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NOTES ON MAGNITUDE FUNCTIONS WITH EQUI-RIPPLE RESPONSE

by

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It may be shown[1] that an equi-ripple approximation to the ideal low-pass magnitude characteristic (Fig. 1) may be written in the forms

$$F(\omega^2) = \frac{K_1 Q(\omega^2)}{R(\omega^2)} = \frac{K_1 \prod_1^r (\omega^2 + z_i^2)}{\prod_1^t (\omega^2 + p_i^2)} \qquad r < t \qquad (1)$$

$$= \epsilon_1 + \epsilon_2 G_1(\omega^2) \qquad (2)$$

or

$$= \frac{1}{1 + \epsilon_3 G_2(\omega^2)} \qquad (3)$$

where $G_1(\omega^2)$ and $G_2(\omega^2)$ are Tchebycheff Rational Functions (TRF) of the form

$$G(\omega^2) = \frac{K_2 N(\omega^2)}{D(\omega^2)} = \frac{K_2 \prod_1^{m+n} (\omega^2 + \omega_i'^2)}{\prod_1^m (\omega^2 + \omega_i''^2)} \qquad (4)$$

$$= \cos\left(2n \cos^{-1}\omega - \sum_1^m 2 \tan^{-1} \frac{c_i \omega}{\sqrt{1-\omega^2}}\right) \qquad (5)$$

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where $K_2 = 2n$ $(n \geqslant 1)$. $G(\omega^2)$ then has m finite poles at $\omega_i''^2 = (1/c_i^2 - 1)$ and n poles at infinity; thus $n_1 = 0$ for $G_1(\omega)$ but $n_2 \geqslant 1$ for $G_2(\omega)$. The choice of Eqs. 2 or 3 depends on whether the zeros or poles of $|T(j\omega)|^2$ are chosen in advance.[2] We now exploit Eqs. 2 and 3 to derive an interesting relation which allows the selection of certain poles and zeros in advance while maintaining the Tchebycheff characteristic.

Differentiating $G(\omega^2)$ it may readily be shown that

$$\left(\frac{dG}{d\omega}\right)^2 - k^2 \frac{(1-G^2)\,P^2(\omega^2)}{(1-\omega^2)\,D^2(\omega^2)} = 0 \tag{6}$$

where (1) $D(\omega^2)$ is given in Eq. 4, (2) $P(\omega^2)$ is a polynomial of degree $2p$ with unity leading coefficient and (3) $k = 2n$ if $n \geqslant 1$. We now can write explicitly

$$\left(\frac{dG_1}{d\omega}\right)^2 - k^2 \frac{(1 - G_1^2)\; P_1^2(\omega^2)}{(1 - \omega^2) \prod_1^t (\omega^2 + p_i^2)^2} = 0 \tag{7}$$

$$\left(\frac{dG_2}{d\omega}\right)^2 - (2n_2)^2 \frac{(1 - G_2^2)\, P_2^2(\omega^2)}{(1 - \omega^2) \prod_1^r (\omega^2 + z_i^2)^2} = 0. \tag{8}$$

If we express $G_1(\omega^2)$ and $G_2(\omega^2)$ in terms of $F(\omega^2)$, Eqs. 7 and 8 become

$$\left(\frac{dF}{d\omega}\right)^2 - \frac{k^2}{1-\epsilon_3^2} \frac{\left[\epsilon_3^2 F^2 - (1-F)^2\right] P_1^2(\omega^2)}{(1-\omega^2)\prod_1^t (\omega^2 + p_i^2)^2} = 0 \qquad (9)$$

$$\left(\frac{dF}{d\omega}\right)^2 - (2n_2)^2 \frac{F^2\left[\epsilon_3^2 F^2 - (1-F)^2\right] P_2^2(\omega^2)}{(1-\omega^2)\prod_1^r (\omega^2 + z_i^2)^2} = 0. \qquad (10)$$

From Eq. 1, we can also write Eq. 10 in the form $\left[\text{since } K_1 = (1/2\epsilon_3 n_2)\right]$

$$\left(\frac{dF}{d\omega}\right)^2 - \frac{1}{\epsilon_3^2} \frac{\left[\epsilon_3^2 F^2 - (1-F)^2\right] P_2^2(\omega^2)}{(1-\omega^2)\prod_1^t (\omega^2 + p_i^2)^2} = 0. \qquad (11)$$

Comparing Eqs. 9 and 11 we conclude

$$k = \left(1 - \epsilon_3^2\right)^{1/2} / \epsilon_3 \qquad (12)$$

$$P_1(\omega^2) = P_2(\omega^2). \qquad (13)$$

Eqs. 12 and 13 imply certain relationships between the zeros and poles of $F(\omega^2)$. For example, since both $G_1(\omega^2)$ and $G_2(\omega^2)$ are TRF, they may be written in the form of Eq. 5 with $n_1 = 0$ and $n_2 \geqslant 1$ respectively. Differentiating according to Eq. 6 we find respectively

$$\frac{\sqrt{1-\epsilon_3^2}}{\epsilon_3}\ \frac{P_1(\omega^2)}{R(\omega^2)} = \sqrt{1-\omega^2}\ \frac{d}{d\omega} \sum_1^t \tan^{-1} \frac{c_i\omega}{\sqrt{1-\omega^2}} \tag{14}$$

$$= \sum_1^t \frac{c_i}{c_i^2 - 1} \left[\frac{1}{\omega^2 + (1/c_i^2 - 1)} \right] \tag{15}$$

and

$$2n_2\ \frac{P_1(\omega^2)}{Q(\omega^2)} = \sum_1^r \frac{c_i'}{c_i'^2 - 1} \left[\frac{1}{\omega^2 + (1/c_i'^2 - 1)} \right] - 2n_2. \tag{16}$$

We can then solve for $P_1(\omega^2)$ in each equation and equate the respective coefficients. There result t nonlinear equations in $r + t$ unknowns so that a Newton-Raphson or similar iterative solution can be attempted. A solution is guaranteed if all r c_i' are known; a unique solution is guaranteed if all t c_i are known (Eqs. 2 and 3). If a combination of c_i and c_i' are known, a solution may not exist; however, the existence of a solution of the coefficient equations <u>is</u> a necessary and sufficient condition for $F(\omega^2)$ to be equi-ripple when certain of its poles and zeros are known.

Another result of interest may be derived from Eqs. 12 and 13. For the special case of $r = 0$ (the all-pole equi-ripple function), we have $p = r = 0$ (from Eq. 16). Eq. 7 then becomes

$$\left(\frac{dG_1}{d\omega}\right)^2 - \frac{1-\epsilon_3^2}{\epsilon_3^2}\ \frac{F^2(1-G_1^2)}{1-\omega^2} = 0 \tag{17}$$

or

$$\frac{\epsilon_3}{\sqrt{1-\epsilon_3^2}} \frac{\sqrt{1-\omega^2}}{\sqrt{1-G_1^{\,2}}} \quad \frac{dG_1}{d\omega} = F. \tag{18}$$

Since $n_1 = 0$, Eqs. 5 and 18 combine to yield

$$\frac{\epsilon_3 \sqrt{1-\omega^2}}{\sqrt{1-\epsilon_3^2}} \frac{d}{d\omega}\left(2\sum_1^t \tan^{-1} \frac{c_i\omega}{\sqrt{1-\omega^2}}\right) = F \tag{19}$$

or

$$\frac{2\epsilon_3}{\sqrt{1-\epsilon_3^2}} \sum_1^t \frac{c_i}{n_2\left[(c_i^{\,2}-1)\omega^2+1\right]} = \sum_1^t \frac{k_i}{\omega^2+p_i^2} \tag{20}$$

so that

$$k_i = \left(\frac{c_i}{c_i^2-1}\right)\left(\frac{2\epsilon_3}{\sqrt{1-\epsilon_3^2}}\right) \tag{21}$$

$$= \frac{2\epsilon_3 p_i \sqrt{1+p_i^{\,2}}}{\sqrt{1-\epsilon_3^2}} \tag{22}$$

Thus we have a simple relation between the residues at the Tchebycheff poles (in ω^2) and the poles themselves. A curious relationship between the Tchebycheff poles and the poles of the equi-ripple group delay function may be derived from the above. It has been shown[3] that

$$\underset{\omega^2 = -\omega_i''^2}{\text{Res}} \left[G(\omega^2)\right] = \frac{c_i^2}{(c_i^2 - 1)^2} \prod_{\substack{k=1 \\ \neq i}}^{t} \frac{c_k + c_i}{c_k - c_i} \qquad (23)$$

when $n = 0$ in Eq. 5. From Eq. 2, we have

$$k_i = \underset{\omega^2 = -p_i^2}{\text{Res}} \left[F(\omega^2)\right] = \frac{\epsilon_3}{1 - \epsilon_3^2} \underset{\omega^2 = -p_i^2}{\text{Res}} \left[G_1(\omega^2)\right] \qquad (24)$$

requiring from Eq. 21,

$$1 - \frac{2\sqrt{1 - \epsilon_3^2}\; c_i}{(c_i^2 - 1)} \prod_{\substack{k=1 \\ \neq i}}^{t} \frac{c_k + c_i}{c_k - c_i} = 0 \quad i = 1, 2, \ldots t. \qquad (25)$$

These equations which must be satisfied by the all-pole equi-ripple magnitude function are similar to the Eq. 3

$$1 + \frac{\epsilon\, c_i^2}{(c_i^2 - 1)^{3/2}} \prod_{\substack{k=1 \\ \neq i}}^{t} \frac{(c_k + c_i)}{(c_k - c_i)} = 0 \quad i = 1, 2, \ldots t \qquad (26)$$

which must be satisfied by the all-pole equi-ripple delay function. However, no explanation has been found for the ease with which the former problem is solved but not the latter.

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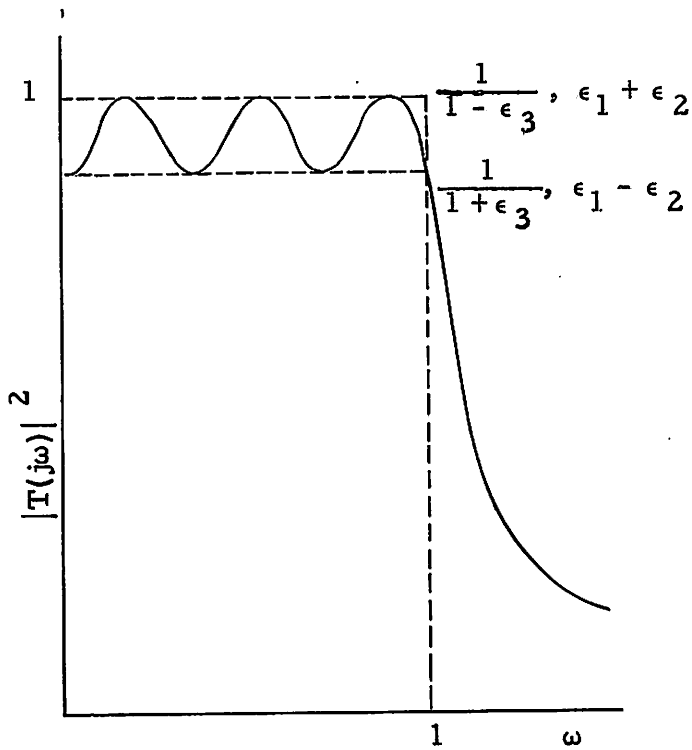

Fig. 1. Definition of tolerances.