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A GENERAL FORMULATION OF THE NYQUIST CRITERION

by

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ABSTRACT

The Nyquist diagram technique is examined under very general assumptions [see (G.1), (G.2), and (G.3)] : in particular, the linear subsystem is represented by a convolution operator [see Eq. (1)]. It is shown that if there are no encirclements of the critical point then the impulse response of the closed-loop system is bounded and absolutely integrable on $[0, \infty)$; it also tends to zero as $t \longrightarrow \infty$. For any initial state, the zero-input response of the closed-loop system is also bounded and goes to zero. If, on the other hand, there is one encirclement of the critical point, then the closed-loop impulse response tends asymptotically to a growing exponential.

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INTRODUCTION

The Nyquist criterion is proved for the single loop feedback case. The purpose of the paper is to demonstrate the extreme generality of the criterion by constructing a proof which requires the least number of assumptions.[†] The main result is stated in the form of a theorem. The hypotheses of this theorem include most cases of engineering interest.

ASSUMPTIONS AND MAIN THEOREM

Following Nyquist[1] we consider the linear time-invariant single-loop feedback system shown in Fig. 1. It will be referred to as the closed-loop system. The block labeled k is a constant gain factor (i.e., independent of time and frequency): if its input is $\eta(t)$, its output is $k\eta(t)$, where k is a fixed positive number. The block labeled G is linear, time-invariant, and nonanticipative (causal) and it satisfies the following conditions:

(G.1) Its input-output relation relating the output y, the zero-input response z and the input ξ is

$$y(t) = z(t) + \int_0^t g(t-\tau)\,\xi(\tau)\,d\tau \qquad \text{for all } t \geq 0. \tag{1}$$

(G.2) For all initial states, the zero-input response is bounded on $[0, \infty)$ and $z(t) \longrightarrow z_\infty$ as $t \longrightarrow \infty$, where z_∞ is a finite number which depends on the initial state. Let $z_M \triangleq \sup_{t \geq 0} |z(t)|$.

[†] The discussion of stability for the case where the transfer functions are not rational is far from trivial. Any reader who doubts this should consider the function defined for $t \geq 0$ by $e^t \sin(e^t)$ and note that its Laplace transform is analytic for all finite s. This example shows that the discussion of stability cannot be settled by "looking at the singularity that is the furthest to the right," which is a legitimate procedure with rational transfer functions.

(G. 3) The unit impulse response g is given by

$$g(t) = 1(t)\,[r + g_1(t)] \qquad (2)$$

where the constant r is nonnegative; $1(t)$ is the unit step function; g_1 is bounded on $[0, \infty)$, is an element of $L^1(0, \infty)$ and $g_1 \longrightarrow 0$ as $t \longrightarrow \infty$. We write

$\mathcal{L}[g(t)] \triangleq G(s) = \frac{r}{s} + G_1(s)$. Let $g_M = \sup_{t \geq 0} |g(t)|$.

For ease of reference, we state formally the main result of this paper:

<u>Theorem</u>. Suppose the linear time-invariant single-loop feedback system shown on Fig. 1 satisfies the conditions (G. 1), (G. 2), and (G. 3). If the Nyquist diagram[+] of $G(s)$ does not encircle or go through the critical point $(-1/k, 0)$, then

(a) the impulse response of the closed-loop system is bounded, tends to zero as $t \longrightarrow \infty$, and is an element of $L^1(0, \infty)$;

(b) for any initial state, the zero-input response of the closed-loop system is bounded and goes to zero as $t \longrightarrow \infty$;

(c) for any initial state and for any bounded input, the response of the closed-loop system is bounded;

(d) let r be positive, then for any input u which tends to a constant u_∞ as $t \longrightarrow \infty$, and for any initial state, the output y tends to u_∞ as $t \longrightarrow \infty$.

If the Nyquist diagram of $G(s)$ encircles the critical point $(-1/k, 0)$ a finite number of times, then the impulse response of the closed-loop system grows exponentially as $t \longrightarrow \infty$.

[+]The Nyquist diagram is the map under G of the imaginary axis from which the interval $[-j\varepsilon, j\varepsilon]$ has been removed and replaced by the semicircle $\{\varepsilon e^{j\theta}: -\pi/2 \leq \theta \leq \pi/2\}$; here ε is taken arbitrarily small.

Comment. It should be stressed that the only assumption that is made concerning the box G is that it fulfills the conditions (G.1), (G.2), and (G.3). Such conditions are often fulfilled by the impulse response of systems described by ordinary differential equations, difference-differential equations, and those whose input-output relation is obtained through the solution of partial differential equations. The latter is the case for distributed circuits and for many control systems.

The analysis to follow applies to all cases where $r \geq 0$. For many circuit applications it turns out that $r = 0$ and that for initial states $z_\infty = 0$. The reader will have no difficulty in inserting the consequent simplifications in the proof.

Analysis. Let u be the bounded input applied to the system and let $u_M \triangleq \sup_{t \geq 0} |u(t)|$. The response of the closed-loop system starting from an arbitrary initial state is given by

$$y(t) = z(t) + k \int_0^t g(t - \tau)[u(\tau) - y(\tau)]\, d\tau \qquad \text{for all } t \geq 0. \tag{3}$$

The theorem will be proved in several steps. First, in order to be able to apply Laplace transform techniques to the integral equation (3) we establish that the solution is of exponential order; second, well known facts concerning Laplace transforms are used to establish the uniqueness of the solution of (3); third, various tools of complex function theory and Fourier analysis are used to establish the properties of the impulse response of the closed-loop system and those of the zero-input response. The proof of the remaining assertions of the theorem follow easily.

Assertion. If (G.1), (G.2), and (G.3) hold and if u is bounded, then

(i) the output y is of exponential order and its Laplace transform, $Y(s)$, is analytic for $\text{Re } s > kg_M$;

(ii) the output y, the solution of (3), is unique.

Proof. From (3) and the definitions of z_M, g_M, and u_M, we get

$$|y(t)| \leq (z_M + kg_M u_M t) + kg_M \int_0^t |y(\tau)| d\tau$$

Hence, by the Gronwall-Bellman inequality,[2,3]

$$|y(t)| \leq b(t) + k \int_0^t b(t - \xi) g_M e^{kg_M \xi} d\xi \qquad \text{for all } t \geq 0 \qquad (4)$$

where $b(t) \triangleq z_M + kg_M u_M t$. Equation (4) implies that y is of exponential order and that its Laplace transform, $Y(s)$, is an analytic function of s for $\text{Re } s > kg_M$. To establish uniqueness suppose there were two responses y_1 and y_2. By subtraction we obtain from (3)

$$y_1(t) - y_2(t) = k \int_0^t g(t - \tau) [y_2(\tau) - y_1(\tau)] d\tau \qquad \text{for all } t \geq 0 \qquad (5)$$

Now g_1 is zero for $t < 0$ and is in $L^1(0, \infty)$; therefore the Laplace transform of g is analytic for $\text{Re } s > 0$ and goes to zero as $|s| \longrightarrow \infty$ with $|\measuredangle s| \leq \pi/2$.[4,9] From (i) y_1 and y_2 are of exponential order; hence taking Laplace transforms of (5) we get

$$Y_1(s) - Y_2(s) = k\, G(s) [Y_2(s) - Y_1(s)] \qquad \text{Re } s > kg_M.$$

Therefore $Y_1(s) - Y_2(s) = 0$ for all s in their domain of definition. By the uniqueness theorem of Laplace transforms,[5] y_1 and y_2 are equal for almost all t in $[0, \infty)$. Since $y_1 - y_2$ is continuous, $y_1(t) = y_2(t)$ for all t in $[0, \infty)$. This completes the proof.

It might be worth noting that since g, u, and z are bounded their restriction to any finite interval, $(0, t)$, is an element of L^2, hence the existence and uniqueness of the solution of (3) may also be established by iterative techniques.[6]

Proof of the Theorem. To prove (a) we recall that, by definition, h is the zero-state response of the system to a unit impulse applied at $t = 0$. By definition of g and from an examination of the configuration of the closed-loop system, to apply a unit impulse at the input of the closed-loop system is equivalent to having an identically-zero input applied to the system but having G start from the state whose zero-input response is kg. Thus

$$h(t) = kg(t) - k \int_0^t g(t - \tau)\, h(\tau)\, d\tau . \tag{6}$$

Let H be the Laplace transform of h; then

$$H(s) = \frac{kG(s)}{1+kG(s)} \qquad \operatorname{Re} s > kg_M . \tag{7}$$

Now, by the principle of the argument,[7] the denominator of $H(s)$ is $\neq 0$ for all $\operatorname{Re} s \geq 0$ if and only if the Nyquist diagram of G does not encircle or go through the critical point $(-1/k, 0)$. By the assumption concerning the Nyquist diagram, the denominator of (7) has no zeros in the closed right half plane. Let us rewrite (7) using (2): if we multiply the numerator and denominator by $s/(s + kr)$ we get

$$H(s) = \frac{k\left(\frac{r}{s} + G_1(s)\right)}{1 + k\left(\frac{r}{s} + G_1(s)\right)} = \frac{\frac{kr}{s + kr} + \frac{ks}{s + kr}\, G_1(s)}{1 + \frac{ks}{s + kr}\, G_1(s)} \tag{8}$$

The denominator may be rewritten as

$$1 + kG_1(s) - k\,\frac{kr}{s + kr}\, G_1(s)$$

Observe that $\mathcal{L}^{-1}[kr/(s + kr)] = 1(t)kr\, e^{-krt}$, which is a function in $L^1(0, \infty)$. Since $g_1 \in L^1(0, \infty)$ and since the product of the transforms of two L' functions is the transform of an L' function, the denominator is of the form "one plus the transform of a function in $L^1(0, \infty)$." The denominator has no zeros in the closed right half plane. The numerator of (8) is also the transform of a function in $L^1(0, \infty)$. Hence, by a theorem of Paley-Wiener,[8] it follows that h is in $L^1(0, \infty)$. Now h is bounded because (6) implies

$$h_M \triangleq \sup_{t \geq 0} |h(t)| \leq kg_M + kg_M \int_0^\infty |h(\tau)| \, d\tau < \infty.$$

To show that h tends to zero as $t \longrightarrow \infty$, observe that (6) implies that

$$(h - kg)\Big|_t^{t+\delta} = -k \int_0^t [g(t + \delta - \tau) - g(t - \tau)] \, h(\tau) d\tau - k \int_t^{t+\delta} g(t + \delta - \tau) h(\tau) d\tau .$$

Therefore, if we remember the form of g specified by (2), for all $t \geq 0$ and all $\delta > 0$,

$$\begin{aligned} &\left| [h(t + \delta) - kg_1(t + \delta)] - [h(t) - kg_1(t)] \right| \\ &\qquad \leq kh_M \int_0^\infty | g_1(\xi + \delta) - g_1(\xi) | \, d\xi + k\delta g_M h_M. \end{aligned} \tag{9}$$

Note that the right hand side of (9) is independent of t. Since $g_1 \epsilon L^1(0, \infty)$, it follows that the first term of the right hand side goes to zero as $\delta \longrightarrow 0$.[9] The same is obviously true of the second term. Consequently (9) implies that $h - kg_1$ is uniformly continuous on $[0, \infty)$. Since h and g_1 are in $L^1(0, \infty)$ so is $h - kg_1$, therefore the uniform continuity implies that $\lim_{t \to \infty} [h(t) - kg_1(t)] = 0$.[11] By (G. 3) it follows that h tends to zero as $t \longrightarrow \infty$.

Let us prove statement (b) of the theorem. The zero-input response of the closed-loop system, z_c, satisfies the equation

$$z_c(t) = z(t) - \int_0^t h(t - \tau) z(\tau) d\tau \qquad \text{for all } t \geq 0 \tag{10}$$

since $h \epsilon L^1(0, \infty)$ and z is bounded, z_c is bounded. It remains to show that z_c goes to zero as $t \longrightarrow \infty$. For this purpose we need only show that the convolution integral tends to z_∞ since, by (G. 2), $z(t) \longrightarrow z_\infty$ as $t \longrightarrow \infty$. The properties of h imply that for any $\varepsilon > 0$ there is a $T(\varepsilon) < \infty$ such that $t > T(\varepsilon)$ implies $|h(t)| < \varepsilon$ and

$\int_{T(\varepsilon)}^{\infty} |h(t)|\, dt < \varepsilon$. The properties of z imply that $|z(t)| \leq z_M < \infty$ for all t and that for any $\varepsilon > 0$ there is a $T'(\varepsilon) < \infty$ such that $t > T'(\varepsilon)$ implies $|z(t) - z_\infty| < \varepsilon$. Rewrite (10)

$$z_c(t) - z(t) = -\int_0^t h(\tau)\,[z(t - \tau) - z_\infty]\,d\tau - z_\infty \int_0^t h(\tau)\,d\tau\,. \tag{11}$$

From these considerations we get the following inequalities: for any $t > T(\varepsilon) + T'(\varepsilon)$

$$\left|z_c(t) - z(t) + z_\infty \int_0^t h(\tau)\,d\tau\right| \leq \int_0^{t - T'(\varepsilon)} |h(\tau)|\,|z(t - \tau) - z_\infty|\,d\tau$$

$$+\int_{t - T'(\varepsilon)}^{t} |h(\tau)|\,|z(t - \tau) - z_\infty|\,d\tau$$

$$\leq \varepsilon \int_0^{t - T'(\varepsilon)} |h(\tau)|\,d\tau + (|z_M| + |z_\infty|) \int_{t - T'(\varepsilon)}^{t} |h(\tau)|\,d\tau\,.$$

Changing the upper limit of integration of both integrals to ∞, we conclude that $t > T(\varepsilon) + T'(\varepsilon)$ implies that

$$\left|z_c(t) - z(t) - z_\infty \int_0^t h(\tau)\,d\tau\right| \leq \varepsilon\left[\int_0^{\infty} |h(\tau)|\,d\tau + |z_M| + |z_\infty|\right]$$

that is

$$\lim_{t \longrightarrow \infty} \left|z_c(t) - z(t) - z_\infty \int_0^t h(\tau)\,d\tau\right| = 0\,. \tag{12}$$

Now since $h \in L^1$ and tends to zero as $t \longrightarrow \infty$,

$$\lim_{t\to\infty} \int_0^t h(t)dt = \lim_{s\to 0} H(s) = 1$$

also by (G. 2),

$$\lim_{t\to\infty} z(t) = z_\infty$$

hence (12) gives

$$\lim_{t\to\infty} z_c(t) = 0.$$

Consider now statement (c) of the theorem. The configuration of the closed-loop system and Eq. (1) imply that the output y starting from an arbitrary initial state of time $t = 0$ and responding to an input u is given by

$$y(t) = z_c(t) + \int_0^t h(t - \tau)u(\tau)d\tau \qquad \text{for all } t \geq 0 \tag{13}$$

where z_c is the closed-loop zero-input response. If u is bounded then y is bounded: this follows from the boundedness of z_c and the fact that h is in $L^1(0, \infty)$. Incidentally, by a previous reasoning $y - z_c$ is uniformly continuous on $[0, \infty)$. Thus statement (c) is established. Since $z_c(t) \longrightarrow 0$ as $t \longrightarrow \infty$, statement (d) is equivalent to the assertion that $u(t) \longrightarrow u_\infty$ implies that $\int_0^t h(t - \tau)u(\tau)d\tau \longrightarrow u_\infty$. This implication has been proved in detail in proving (b). Therefore statement (d) holds.

Suppose now that the Nyquist diagram encircles $(-1/k, 0)$ a finite number of times. Since $G(s) \longrightarrow 0$ as $|s| \longrightarrow \infty$ with $|\measuredangle s| \leq \pi/2$ and since G is analytic in the open right half plane, the principle of the argument[7] shows that $1 + kG(s)$ has a finite number of zeros in the open right half plane. For simplicity of notation we shall write the following expressions assuming that each pole is simple. By a partial fraction expansion we get

$$H(s) = \sum_{\nu=1}^{n} \frac{b_\nu}{s - s_\nu} + H_1(s) \qquad \operatorname{Re} s_\nu > 0, \ \nu = 1, 2, \ldots, n.$$

where $H_1(s)$ is analytic for $\operatorname{Re} s > 0$. It can be easily verified that the behavior of $H(\sigma + j\omega)$ as $\omega \longrightarrow \infty$ satisfies the conditions of Doetsch's theorem.[10] Therefore we conclude that

$$h(t) \sim b_1 e^{s_1 t} \qquad \text{as } t \longrightarrow \infty, \ \operatorname{Re} s_1 > 0$$

where s_1 is the zero of $1 + kG(s)$ which has the largest real part. (If there are several such zeros, then the right hand side must include the appropriate sum.) This completes the proof of the theorem.

For some applications it may be useful to be able to relate the norm of the output y to that of the input u and the zero-input response z.

Corollary. Let (G.1), (G.2), (G.3) hold and the Nyquist diagram satisfy the condition of the theorem. If, for some number $p \geq 1$, both z and u are elements of $L^p(0, \infty)$, then

$$\|y\|_p \leq (1 + \|h\|_1) \|z\|_p + \|h\|_1 \cdot \|u\|_p . \tag{14}$$

When $p = \infty$, if we let $y_M = \sup_{t \geq 0} |y(t)|$, then, using previous notations,

$$y_M \leq (1 + \|h\|_1) z_M + \|h\|_1 u_M .$$

Proof. Observe that, for any $p \geq 1$, if h is in $L^1(0, \infty)$ and z is in $L^p(0, \infty)$, then $h * z$ is also in $L^p(0, \infty)$ and $\|h * z\|_p \leq \|h\|_1 \cdot \|z\|_p$.[12] The inequalities above follow directly from the application of this fact to (10) and (12).

CONCLUSIONS

Under very general assumptions pertaining to the open loop system we have shown that if the Nyquist diagram satisfies the nonencirclement conditions then the zero-input response, the impulse response, and the complete response have all the usual properties associated with stable systems. The inequality (14) shows that if z is in L', then for all $p \geq 1$ (including $p = \infty$) the closed loop system is L_p-stable in the sense of I. W. Sandberg.[13] The results obtained here are essential for some recent extensions of Popov's Criterion.[6]

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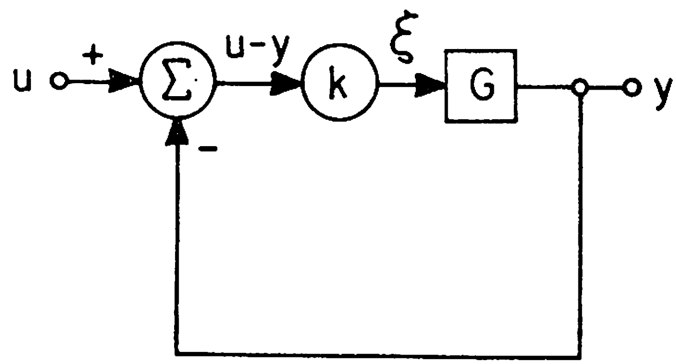

Fig. 1. Single-loop feedback system under consideration: the gain factor k is positive and the linear time-invariant subsystem G is characterized by a convolution operator (see Eq. 1.).