H^o(r_o) \cdot \underset{\sim}{T} = H^i(r_o) \cdot \underset{\sim}{T}\,. \qquad (22)$$

(c) The current densities at the poles of the sphere vanish, if the antenna is an open circuit at the poles.

$$H^o(r_o) \cdot (\underset{\sim}{T} \times \underset{\sim}{r}) = H^i(r_o) \cdot (\underset{\sim}{T} \times \underset{\sim}{r}) \text{ at } \begin{cases} \theta = 0 \\ \theta = \pi \end{cases}. \qquad (23)$$

The boundary condition (21) requires

$$A_n^{\,o} x_n^{\,o} = A_n^{\,i} x_n^{\,i} \;, \tag{24}$$

and

$$B_n^{\,o} W_n^{\,o} = B_n^{\,i} W_n^{\,i} \;. \tag{25}$$

The condition (22), for the antenna of Fig. 1a for example, requires

$$H_\theta^{\,o} + aH_\varphi^{\,o} = H_\theta^{\,i} + aH_\varphi^{\,i} \;. \tag{26}$$

Thus, upon substitution of the corresponding components of $\underset{\sim}{H}$ in (26) and using (24) and (25), we obtain

$$a\left(W_{n+1}^{o} - W_{n+1}^{i}\frac{x_{n+1}^{o}}{x_{n+1}^{i}}\right)\frac{(n+m+1)(n+2)}{2n+3}\,A_{n+1}^{o} + jm\left(W_n^{o} - W_n^{i}\frac{x_n^{\,o}}{x_n^{\,i}}\right)$$

$$A_n^{\,o} - a\left(W_{n-1}^{o} - W_{n-1}^{i}\frac{x_{n-1}^{o}}{x_{n-1}^{i}}\right)\frac{(n-1)(n-m)}{2n-1}\,A_{n-1}^{o} + j\frac{1}{Z_o}\left(x_{n+1}^{o} - \right.$$

$$\left. - x_{n+1}^{i}\frac{W_{n+1}^{o}}{W_{n+1}^{i}}\right)\frac{(n+m+1)(n+2)}{2n+3}\,B_{n+1}^{o} + \frac{am}{Z_o}\left(x_n^{\,o} - x_n^{\,i}\frac{W_n^{\,o}}{W_n^{\,i}}\right)$$

$$B_n^{\,o} - j\frac{1}{Z_o}\left(x_{n-1}^{o} - x_{n-1}^{i}\frac{W_{n-1}^{o}}{W_{n-1}^{i}}\right)\frac{(n-1)(n-m)}{2n-1}\,B_{n-1}^{o} =$$

$$= 0\;; \quad (n = m,\ m+1,\ m+2, \ldots)\;. \tag{27}$$

Because of (22), condition (23) may be simplified to

$$\left.\begin{array}{c} H_\varphi^{\,o} = H_\varphi^{\,i} \\ \text{or} \\ H_\theta^{\,o} = H_\theta^{\,i} \end{array}\right. \quad \text{at} \quad \left\{\begin{array}{l} \theta = 0 \\ \\ \theta = \pi \end{array}\right. \tag{28}$$

The coefficients $A_n^{\,o\,i}$ and $B_n^{\,o\,i}$ can be obtained by assuming a converging-series solution for the field so that those terms in

(28) of order greater than a sufficiently large number N can be neglected.

Now, let $n = m$ in (17) and (28). We obtain two equations involving A^o_{m+1}, $A_m^{\ o}$, B^o_{m+1}, and $B_m^{\ o}$, so that A^o_{m+1} and B^o_{m+1} may be solved in linear combinations of A^o_m and $B_m^{\ o}$. Substituting those values of $A^{\ o}_{m+1}$ and B^o_{m+1} into (17) and (27) for $n=m+1$, we get A^o_{m+2} and B^o_{m+2} as linear functions in A^o_m and B^o_m. Hence, in principle, each of the coefficients $A_n^{\ o}$ and $B_n^{\ o}$ may be resolved to linear functions in A^o_m and B^o_m. When the coefficients in (28) are replaced by their corresponding functions in A^o_m and B^o_m, the values A^o_m and B^o_m are obtained and consequently the remaining coefficients.

5. Results

(a) $m = 0$

In this case the gap field E_o is uniform in ϕ. For circumferential field excitation around the equator of the antenna of Fig. 1a, the electric field along the spiral is given by

$$\underset{\sim}{E} \cdot \hat{T} = (1+a^2)^{-1/2}\left[E_\theta + aE_\phi\right] = E_o \delta\left(\theta - \frac{\pi}{2}\right) \tag{29}$$

Applying the relation

$$P_n^1 = \sin\theta \frac{dP_\eta(\eta)}{d\eta} \tag{30}$$

we get from (15)

$$\sum_{n=1}^{\infty} (jZ_o A_n^{oi} X_n^{oi} - aB_n^{oi} W_n^{oi}) P_n^1(\cos\theta) = E_o(1+a^2)\delta\left(\theta - \frac{\pi}{2}\right) \tag{31}$$

Expanding the right-hand side of (31) in $P_n^1(\cos\theta)$ and equating (31) term by term, the following relation is obtained:

$$jZ_o A_n^{oi} X_n^{oi} - aB_n^{oi} W_n^{oi} = E_o(1+a^2)\frac{2n+1}{2n(n+1)} P_n^1(o) \tag{32}$$

The boundary condition (26) gives

$$\frac{jB_n^{\,o}X_n^{\,o}}{Z_o} + a\,A_n^{\,o}W_n^{\,o} = \frac{jB_n^{\,i}X_n^{\,i}}{Z_o} + a\,A_n^{\,i}X_n^{\,i} \tag{33}$$

Hence, the values of $A_n^{\,o}$ and $B_n^{\,o}$ may be obtained from (24), (25), (32), and (33), i.e.,

$$A_n^{\,o} = -\frac{j}{Z_o}\left[X_n^{\,o} - a^2\,\frac{W_n^{\,i}X_n^{\,i}}{W_n^{\,o}}\right]^{-1} \frac{(2n+1)}{2n(n+1)}\;E_o(1+a^2)^{1/2}P_n^1(o) \tag{34}$$

$$B_n^{\,o} = -\left[a\,X_n^{\,i} + \frac{W_n^{\,o}X_n^{\,o}}{a\,W_n^{\,i}}\right]^{-1} \frac{(2n+1)}{2n(2n+1)}\;E_o(1+a^2)^{1/2}P_n^1(o) \tag{35}$$

At very low frequencies (kr_o small) the series converges very fast, so that the first term is sufficient to represent the solution. In this case, the approximate formulas for the radial functions of small arguments may be used to obtain $A_1^{\,o}$ and $B_1^{\,o}$, which are found to be

$$A_1^{\,o} \approx -\frac{3E_o}{4Z_o}\left(\frac{(1+a^2)^{1/2}}{(kr_o)^{-2}+a^2}\right) kr_o \tag{36}$$

$$B_1^{\,o} \approx j\,\frac{3aE_o}{8}\left(\frac{(1+a^2)^{1/2}}{(kr_o)^{-2}+a^2}\right)(kr_o)^2 \tag{37}$$

The far fields are

$$E_\theta = \frac{3}{4}\frac{E_o}{Z_o}\left(\frac{(1+a^2)^{1/2}}{(kr_o)^{-2}+a^2}\right) kr_o\,\frac{e^{-jkr}}{kr}\,\sin\theta \tag{38}$$

$$E_\phi = \frac{j3aE_o}{8}\left(\frac{(1+a^2)^{1/2}}{(kr_o)^{-2}+a^2}\right)(kr_o)^2\,\frac{e^{-jkr}}{kr}\,\sin\theta \tag{39}$$

Which are just dipole patterns with ratio

$$\frac{E_\phi}{E_\theta} = j\frac{a}{2}\,(kr_o) \doteq jS \tag{40}$$

When $S=1$, we have circularly polarization for all θ. The effect of the spiral parameter $\underline{a}$ in the radiation pattern is quite clearly shown in (38) and (39), i. e., the pattern reduces to that of an electric dipole when $a \rightarrow 0$, and that of a magnetic dipole when $a \rightarrow \infty$. Computated results of the complete expression for the fields using as many terms as was necessary for convergence, are shown in Figs. (2) through (4). The low frequency approximations are good for $kr_o < 1$.

(b) $m=1$

Computed results for the antenna of Fig. 1a, with circumferential field excitation of $e^{j\phi}$ around the equator are shown in Figs. (5) to (7) for various parameters. It is of interest to note that in this case, the fields at low frequencies are essentially transverse magnetic, and the ϕ-component of the electric field is practically omnidirectional. The dual of this solution is a TE field. A simplified solution of the antenna may be obtained when the excitation is transverse electric. This can be considered as a rough approximation to the radiation from a "spiral slot" on a conducting sphere at low frequencies. In this case, the recurrence relations (17) are reduced to

$$a_o W^o_{n+1} \frac{(n+m+1)(n+2)}{2n+3} B^o_{n+1} - jmW_n{}^o B_n{}^o - a_o W^o_{n-1} \frac{(n-1)(n-m)}{2n-1} = $$
$$= 0 . \tag{41}$$

Let $m=1$, and start with $n=1$, and we have

$$B_2{}^o W_2{}^o = j\frac{0.777}{a} W_1{}^o B_1{}^o \ldots, \text{ etc.} \tag{42}$$

An interesting feature of this recurrence relation is that the products $B_n^{\,o}W_n^{\,o}$ are independent of the size of the sphere. Therefore, the largest $B_n^{\,o}$ must be associated with the smallest $W_n^{\,o}$. At low frequencies $W_1^{\,o} \ll W_2^{\,o} \ll \ldots \ll W_n^{\,o}$, so TE_{11} is the dominant mode. Hence, the low-frequency far fields of this antenna are

$$E_\theta = -j\,B_1^{\,o}\,\frac{e^{-jkr}}{r}\,e^{j\varphi}\,; \tag{43}$$

$$E_\varphi = B_1^{\,o}\cos\theta\,\frac{e^{-jkr}}{r}\,e^{i\varphi}\,. \tag{44}$$

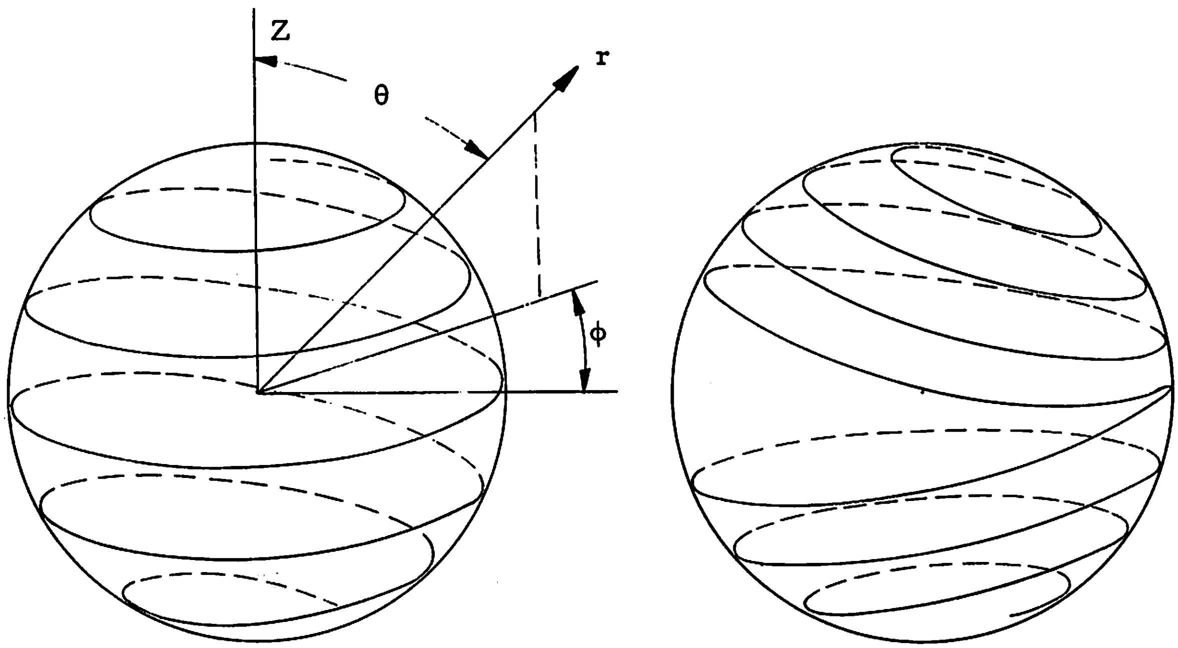
A similar result may be obtained when the recurrence relations (20) are used. That is, the case when the antenna is made of two oppositely sensed spiral hemispheres, which has been experimentally investigated by Riblet.[3] The boundary condition of the antenna in Riblet's investigation can be shown to be that of (18). We notice that (43) and (44) represent an omnidirectional E_θ in the sense of a "time average," circularly polarized fields in the polar directions, and linearly polarized fields in the equatorial plane. Those are essentially the results obtained by Riblet.

---

3. Riblet, H. B., "A Broadband Spherical Satellite Antenna," Proc. IRE, 48, 631, Apr. 1960.

ACKNOWLEDGMENT

The authors wish to express their thanks to Prof. R. H. Rumsey, who initiated and encouraged the authors' interest in antennas, and to Prof. D. J. Angelakos for valuable discussions.



$$\phi = \frac{a}{2} \log \frac{1 - \cos\theta}{1 + \cos\theta}$$

$$\frac{d\phi}{d\theta} = \frac{a}{\sin\theta}$$

(a)

$$\phi = b \log \sin\theta$$

$$\frac{d\phi}{d\theta} = \frac{b\cos\theta}{\sin\theta}$$

(b)

Fig. 1

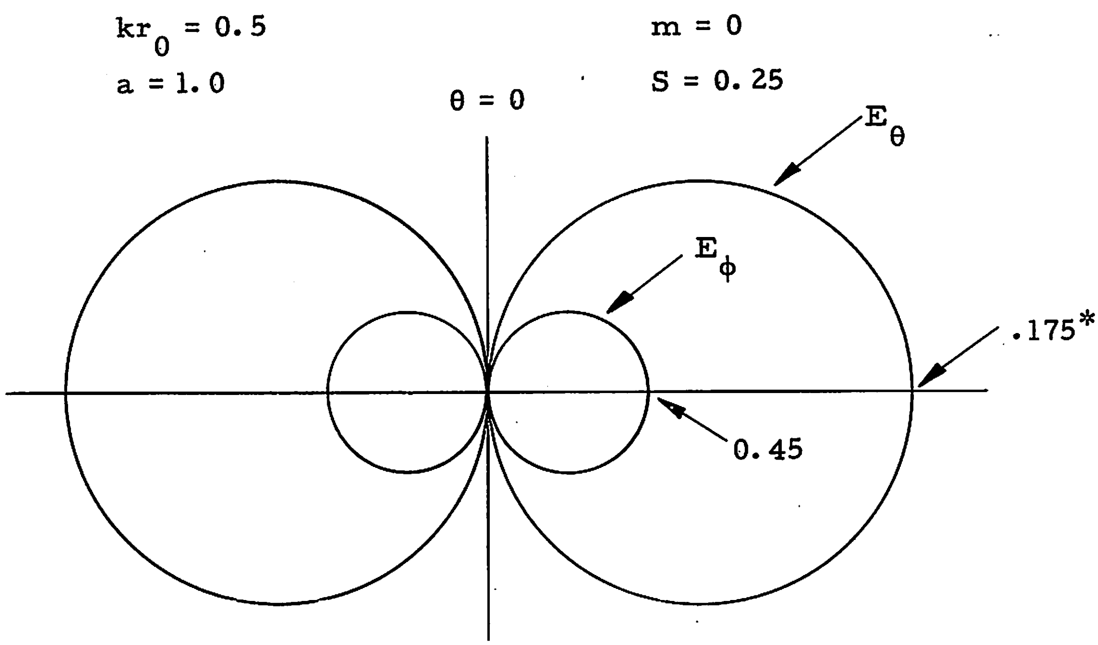


Fig. 2

* Amplitude A of the far field $AE_0 \frac{e^{-jkr}}{r}$

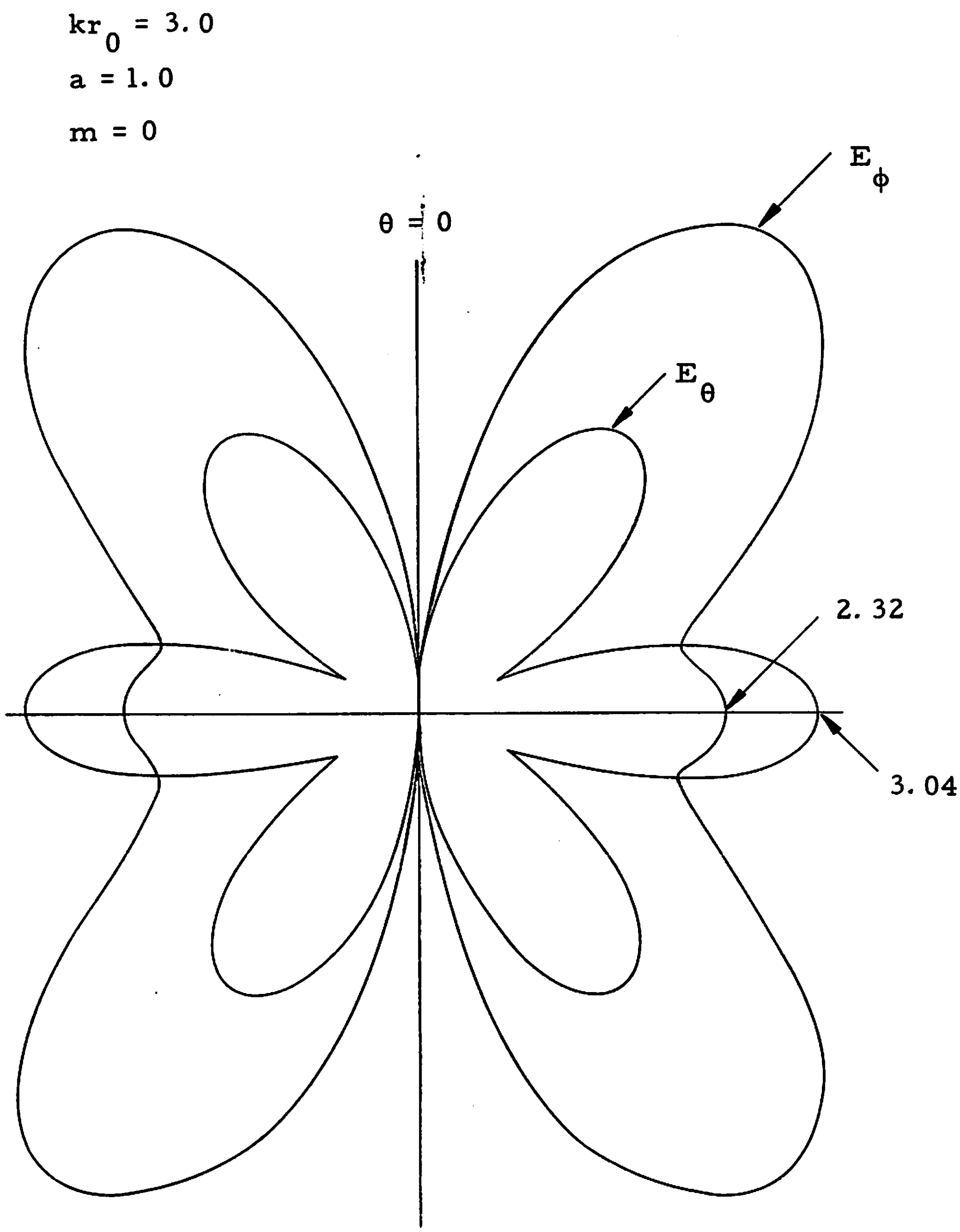


Fig. 3

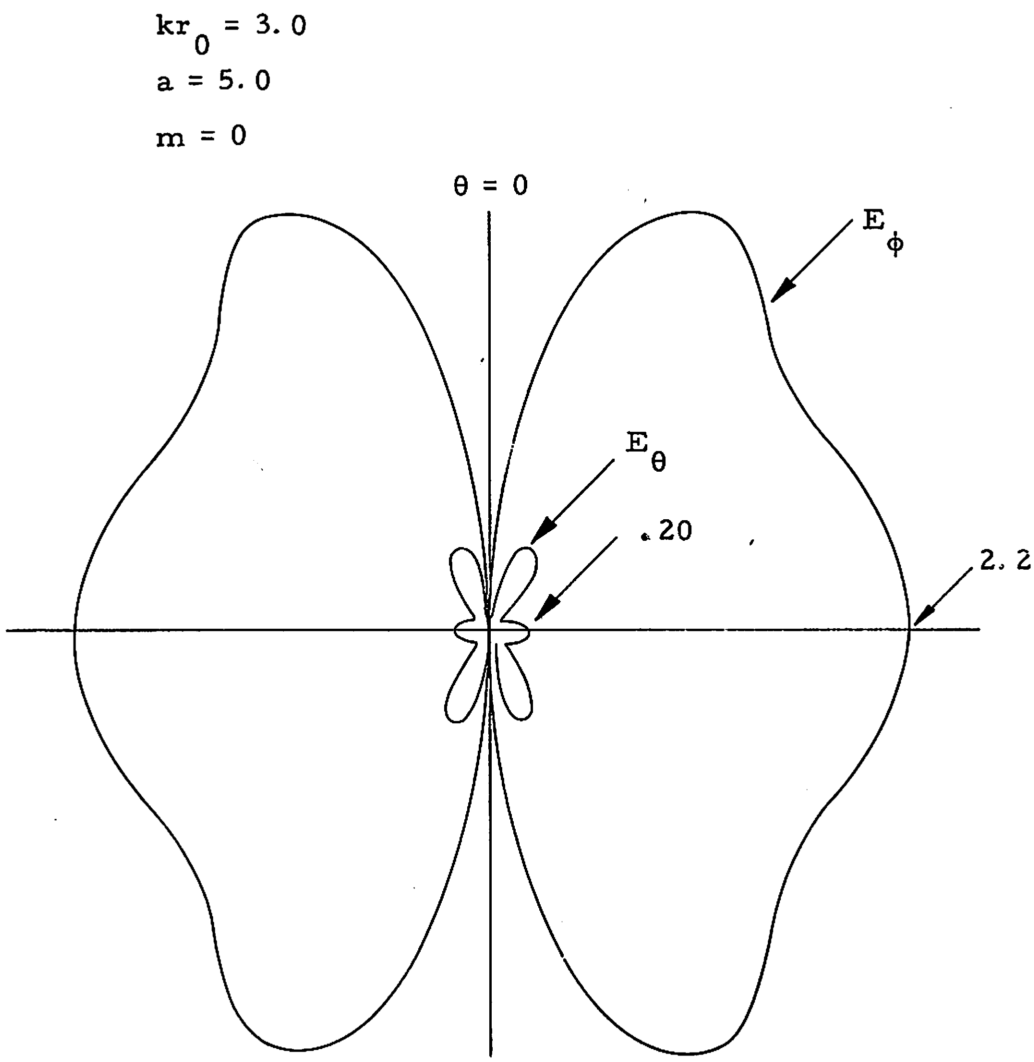


Fig. 4

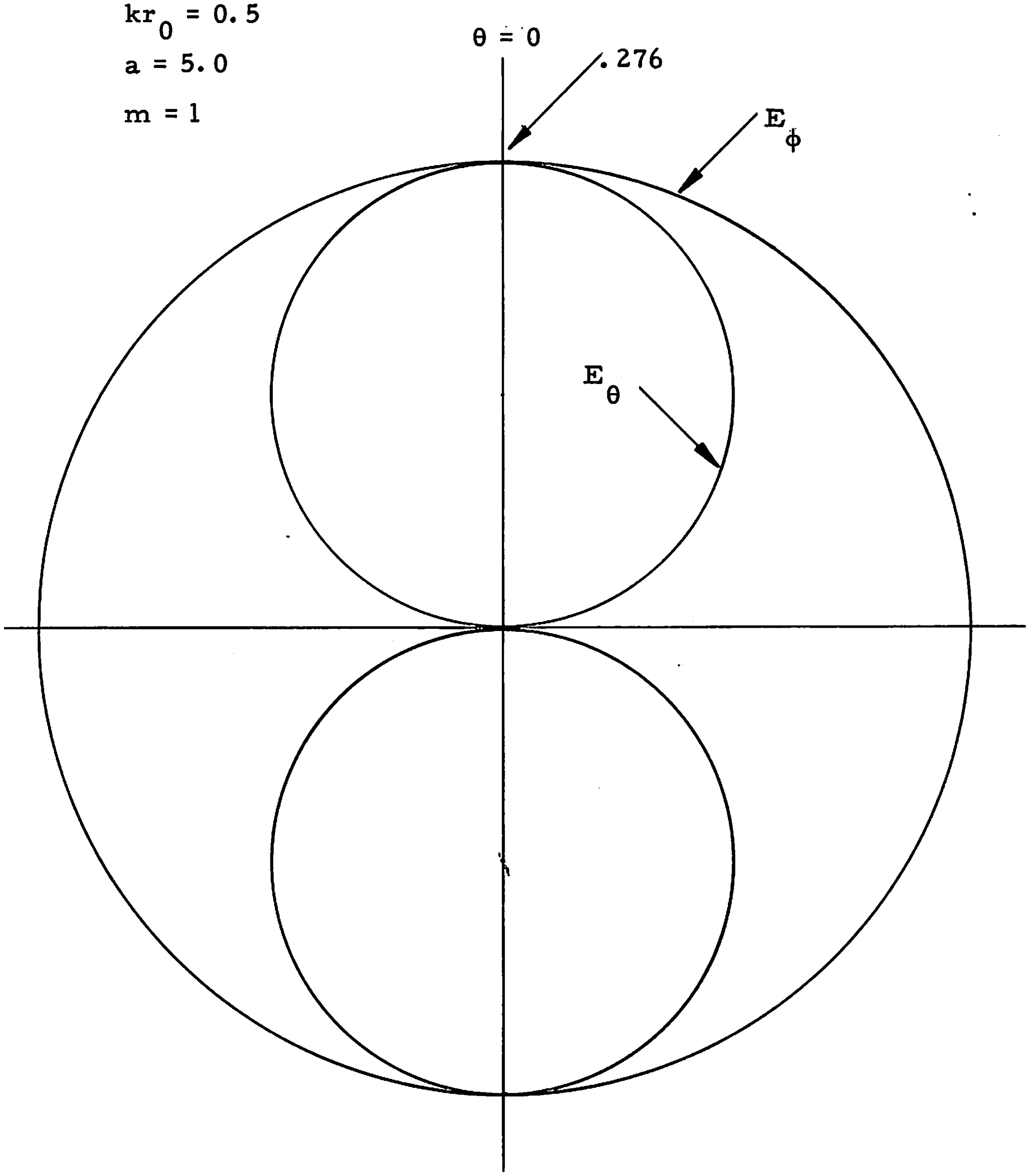


Fig. 5

$kr_0 = 0.5$

$a = 1.0$

$m = 1$

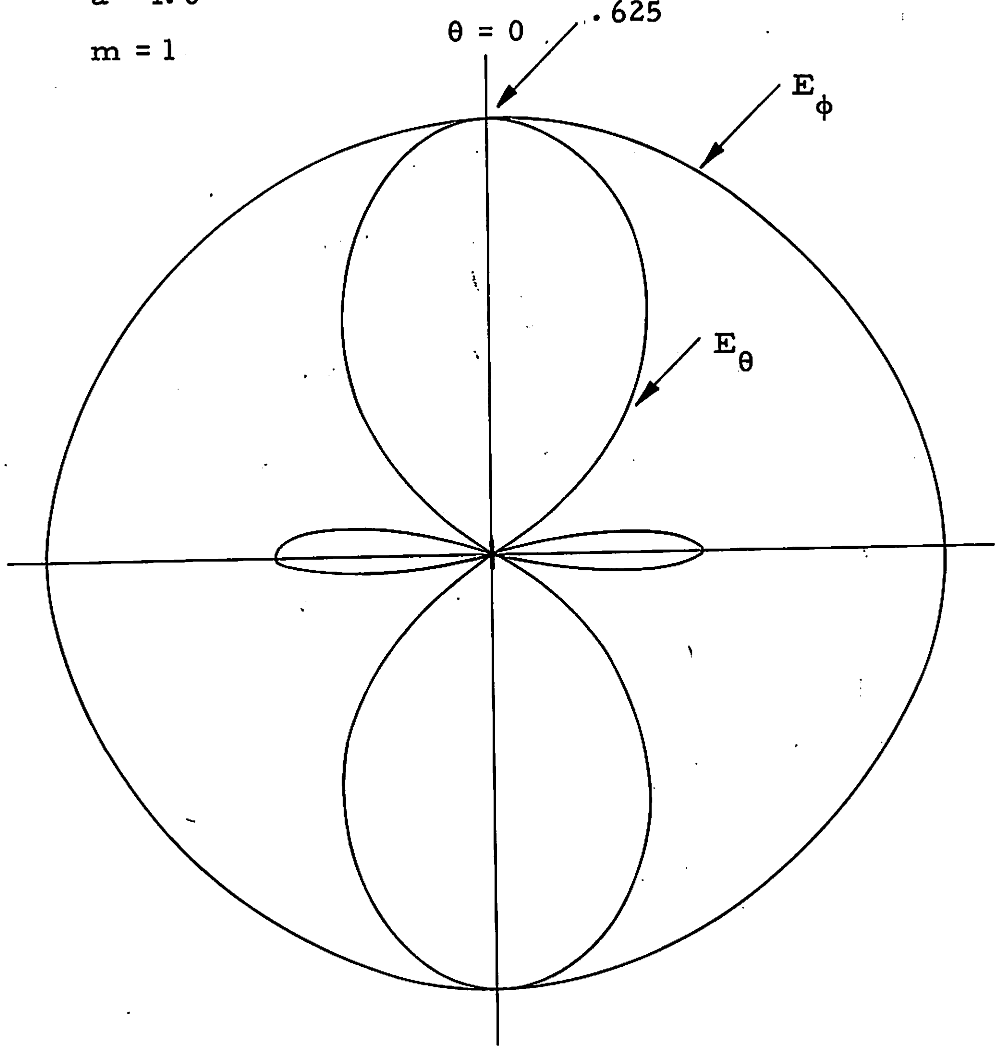


Fig. 6

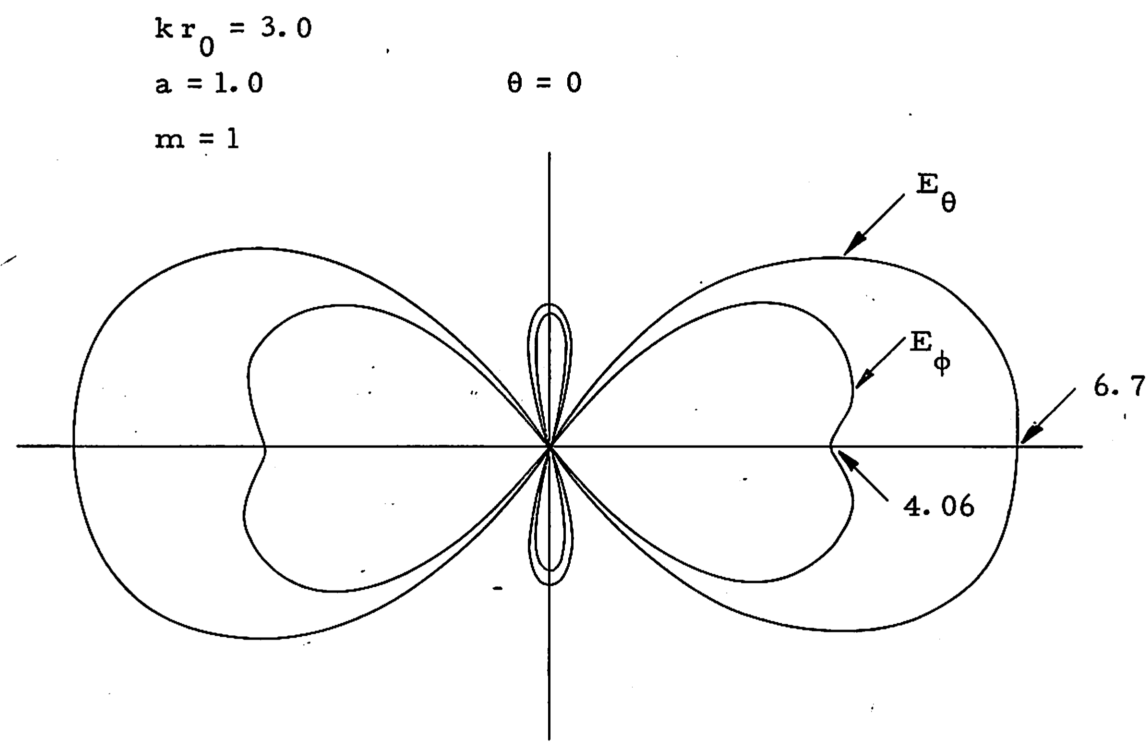


Fig. 7