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PROOF OF THE GENERATION RULE FOR THE STABILITY CONSTRAINTS IN LINEAR DISCRETE SYSTEMS

by

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In a preceding publication the generation rule has been presented without the detail proof.[1] In this short not this proof is constructed and is presented by the following theorem.

Theorem: Given $a_0, a_1, \ldots, a_m$, define the substitution function by the following rules:

1) $a_i = b_i$, $i = 0, 1, \ldots, m$

2)
$$\left.\begin{array}{ll} S(a_0) = a_0^2 - b_m^2 & S(b_m) = a_0 b_{m-1} - a_1 b_m \\ S(a_1) = a_0 a_1 - b_{m-1} b_m & S(b_{m-1}) = a_0 b_{m-2} - a_2 b_m \\ \vdots & \vdots \\ & S(b_2) = a_0 b_1 - a_{m-1} b_m \\ S(a_{m-1}) = a_0 a_{m-1} - b_1 b_m & S(b_1) = a_0 b_0 - a_m b_m \, . \end{array}\right.$$

3) If $P(a_0, a_1, \ldots, a_k, \; b_m, b_{m-1}, \ldots, b_{m-k})$ is a polynomial, let $S\{P(a_0, a_1, \ldots, a_k, \; b_m, b_{m-1}, \ldots, b_{m-k})\}$

$$= P\left[S(a_0), \ldots, S(a_k), \; S(b_m), \ldots, S(b_{m-k})\right] .$$

Then, the stability constraints are

$$a_0^2 - b_m^2 \Big|_{b_k = a_k} , \quad S(a_0^2 - b_m^2) \Big|_{b_k = a_k} , \quad \ldots , \quad S^{m-1}(a_0^2 - b_m^2) \Big|_{b_k = a_k} .$$

Proof: It is easy to check the above for $m = 2$, assume it is valid for $m = n-1$, then it suffices to prove it is valid for $m = n$. The table begins:

$$
\begin{array}{lccccc}
 & a_0 & a_1 & \cdots & a_{n-1} & a_n \\
 & a_n & a_{n-1} & \cdots & a_1 & a_{n-1} \\
\hline
3^{rd}\text{ row} & a_0^2 - a_n^2 & a_0a_1 - a_{n-1}a_n & \cdots a_0a_{n-2} - a_2a_n & a_0a_{n-1} - a_1a_n & \\
 & a_0a_{n-1} - a_1a_n & \cdots & \cdots \quad \cdots & a_0^2 - a_n^2 . &
\end{array}
$$

Note that the manner of generating the rest of the table is the same as obtaining the whole table for $m = n - 1$. Thus, we can utilize the induction hypothesis as follows:

1. Define

$$
\begin{aligned}
a_0^{(1)} &= a_0^2 - b_n^2 & &= b_0^{(1)} \\
a_1^{(1)} &= a_0a_1 - b_{n-1}b_n & &= b_1^{(1)} = a_0b_1 - a_{n-1}b_n \\
a_2^{(1)} &= a_0a_2 - b_{n-2}b_n & &= b_2^{(1)} = a_0b_2 - a_{n-2}b_n \\
&\vdots \\
a_{n-2}^{(1)} &= a_0a_{n-2} - b_2b_n & &= b_{n-2}^{(1)} = a_0b_{n-2} - a_2b_n \\
a_{n-1}^{(1)} &= a_0a_{n-1} - b_1b_n & &= b_{n-1}^{(1)} = a_0b_{n-1} - a_1b_n
\end{aligned}
$$

or in a compact form,

$$a_i^{(1)} = S(a_i), \quad i = 0, 1, \ldots, n-1$$

$$b_i^{(1)} = S(b_{i+1}), \quad i = 1, 2, \ldots, n-1 .$$

One readily notices that

$$a_0^{(1)}, a_1^{(1)}, \ldots, a_{n-1}^{(1)}$$

are just the 3rd row in the table.

2. The substitute function corresponding to the rest of the table, in accordance to the theorem is:

$$\left.\begin{aligned}
S_1(a_0^{(1)}) &= (a_0^{(1)})^2 - (b_{n-1}^{(1)})^2 \\
S_1(a_1^{(1)}) &= a_0^{(1)} a_1^{(1)} - b_{n-2}^{(1)} b_{n-1}^{(1)} \\
&\vdots \\
S_1(a_{n-3}^{(1)}) &= a_0^{(1)} a_{n-3}^{(1)} - b_2^{(1)} b_{n-1}^{(1)} \\
S_1(a_{n-2}^{(1)}) &= a_0^{(1)} a_{n-2}^{(1)} - b_1^{(1)} b_{n-1}^{(1)}
\end{aligned}\right\}
\begin{aligned}
S_1(b_{n-1}^{(1)}) &= a_0^{(1)} b_{n-2}^{(1)} - a_1^{(1)} b_{n-1}^{(1)} \\
S_1(b_{n-2}^{(1)}) &= a_0^{(1)} b_{n-3}^{(1)} - a_2^{(1)} b_{n-1}^{(1)} \\
&\vdots \\
S_1(b_2^{(1)}) &= a_0^{(1)} = b_1^{(1)} - a_{n-2}^{(1)} b_{n-1}^{(1)} \\
S_1(b_1^{(1)}) &= a_0 b_0^{(1)} - a_{n-1}^{(1)} b_{n-1}^{(1)} .
\end{aligned}$$

Then, the rest of the stability constraints are,

$$(a_0^{(1)})^2 - (b_{n-1}^{(1)})^2 ,\ S_1 \left[(a_0^{(1)})^2 - (b_{n-1}^{(1)})^2\right], \ldots, S_1^{n-2}\left[(a_0^{(1)})^2 - (b_{n-1}^{(1)})^2\right].$$

To prove the theorem, we have to show that the following two sequences are identical:

$$\begin{aligned}
&a_0^{(1)} && = a_0^2 - b_n^2 \\
&(a_0^{(1)})^2 - (b_{n-1}^{(1)})^2 && = S(a_0^2 - b_n^2) \\
&S_1\left[(a_0^{(1)})^2 - (b_{n-1}^{(1)})^2\right] && = S^2(a_0^2 - b_n^2) \\
&\vdots \\
&S_1^{n-2}\left[(a_0^{(1)})^2 - b_{n-1}^{(1)})^2\right] && = S^{n-1}(a_0^2 - b_n^2) .
\end{aligned}$$

It is easy to check the first two relationships. To show the rest, we may note that ignoring superscripts $S(a_i)$ and $S_1(a_i^{(1)})$ are identical except the subscripts of the "b" terms in $S(a_i)$ are one greater than the corresponding ones in $S_1(a_i^{(1)})$. The same situation holds for $S(b_{i+1})$ and $S_1(b_i^{(1)})$, i.e., the subscripts of the "b" terms in $S(b_{i+1})$ are one greater than the corresponding ones in $S_1(b_i^{(1)})$. From these two facts, it follows that (except for superscripts and the difference in the "b" subscripts), $S_1^k\left[(a_0^{(1)})^2 - (b_{n-1}^{(1)})^2\right]$ and $S^k\left[a_0^2 - b_n^2\right]$ are the same. Therefore, to verify that the above sequences are the same it is only necessary to note from the previous definition that,

$$S(a_i) = a_i^{(1)}\ , \quad i = 0, 1, \ldots, n-1$$

$$S(b_{i+1}) = b_i^{(1)}, \quad i = 1, 2, \ldots, n-1\ .$$

Thus, the theorem has been demonstrated.

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REFERENCES

1. E. I. Jury, "On the generation of the stability constraints in linear discrete systems," IEEE, PTGAC, Vol. AC-8, No. 2, pp. 184, April 1963.

2. E. I. Jury, "On the roots of a real polynomial inside the unit circle and a stability criteria for linear discrete systems," presented at 2nd Congress, Basle, Switzerland, August 1963; to be published in the Congress Proceedings in 1964.