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THE GENERATION OF THRESHOLD NETS BY INTEGER LINEAR PROGRAMMING

by

M. A. Breuer

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M. A. Breuer

Department of Electrical Engineering
University of California
Berkeley, California

This correspondence deals with the use of integer linear programming in designing a combinatorial network of threshold gates capable of evaluating a partially specified Boolean function G. Cameron[1] uses integer programming to find the N-realization of a Boolean function, where N is minimal. This paper is meant as an extension to his work, and much of the same notation will be used.

Unfortunately, due to such factors as tolerance limitations and the fan-in and fan-out constraints, the minimal realization is usually physically unrealizable. In the formulation to be presented, these factors can be considered. The circuit will consist of $R + 1$ threshold gates, where $R \geq \|\delta^M\|$; $\|\delta^M\|$ is the minimal number of n-dimensional hyperplanes required to partition the Boolean n-dimensional vector space such that all of the true vertices (T) of G lie in one half space, and all of the false vertices (F) lie in the other. The main problem to be solved here is that of determining the R hyperplanes, from which the value of

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the weights in the network are readily ascertained. If no circuit limitations are imposed, then $R = \| \delta^M \|$. In Cameron's synthesis procedure, the output of one threshold gate can be used, along with the original set of input variables, as inputs to another threshold gate. Due to this additional versatility, we have that $N \leq \| \delta^M \|$.

Let the n Boolean arguments of G be $a_k \epsilon \{0, 1\}$, $k = 1, 2, \ldots, n$, and let the j-th assignment of values to the a_k be denoted by a Boolean column vector $\underline{A}_j = (a_0, a_1, \ldots, a_n)$, where $a_0 = 1$. Then define $T = \{\underline{A}_j | G(\underline{A}_j) = 1\}$, $F = \{\underline{A}_j | G(\underline{A}_j) = 0\}$, where G is not a function of a_0. Let $\underline{w} = [w_{ik}]$ be an $m \times (n + 1)$ matrix, where $w_{ik} = w'_{ik} - w''_{ik}$. The $w_{ik}(i > 0)$ are the unknown weights in the threshold gates associated with the Boolean variables, and the w_{0k} are the unknown threshold levels. Let $\|\underline{w}\|$ be the number of rows in a matrix $\underline{w}$, and let $\tau = \|T\|$.

The conjunction of n Boolean variables is a 1-realizable function. Assume G is expressed as the logical disjunction of β conjunctive clauses, where β is minimal. Hence G is at least $\beta + 1$ realizable. Let $m = \beta = \|\delta\|$. We now wish to find a minimal $\underline{w}$ matrix, denoted by $\underline{w}^M$, where $\underline{w}^M$ consists of a subset of the rows of $\underline{w}$, and where the following three conditions hold:

$$\text{a)} \quad \underline{w}^M \underline{A}_j > \underline{0} \; ; \; \forall \underline{A}_j \in T \qquad \text{(1a)}$$

$$\text{b)} \quad \underline{w}^M \underline{A}_j \leq \underline{0} \; ; \; \forall \underline{A}_j \in F \qquad \text{(1b)}$$

$$\text{c)} \quad \|\underline{w}^M\| \text{ is minimal} \qquad \text{(1c)}$$

where $\underline{0}$ is the zero vector. We denote this minimal value of $\|\underline{w}^M\|$ by $\|\delta^M\|$, and we indicate the i-th row of $\underline{w}$ as $\underline{w}_i = (w_0, w_1, \ldots, w_n)$.

Define the variable $\sigma_i \epsilon \{0, 1\}$, where if $\sigma_i = 1$, then the i-th row of $\underline{w}$ is a row of $\underline{w}^M$. Otherwise $\sigma_i = 0$ and then the i-th row of $\underline{w}$ is not in $\underline{w}^M$. Hence $\|\delta^M\| = \Sigma_{i=1}^m \sigma_i$.

For all $\underline{A}_j \epsilon F$ we require that

$$\underline{w}_i \cdot \underline{A}_j \leq K\delta_{ij} < K + \underline{w}_i \cdot \underline{A}_j \tag{2*}$$

for i = 1, 2, ..., m, where $\delta_{ij} \epsilon \{0, 1\}$, and where K is an upper bound on $|\underline{w}_i \cdot \underline{A}_j|$ for all $\underline{A}_j$ and $\underline{w}_i$. Therefore $\delta_{ij} = 1$ if $\underline{w}_i \cdot \underline{A}_j > 0$; otherwise $\delta_{ij} = 0$.

For all $\underline{A}_j \epsilon T$ we require the following conditions. Let $\gamma_{ij} = 1$ if $\underline{w}_i \cdot \underline{A}_j > 0$; otherwise $\gamma_{ij} = 0$. Let

$$\sum_{j \epsilon \widetilde{T}} \gamma_{ij} = \beta_i \tag{3*}$$

where $\widetilde{T} = \{j | \underline{A}_j \epsilon T\}$. Then β_i is the number of $\underline{A}_j$'s which satisfy the inequality $\underline{w}_i \cdot \underline{A}_j > 0$. Let $\sigma_i = 1$ if $\beta_i = \tau$, $\sigma_i = 0$ otherwise. Note that $\beta_i \leq \tau$.

The objective function is

$$\sum_{i=1}^{m} \sigma_i = z \text{ (min)}. \tag{4*}$$

In order to ensure that condition b) holds, we require that

$$\sum_{i=1}^{m} (1 - \delta_{ij})\sigma_i \geq 1 \tag{5}$$

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* The equations marked with an asterisk are part of the integer program.

for every j such that $\underline{A}_j \epsilon F$. To satisfy this inequality, for say $j = j_o$, there must exist an $i(j_o)$, such that $\sigma_{i(j_o)} = 1$ and $\delta_{i(j_o)j_o} = 0$. This condition states that the $i(j_o)$ row of $\underline{w}$ is in $\underline{w}^M$ and that $\underline{w}_{i(j_o)} \cdot \underline{A}_{j_o} \leq 0$. However, $\underline{w}_i \cdot \underline{A}_j > 0$ for all $\underline{A}_j \epsilon T$.

The inequalities required to define γ_{ij} are

$$\frac{\underline{w}_i \cdot \underline{A}_j}{1 + K} \leq \gamma_{ij} < 1 + \frac{\underline{w}_i \cdot \underline{A}_j}{1 + K} . \qquad (6^*)$$

σ_i is defined by the inequality

$$\sigma_i \leq \beta_i / \tau . \qquad (7^*)$$

Finally, in order to convert (5) into a linear form, we let ϕ_{ij} be defined by the inequalities

$$0.5[1 + \sigma_i - \delta_{ij}] \geq \phi_{ij} \geq \sigma_i - \delta_{ij}. \qquad (8^*)$$

Hence $\phi_{ij} = (1 - \delta_{ij})\sigma_i$ and we now require that

$$\sum_{i=1}^{m} \phi_{ij} \geq 1 \qquad (9^*)$$

for every j such that $\underline{A}_j \epsilon F$.

This completes the integer linear programming formulation for finding the $\underline{w}^M$ matrix, and hence the $\| \delta^M \| + 1$ realization of G. The implementation of G is shown in Fig. 1, where the indices are now with reference to $\underline{w}^M$ rather than to the $\underline{w}$ matrix.

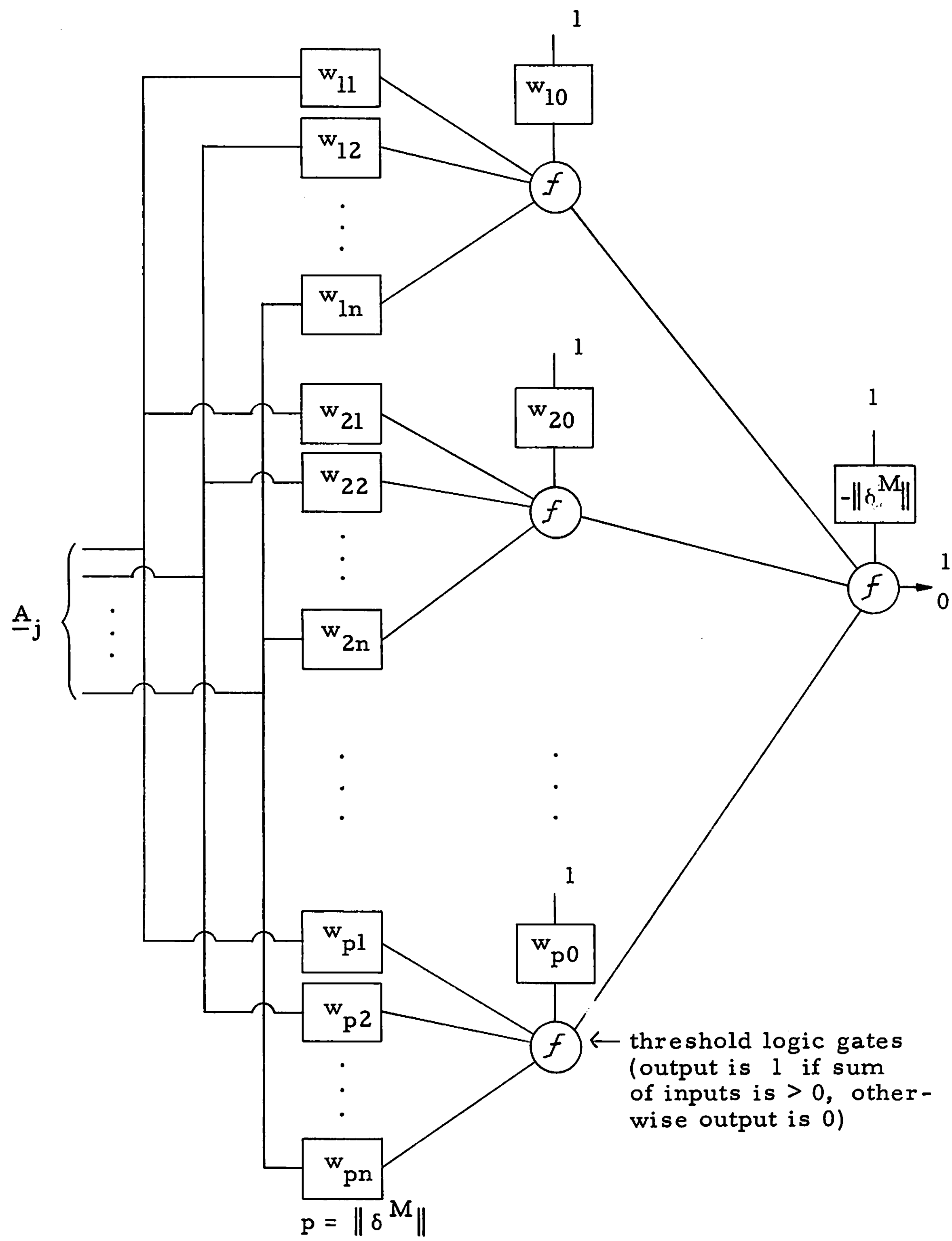

Fig. 1. $\|\delta^M\| + 1$ realization of G.

Another procedure for finding $\underline{w}^M$ is the iterative approach of Cameron. That is, one can assume a $\underline{w}$ system, where on the i-th iteration $\|\underline{w}\| = s(i)$. If no feasible solution exists, the procedure is repeated with $s(i + 1) = s(i) + 1$. Begin with $s(1) = 1$. The final $\underline{w}$ is $\underline{w}^M$.

We will now introduce a few circuit constraints into the formulation. In the following, it is assumed that the iterative approach is being employed. Let $w_{ik} \leq \alpha_{ik}$ and $-w_{ik} \leq \alpha_{ik}$. Hence $|w_{ik}| \leq \alpha_{ik}$. It is usually the case that the sum of the absolute value of the weights associated with a threshold gate must be less than or equal to some upper bound (UB). This condition can be included in the program by the addition of the constraints

$$\sum_{k=0}^{n} \alpha_{ik} \leq \text{UB}.$$

Also, if it is desired to find the minimal integer realization of G, then the objective function becomes

$$\sum_{i=1}^{m} \sum_{k=0}^{n} \alpha_{ik} = z\ (\min).$$

In the final solution we have $\alpha_{ik} = |w_{ik}|$.

Finally, it may be desirable to restrict the number of inputs to the threshold gates to p or less, where $p \leq n$. Note that Cameron's N-realization uses $n+N-1$ inputs to the last gate.

Let

$$\alpha_{ik} \geq \nu_{ik} \geq \alpha_{ik}/L$$

where L is an upper bound on α_{ik}. Then $\nu_{ik} = 0$ if $\alpha_{ik} = 0$ ($|w_{ik}| = 0$)

and $\nu_{ik} = 1$ if $\alpha_{ik} > 0$ ($|w_{ik}| > 0$). Then the restriction of p or less inputs is realized by the inequality

$$\sum_{k=0}^{n} \nu_{ik} \leq p.$$

If the i-th variable can be used as an input in q or less threshold gates, then the inequality

$$\sum_{i=1}^{m} \nu_{ik} \leq q$$

is required. Note that due to these additional constraints, it may be necessary to use R + 1 gates, where $R > \| \delta^M \|$. Finally, the previous conditions can be included, with some modification, in the noniterative approach first presented. It may be necessary to take $m > \beta$.

REFERENCE

1. S. H. Cameron, "The generation of minimal threshold nets by an integer program," IEEE Trans. on Electronic Computers, Vol. EC-13, No. 3, pp. 299-302; June 1963.