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SIMPLIFICATION OF THE INTEGER PROGRAMMING FORMULATION OF THE COVERING PROBLEM

by

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ABSTRACT

In this paper the covering problem, examples of which can be found in many fields of study, is first defined. This problem is equivalent to the following integer program: find $P_j \epsilon \{0, 1\}$, such that

$$\sum_{j=1}^{n} a_{ij} P_j \geq 1 \qquad \text{(I)}$$

for $i=1, 2, \ldots, m$, and such that $\sum_{j=1}^{n} c_j P_j = Z \text{ (min)}$, where $a_{ij} \epsilon \{0, 1\}$.

The purpose of this paper is to look into various procedures for reducing the number of inequalities and variables in the system of constraints (I) prior to solving for the optimal solution. A new procedure for combining sets of inequalities, called proper replacement reduction, is outlined. Theorems dealing with the properties of (I) and the proper replacement system are presented, along with two synthesis procedures and a partial table of pair-wise proper replacement systems. A correspondence between (I) and Boolean threshold switching functions is shown.

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THE COVERING PROBLEM

Given a set V of m elements $V = \{v_1, \ldots, v_m\}$ and a set P of n sets $P = \{p_1, \ldots, p_n\}$, where for each $v_i \in V$ there exists a p_j such that $v_i \in p_j$. Associate with each p_j a constant cost c_j.

The covering problem consists of a) finding a set P' such that $P' \subseteq P$ and where for each $v_i \in V$ there exists a p_j such that $v_i \in p_j$ and $p_j \in P'$, b) and where the costs associated with the elements in P' minimize some linear objective function.

There are numerous examples of covering type problems. For example, let the v_i represent demands and the p_j represent resources. Then it is desired to schedule the use of the resources such that each demand is met and the total cost is minimal. Also, the minimization of Boolean switching functions is a covering problem.

The covering problem will now be formulated as an integer linear program (I. L. P.), which can thus be solved by employing Gomory's[1] algorithm.

Let $a_{ij} = 1$ if $v_i \in p_j$, otherwise $a_{ij} = 0$. Associate with each element p_j a bi-valued variable $P_j \in \{0, 1\}$, where $P_j = 1$ implies that $p_j \in P'$, and $P_j = 0$ implies that $p_j \notin P'$. The necessary and sufficient constraints for the I. L. P. are $P_j \leq 1$ and $\sum_{j=1}^{n} a_{ij} P_j \geq 1$, for $i = 1, 2, \ldots, m$.

One possible objective function is $\sum_{j=1}^{n} c_j P_j = Z\,(\min)$. In matrix notation we wish to find a $\underline{P}$ with integer components which satisfies

$$\underline{P} \leq \underline{1} \qquad \text{(1a)}$$

$$\underline{A}\underline{P} \geq \underline{1} \qquad \text{(1b)}$$

$$\underline{c} \cdot \underline{P} = Z\,(\min) \qquad \text{(1c)}$$

where $\underline{A} = [a_{ij}]$ is an $m \times n$ (0, 1) matrix, $\underline{P} = (P_1, \ldots, P_n)$ is a column vector where the variables $P_j \epsilon \{0, 1\}$, $\underline{c} = (c_1, \ldots, c_n)$ is a row vector, and $\underline{1} = (1, \ldots, 1)$ is an $m \times 1$ column vector. In all that follows, we will assume that $c_j > 0$ for all j. Our primary interest will be to simplify the system of inequalities (1b) prior to employing Gomory's I. L. P. algorithm. This is desirable in order to 1) reduce the size of large problems so that they will fit the bounds imposed by the I. L. P. routine, and 2) decrease the computation time. We will first review a few system properties and elementary simplification techniques, after which the concept of proper replacement reduction will be introduced.

A. SYSTEM PROPERTIES

<u>Theorem</u>: In the optimal solution of the L. P. defined by (1b) and (1c) with all $c_j > 0$ we have $0 \leq P_j \leq 1$ for all j. Therefore the inequalities (1a) are not required.

<u>Theorem</u>: All integer solutions to (1b) and (1c) are vertices of the convex hull defined by (1b) and (1a).

Finally, through examples it can be shown that all optimal solutions to (1) are not always integer solutions.

B. ELEMENTARY SIMPLIFICATION TECHNIQUES

1. If $a_{ij} = 1$ and $a_{ik} = 0$ for all $k \neq j$, then $P_j = 1$. Hence the j-th variable can be deleted from the system, as well as all inequalities q for which $a_{qj} = 1$.

2. Let $\Gamma \subseteq \Delta \subseteq \{1, 2, \ldots, n\}$. If $a_{\gamma j} = 1$ iff $j \epsilon \Gamma$ and $a_{\delta j} = 1$ iff $j \epsilon \Delta$, then the δ-th constraint in (1b) may be deleted without affecting the results.

3. Let $\Theta \subseteq \Omega \subseteq \{1, 2, \ldots, m\}$. Let $a_{i\theta} = 1$ iff $i \epsilon \Theta$ and $a_{i\omega} = 1$ iff $i \epsilon \Omega$. If $c_\theta > c_\omega$, then $P_\theta = 0$ in all optimal solutions. Hence the variable P_θ may be deleted from (1b) and (1c).

If $c_\theta = c_\omega$ and $\vdash\!\Theta\!\dashv \subset \Omega$, we set $P_\theta = 0$ since P_ω "does more for us" than P_θ does. If $\vdash\!\Theta\!\dashv = \Omega$, either P_θ or P_ω may be deleted.

Cobham[2] has shown that the end result after exhaustively applying these simplification procedures results in a unique system, independent of the order in which the procedures are applied. A new proof of this fact is obtained here, as a result of corollary C-5.

At this point, the system may be solved by employing I. L. P. However, it is possible to still further reduce the number of constraints by making use of the fact that for all j, $P_j \epsilon \{0, 1\}$ in all solutions of (1b) and (1c).

C. PROPER REPLACEMENT (P.R.) REDUCTION (THEORY)

This section deals with new procedures for combining linear inequalities.

Definitions:

1. Matrix $\underline{A}$ is said to be in reduced form if the application of the elementary simplification techniques 1 and 2 leads to no further reductions. Note that a necessary and sufficient condition for a matrix $\underline{A}$ to be reduced is that

   a) every row of $\underline{A}$ contains at least two 1's.

   b) for each possible pair (i_1, i_2), $i_1 \neq i_2$, there exists a j_1 and a j_2 such that $a_{i_1j_1} = (1 - a_{i_2j_1}) = 1$ and

   $a_{i_1j_2} = (1 - a_{i_2j_2}) = 0$.

2. The i-th row of a matrix, say $\underline{A}$, is denoted by $\underline{a}_i = (a_{i_1}, a_{i_2}, \ldots, a_{i_n})$.

In the following definitions and theorem, we have $\underline{P} = (P_1, \ldots, P_n)$, where $P_j \epsilon \{0, 1\}$ only.

3. Definition of Proper Replacement (P. R.): Given the α system $\underline{B}\underline{P} > \underline{b}$, where $\underline{B} = [b_{ij}]$ is an $m' \times n$ constant matrix ($\|\alpha\| = m'$), $\underline{b} = (b_1, \ldots, b_{m'})$ is a constant column vector, and $\mathcal{A} = \{\underline{P} \mid \underline{B}\underline{P} \geq \underline{b}\}$ is the solution space associated with α. Similarly let the δ system be $\underline{D}\underline{P} \geq \underline{d}$, where $D = [d_{ij}]$ is an $m'' \times n$ constant matrix ($\|\delta\| = m''$), $\underline{d} = (d_1, \ldots, d_{m''})$ is a constant column vector, and let $\mathcal{D} = \{\underline{P} \mid \underline{D}\underline{P} \geq \underline{d}\}$. Then the δ system is said to be a P.R. system for α iff $\|\delta\| < \|\alpha\|$, and $\mathcal{A} = \mathcal{D}$. Note that the P.R. relation is transitive.

All of the systems considered will either be of the form given by

$$\underline{A}\,\underline{P} \geq \underline{1} \tag{1b}$$

or a P.R. to some system given by (1b), where $\underline{A}$ is in reduced form.

For example, the δ system

$$P_1 + P_2 + P_3 \geq 2 \tag{2}$$

is a P.R. for the α system

$$\begin{aligned} P_1 + P_2 \phantom{{}+P_3} &\geq 1 \\ P_2 + P_3 &\geq 1 \\ P_1 \phantom{{}+P_2} + P_3 &\geq 1 . \end{aligned}$$

Note that for the objective function

$$c_1P_1 + c_2P_2 + c_3P_3 = Z \text{ (min)} \tag{3}$$

with $0 < c_1 < c_2 < c_3$, the vector $\underline{P} = (2, 0, 0)$ is an allowable optimal solution to the L.P. defined by (2) and (3). Hence, to ensure that for all j, $P_j \leq 1$, upper bounds are used in the simplex computational procedure, as outlined in Chapter 18 in Dantzig.[3] This procedure is not difficult to implement, does not significantly slow down the computation, and eliminates the necessity of including in the system the constraints $P_j \leq 1$, for $j = 1, 2, \ldots, n$.

Two systems of inequalities are equivalent if they have identical solution spaces. Note that if δ is a P.R. system for α, the α and δ systems need not be equivalent. For example $P_1 = P_2 = P_3 = 1/2$ does not satisfy the δ system (2), but does satisfy the α system.

Some properties of the α and δ systems are now derived.

Theorem 1: Let $\underline{P}' = (P_1', \ldots, P_n')$ and $\underline{P}'' = (P_1'', \ldots, P_n'')$ be two constant vectors, with $P_{j_0}' = 0$, $P_{j_0}'' = 1$, $P_j' = P_j''$ for all $j \neq j_0$. Then $\underline{P}' \epsilon\, \mathcal{D} \Rightarrow \underline{P}'' \epsilon\, \mathcal{D}$.

This result is a direct consequence of the fact that the δ system is a P.R. for some α system of the form (1b).

Theorem 2: Let $\underline{A}^i$ be the matrix $\underline{A}$ with the i-th row of $\underline{A}$ deleted. Then for the systems $\underline{A}\underline{P} \geq \underline{1}$ and $\underline{A}^i\underline{P} \geq \underline{1}$, we have $\mathcal{A} \subset \mathcal{A}^i \triangleq \{\underline{P} \mid \underline{A}^i\underline{P} \geq \underline{1}\}$. This result follows from the fact that $\underline{A}$ is reduced.

Theorem 3: If δ is a P.R. to some system of the form (1b), then there exist no vectors $\underline{P}^a$, $\underline{P}^b$, $\underline{P}^c$ and $\underline{P}^d$ and no i such that

$$\underline{d}_i \cdot \underline{P}^a < d_i \tag{4}$$

$$\underline{d}_i \cdot \underline{P}^b < d_i \tag{5}$$

$$\underline{D}\,\underline{P}^c \geq \underline{d}, \text{ i.e., } \underline{P}^c \epsilon\, \mathcal{D} \tag{6}$$

$$\underline{D}\,\underline{P}^d \geq \underline{d}, \text{ i.e., } \underline{P}^d \epsilon\, \mathcal{D} \tag{7}$$

and $$\underline{P}^c + \underline{P}^d \leq \underline{P}^a + \underline{P}^b. \tag{8}$$

Proof: Assume δ exists, and that (4) through (8) are satisfied. Adding (4) and (5) gives $\underline{d}_i \cdot (\underline{P}^a + \underline{P}^b) < 2d_i$. Adding the i-th rows of (6) and (7) we have

$$\underline{d}_i \cdot (\underline{P}^c + \underline{P}^d) \geq 2d_i. \tag{9}$$

Now δ is the P.R. system for some system α of the form (1). Hence $\underline{P}^c$ and $\underline{P}^d$ are elements of $\mathcal{A}$ and $\mathcal{D}$. If $P_j^a + P_j^b > P_j^c + P_j^d$, then by Theorem 1, P_j^c or P_j^d or both can be changed from 0 to 1. We again call the new vectors formed by this change $\underline{P}^c$ and $\underline{P}^d$, and they are still solutions to the α and δ systems. By repeated application of this operation we can construct the new vectors such that $\underline{P}^a + \underline{P}^b = \underline{P}^c + \underline{P}^d$, hence $\underline{d}_i \cdot (\underline{P}^c + \underline{P}^d) = \underline{d}_i \cdot (\underline{P}^a + \underline{P}^b) < 2d_i$ which contradicts (9). Hence if δ exists, no such set of vectors $\underline{P}^a$, $\underline{P}^b$, $\underline{P}^c$ and $\underline{P}^d$ exists.

Definition: Let the single inequality $\sum_{j=1}^{n} d_{1j} P_j \geq b_1$ be the δ system for α. Then this system is said to be minimal integer if there does not exist another δ system for α of the form $\sum_{j=1}^{n} d_{1j}' P_j \geq b_1'$, where b_1, b_1', d_{1j} and d_{1j}' for $j = 1, \ldots, n$ are all integers, and $|b_1'| + \sum_{j=1}^{n} |d_{1j}'| < |b_1| + \sum_{j=1}^{n} |d_{1j}|$.

Theorem 4: Let α consist of the two inequalities*

$$P_1 + P_2 + \ldots + P_s \qquad\qquad + P_{t+1} + \ldots + P_u \geq 1 \tag{10}$$

$$P_{s+1} + \ldots + P_t + P_{t+1} + \ldots + P_u \geq 1 \tag{11}$$

where $u \leq n$.

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* Inequalities (10) and (11) represent the most general form of a pair of inequalities from $\underline{A}\,\underline{P} \geq \underline{1}$, except for the labelling of the variables.

Case 1. If $s + 1 = t$, then a minimal integer P.R. exists, and it is

$$P_1 + P_2 + \dots + P_s + sP_t + (s+1)P_{t+1} + (s+1)P_{t+2} + \dots + (s+1)P_u \geq (s+1).$$

Case 2. If $s \geq 2$ and $s + 2 \leq t$, then no P.R. exists.

Proof: Assume the P.R. to be of the form $\sum_{j=1}^{n} a_j P_j \geq b$.

Case 1. We require $\sum_{j=1}^{s} a_j < b$, since the evaluation $P_1 = P_2 = \dots = P_s = 1$, $P_t = P_{t+1} = \dots = P_u = 0$ does not satisfy (11). However, we require that $a_{s+1} + a_j \geq b$, for $j \in \{1, 2, \dots, s\}$, and also that $a_{s+1} < b$. Therefore these a_j cannot be zero, and it is seen that for a minimal integer solution, $a_j = 1$ for $j \in \{1, 2, \dots, s\}$, hence $b = s + 1$ and $a_{s+1} = s$. Also $a_j = b = s + 1$ for $j \in \{t+1, \dots, u\}$ and $a_j = 0$ for $j \in \{u+1, u+2, \dots, n\}$.

Case 2. We require $a_1 + a_s < b$ and $a_{s+1} + a_t < b$. Adding we have $a_1 + a_s + a_{s+1} + a_t < 2b$. We require $a_1 + a_{s+1} \geq b$ and $a_s + a_t \geq b$. Adding we have $a_1 + a_s + a_{s+1} + a_t \geq 2b$ which contradicts the previous statement. Hence the P.R. is not realizable.

Theorem 4 states that in a system of the form (1), a pair of inequalities will not have a P.R. iff they are of the form

$$\begin{aligned} \dots + 1P_i + \dots + 1P_j + \dots + 0P_k + \dots + 0P_\ell + \dots &\geq 1 \\ \dots + 0P_i + \dots + 0P_j + \dots + 1P_k + \dots + 1P_\ell + \dots &\geq 1. \end{aligned} \tag{12}$$

Corollary 1: For a system α defined by (1), if there exists a set of column indices J and two rows i_1 and i_2 such that for all $j \in J$, $a_{i_1 j} = a_{i_2 j} = 1$ and $\sum_{j=1}^{n} a_{i_1 j} \leq ||J|| + 1$, then a P.R. system δ exists.

The following theorem gives a lower bound for the number of inequalities in δ.

Theorem 5: Given the α system $\underline{A}\,\underline{P} \geq \underline{1}$. Let $U = \{i \mid \forall\, k \neq i$, inequalities i and k in α do not have a P.R.$\}$, where $\| U \| \triangleq K \leq m$. Then if a P.R. system δ exists, we have $\| \delta \| \geq K$.

Proof: It is obvious that if $K < m$, then a δ exists. If δ exists and $\| \delta \| < K - 1$, then by addition of a sufficient number of any of the inequalities in α, we can make $\| \delta \| = K - 1$. Hence, assume $\| \delta \| = K - 1$. For each $i \in U$ let $\underline{P}^i = (P_1{}^i, \ldots, P_n{}^i)$, where $P_j{}^i = 0$ if $a_{ij} = 1$, otherwise $P_j{}^i = 1$. Hence $\underline{a}_i \cdot \underline{P}^i = 0$ and $\underline{P}^i \notin \mathcal{D}$.

Now for all $k \neq i$, since inequalities i and k do not have a P.R., we have from Theorem 4 that there exists j_{k_1} and j_{k_2} such that

$a_{ij_{k_1}} = (1 - a_{kj_{k_2}}) = a_{ij_{k_1}} = (1 - a_{ij_{k_2}}) = 0$. Hence $\underline{a}_k \cdot \underline{P}^i \geq 2$ for all $k \neq i$.

Now, since $\underline{P}^i \notin \mathcal{D}$, there exists an $\ell \ni \underline{d}_\ell \cdot \underline{P}^i < d_\ell$. Now there are K conditions of this type, corresponding to the K elements of U, and only K - 1 choices for ℓ. Hence there exist an i_1, i_2 and an ℓ_0 such that $\underline{d}_{\ell_0} \cdot \underline{P}^{i1} < d_{\ell_0}$ and $\underline{d}_{\ell_0} \cdot P^{i2} < d_{\ell_0}$. Relabel the variables P_1, P_2, $\ldots$, P_n so that $\underline{P}^{i1} = (1, 1, 0, 0, P_5{}^{i1}, P_6{}^{i1}, \ldots, P_n{}^{i1})$ and $\underline{P}^{i2} = (0, 0, 1, 1, P_5{}^{i2}, \ldots, P_n{}^{i2})$. Consider the vector $\underline{P}^c \triangleq (1, 0, 1, 0, P_5{}^{i1}, \ldots, P_n{}^{i1})$. Since $P_2{}^c = 0$, we have $\underline{a}_k \cdot \underline{P}^c \geq 1$ for all $k \neq i$, rather than ≥ 2. But since $P_3{}^c = 1$, we have $\underline{a}_{i_1} \cdot \underline{P}^c = 1$. Hence $\underline{P}^c \in \mathcal{D}$. Similarly, if $\underline{P}^d \triangleq (0, 1, 0, 1, P_5{}^{i2}, \ldots, P_n{}^{i2})$, then $\underline{P}^d \in \mathcal{D}$. But by construction, $\underline{P}^{i1} + \underline{P}^{i2} = \underline{P}^c + \underline{P}^d$; and by Theorem 3 with $i = \ell_0$, $i_1 = a$, $i_2 = b$, δ does not exist. Hence we have $\| \delta \| \geq K$.

Corollary 2: If $K = m$ then a P.R. system δ does not exist.

Proof: By definition $\|\delta\| < \|\alpha\| = m$. From Theorem 5, $\|\delta\| > K = m$ which is a contradiction.

Note that K cannot equal $m-1$, since if U contains $m - 1$ elements, it also contains m elements.

Theoream 6: Assume a P.R. system δ exists for the α system given by (1), with $U = \{i\}$ and hence $K = 1$. Then in the system there exists a k such that if $a_{ij_2} > a_{ij_1}$, then $d_{kj_2} > d_{kj_1}$, and if $d_{kj_2} > d_{kj_1}$, then $a_{ij_1} \le a_{ij_2}$.

Proof: Let $\underline{P}^i = (P_1^{\,i}, \ldots, P_n^{\,i})$ where $P_j^{\,i} = 0$ if $a_{ij} = 1$, $P_j^{\,i} = 1$ otherwise. Hence $\underline{a}_i \cdot \underline{P}^i = 0$ and for all $k \ne i$, $\underline{a}_k \cdot \underline{P}^i \geq 2$ since $\underline{A}$ is reduced. Since $\underline{P}^i \notin \mathcal{A}$ or $\mathcal{D}$ we have that there exists a k such that $\underline{d}_k \cdot \underline{P}^i < d_k$. Let $\underline{P}' = (P_1', \ldots, P_n')$ where $P_{j_1}' = 0$ for some j_1 such that $a_{ij_1} = 0$, $P_{j_2}' = 1$ for some j_2 such that $a_{ij_2} = 1$, and $P_j' = P_j^{\,i}$ for all $j \ne j_1$ and $j \ne j_2$. Now by construction $\underline{A}\,\underline{P}' \geq \underline{1}$, hence $\underline{d}_k \cdot \underline{P}' \geq d_k$. But $\underline{d}_k \cdot \underline{P}' - \underline{d}_k \cdot \underline{P}^i = d_{kj_2} - d_{kj_1} > 0$, hence $d_{kj_2} > d_{kj_1}$ for $(a_{ij_2} = 1) > (a_{ij_1} = 0)$. Therefore, if $d_{kj_2} > d_{kj_1}$, we have that either $a_{ij_2} = (1 - a_{ij_1}) = 1$, or $a_{ij_1} = a_{ij_2} = 1$ or 0.

Theorem 7: For a given α system of the form (1 ,, let the i-th inequality in its P.R. δ system be

$$\sum_{j=1}^{n} d_{ij} P_j \geq d_i \tag{13}$$

where $d_i > 0$. Then

$$\text{a)} \quad \sum_{j=1}^{n} d_{ij} > d_i \tag{14}$$

$$\text{b)} \quad \sum_{j=1}^{n} d_{ij} - \max_{j} (d_{ij}) = \sum_{\substack{j=1 \\ j \neq j_M}}^{n} d_{ij} \geq d_i \tag{15}$$

where for all j, $d_{ij_M} \geq d_{ij}$.

<u>Proof:</u>

a) Let $\underline{P}' = (1, \ldots, 1)$ which is a solution to α. Substituting into (13) we get $\sum_{j=1}^{n} d_{ij} \geq d_i$. Assume $\sum_{j=1}^{n} d_{ij} = d_i$. Since $d_i > 0$, there exists a j_0 such that $d_{ij_0} > 0$. Hence $\sum_{\substack{j=1 \\ j \neq j_0}}^{n} d_{ij} < d_i$, and therefore $\underline{P}^{j_0} = (\underset{1}{1}, 1, \underset{\ldots}{\ldots}, 1, \underset{j_0}{0}, 1, \underset{\ldots}{\ldots}, \underset{n}{1}) \notin \mathcal{D}$. But $\underline{P}^{j_0} \in \mathcal{A}$ since every row in $\underline{A}$ has at least two 1 entries. Since $\mathcal{D} = \mathcal{A}$ the assumption that $\sum_{j=1}^{n} d_{ij} = d_i$ is contradicted, and hence $\sum_{j=1}^{n} d_{ij} > d_i$.

b) Letting $j_0 = j_M$, the desired result is obtained.

<u>Definition:</u> A Boolean function $\mathcal{P}(P_1, \ldots, P_n)$ is said to be a <u>threshold function</u> if there exist real numbers $d_{11}, d_{12}, \ldots, d_{1n}$, and d_1 so that

$$(\mathcal{P} = 1) \Longleftrightarrow \sum_{j=1}^{n} d_{1j} P_j \geq d_1 \tag{16}$$

or equivalently

$$(\mathcal{P}=0) \Longleftrightarrow \sum_{j=1}^{n} d_{1j} P_j < d_1 \tag{17}$$

where $P_j \epsilon \{0, 1\}$. The inequality in (16) is said to be the 1-realization of $\mathcal{P}$.

Corresponding to the α system defined by (1) one can associate a Boolean function $\mathcal{P}_\alpha$ having the solution space $\mathcal{A}$. To construct such a function, let the Boolean disjunctive clause $w_i \triangleq \bigvee_{j=1}^{n} a_{ij} P_j$. Hence w_i is true ($w_i = 1$) for all $\underline{P}$ satisfying the i-th inequality of α. Therefore $\mathcal{P}_\alpha = \bigwedge_{i=1}^{m} w_i$, i.e., $\mathcal{P}_\alpha$ is the common intersection (logical conjunction) of the set of solutions satisfying the 1-st, 2-nd, ..., and m-th inequalities of $\underline{A}$. Also, we have $\overline{\mathcal{P}}_\alpha = \bigvee_{i=1}^{m} \overline{w}_i$, where $\overline{w}_i = \bigwedge_{j=1}^{n} a_{ij} \overline{P}_j$ (a barred variable indicates logical negation).

Now $\mathcal{A}$ is a subset of the vertices of an n-dimensional unit cube. By definition a Boolean function is a subset of the vertices of an n-dimensional unit cube. Hence $\mathcal{P}_\alpha \equiv \mathcal{A}$.

Since $\overline{\mathcal{P}}_\alpha$ is true for all $\underline{P} \notin \mathcal{A}$ it has the solution space $\overline{\mathcal{A}} \triangleq \{\Omega - \mathcal{A}\}$, where Ω is the entire space of (0, 1) n-tuples.

For example, corresponding to the matrix $\underline{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

we have $\mathcal{P}_\alpha = (P_1 \vee P_2)(P_1 \vee P_3 \vee P_4)(P_2 \vee P_4)$ and $\overline{\mathcal{P}}_\alpha = \overline{P}_1\overline{P}_2 \vee \overline{P}_1\overline{P}_3\overline{P}_4 \vee \overline{P}_2\overline{P}_4$.

It is seen that $\mathcal{P}_\alpha$ is a positive function and hence is unate. This is a necessary condition for $\mathcal{P}_\alpha$ to be a threshold function. From the definition of threshold functions and P.R. systems we have:

Theorem 8: If α is of the form (Ib) and has a P.R. system δ, then $\|\delta\| = 1$ iff $\mathcal{P}_\alpha$ is a threshold function.

In Table II of Winder,[4] all threshold functions of six or less variables are listed. From this table, it is quite simple to find δ, with $\|\delta\| = 1$, if $\mathcal{P}_\alpha$ is a threshold function and $n \leq 6$. If $\mathcal{P}_\alpha$ is not in the table, then it is known that $\|\delta\| \geq 2$.

From Theorem 4 it follows that all functions of the form $\mathcal{P}_\alpha = (P_1 \vee P_2 \vee \ldots \vee P_s)\ P_{s+1} \vee P_{s+2} \vee \ldots \vee P_u$ are threshold.

Definitions: A Boolean function is negative if every variable in it appears in its negated form. A clause is the logical conjunction of Boolean variables. A negative Boolean function $\bar{\mathcal{P}}$ is in reduced normal form iff $\bar{\mathcal{P}} = \bigvee_{i=1}^{m} \bar{w}_i$, and where there exist no two negative clauses $\bar{w}_{i_1}$ and $\bar{w}_{i_2}$ of $\bar{\mathcal{P}}$ where $\bar{w}_{i_1} = \bar{w}_{i_2}\bar{w}$ and where $\bar{w}$ is a negative clause not identically 1. Note that if this condition is violated, then we have $\bar{w}_{i_1} \vee \bar{w}_{i_2} = \bar{w}_{i_2}(\bar{w} \vee 1) \equiv \bar{w}_{i_2}$.

Theorem 9: If the α system is of the form (1), then $\bar{\mathcal{P}}_\alpha$ is in reduced normal form.

Proof: From condition b) of the definition of a reduced matrix we have that for each pair of row indices (i_1, i_2), $i_1 \neq i_2$, there exists a pair of column indices (j_1, j_2) such that $\bar{P}_{j_1}$ is a literal in $\bar{w}_{i_1}$ but not in $\bar{w}_{i_2}$, and $\bar{P}_{j_2}$ is a literal in $\bar{w}_{i_2}$ but not in $\bar{w}_{i_1}$. Hence there exists no $\bar{w}$ such that $\bar{w}_{i_1} = \bar{w}_{i_2}\bar{w}$.

Since the first condition of the definition of a reduced matrix is not used in the proof of Theorem 9, most of the following results will be true for the case where w_i consists of a single literal.

Theorem 10: If a negative (positive) Boolean function $\bar{\mathcal{P}}(\mathcal{P})$ is in reduced normal form, then every clause is an essential prime implicant (e.p.i.).

Proof: The fact that $\bar{\mathcal{P}} = \bigvee_{i=1}^{m} \bar{w}_i$ is in reduced normal form is equivalent to saying that each clause in $\bar{\mathcal{P}}$ is a prime implicant (p.i.). To show that each clause is an e.p.i. we construct a minterm (vertex) v_i such that $(v_i = 1) \Rightarrow (\bar{w}_i = 1)$ and $(v_i = 1) \Rightarrow (\bar{w}_k = 0)$ for all $k \neq i$. If this can be done, then $\bar{w}_i$ is an e.p.i. Let $v_i = \bigwedge_{j=1}^{n} P_j^{\,i}$, where $P_j^{\,i} = \bar{P}_j$ if $\bar{P}_j$ is a literal in $\bar{w}_i$, and $P_j^{\,i} = P_j$ otherwise. Hence $(v_i = 1) \Rightarrow (\bar{w}_i = 1)$. Since all $\bar{w}_i$'s are p.i.'s, for all $k \neq i$ there exists a j_k such that $\bar{P}_{j_k}$ is a literal in $\bar{w}_k$ but not in $\bar{w}_i$. Hence $(v_i = 1) \Rightarrow (\bar{w}_k = 0)$. Since such a v_i can be constructed for all i, each $\bar{w}_i$ is an e.p.i.

Corollary 3: If $\bar{\mathcal{P}}$ is a negative (positive) reduced Boolean function, then it has a unique reduced normal form.

Corollary 4: For the α system defined by (1), there exists no matrix $\underline{A}' = [a_{ij}']$, $a_{ij}' \epsilon \{0, 1\}$, of dimension $m' \times n$, where $m' < m$, $\underline{A}'\underline{P} \geq 1$, and such that $\mathcal{A} = \mathcal{A}'$.

Proof: From Theorem 10, all m clauses of $\bar{\mathcal{P}}_\alpha$ are e.p.i.'s, and therefore represent the fewest number of clauses which can cover the elements of $\mathcal{A}$. Since each clause in $\bar{\mathcal{P}}_\alpha$ corresponds to a row in $\underline{A}$, Corollary 4 follows.

Corollary 5: Let the α system be of the form (1). Let $\underline{A}' = [a_{ij}']$ be an $m \times n$ matrix, with $a_{ij}' \epsilon \{0, 1\}$, $\underline{A}'\underline{P} \geq \underline{1}$, and $\mathcal{A}' = \mathcal{A}$. Then $\underline{A}'$ differs from $\underline{A}$ by at most an interchange of rows.

Proof: Since $\underline{A}$ is a reduced matrix, $\overline{\mathcal{P}}_\alpha$ is unique up to the order of its clauses. Since $\mathcal{Q}' = \mathcal{Q}$, every clause in $\overline{\mathcal{P}}_\alpha$ corresponds to a row in $\underline{A}'$, and the result follows.

As a consequence of Theorem 1 or from the fact that $\mathcal{P}$ is a positive function we have:

Corollary 6: Let $\mathcal{P}_j^1 \triangleq [\mathcal{P} \mid (P_j = 1)]$ and $\mathcal{P}_j^0 = [\mathcal{P} \mid (P_j = 0)]$. Then $\mathcal{P}_j^0 \subset \mathcal{P}_j^1$.

Theorem 11: Let the α system of the form (1) have a P.R. system δ. If there exists a $d_{ij} < 0$, then there also exists a P.R. system $\underline{H}\,\underline{P} \geq \underline{h}$, where $\underline{H} = [h_{ij}]$ is an $m''' \times n$ matrix, $m''' \leq m''$, and where $h_{ij} \geq 0$ for all i, j.

Proof: For simplicity, assume $d_{11} < 0$. Construct a system $\underline{D}'\underline{P} \geq \underline{d}'$, where for $i > 1$ the i-th rows of $\underline{D}$ and $\underline{D}'$ are identical, and $d_i = d_i'$. Let the first row of D' be $\underline{d}_1' = (0, d_{12}, d_{13}, \ldots, d_{1n})$, and set $d_1' = d_1 + |d_{11}|$. We now prove that $\mathcal{D}' = \mathcal{D}$.

Let $\mathcal{Q}_j^1 = \{\underline{P} \mid \underline{A}\,\underline{P} \geq \underline{1}$), and $P_j = 1\}$, and $\mathcal{Q}_j^0 = \{\underline{P} \mid \underline{A}\,\underline{P} \geq \underline{1}$, and $P_j = 0\}$. $\mathcal{D}_j^1$ and $\mathcal{D}_j^0$ are defined similarly.

From Theorem 1 we have that $\mathcal{Q}_j^0 \subset \mathcal{Q}_j^1$. Now $\mathcal{D}_j^1 = \mathcal{Q}_j^1$ and $\mathcal{D}_j^0 = \mathcal{Q}_j^0$ because $\mathcal{D}_j^1 \cup \mathcal{D}_j^0 = \mathcal{D}$, and $\mathcal{D}_j^1 \cap \mathcal{D}_j^0 = \phi$. The same holds for $\mathcal{Q}$. Let $\underline{P}_1^1 = (1, P_2, \ldots, P_n)$ and $\underline{P}_1^0 = (0, P_2, \ldots, P_n)$. Then $\underline{d}_1 \cdot \underline{P}_1^1 \geq d_1$ is equivalent to the inequality

$$\sum_{j=2}^{n} d_{1j} \cdot P_j \geq d_1 + |d_{11}| > d_1 \tag{18}$$

which is equivalent to the inequality $\underline{d}_1' \cdot \underline{P}_1^1 \geq d_1 + |d_{11}|$. Also $\underline{d}_1 \cdot \underline{P}_1^0 \geq d_1$.

is equivalent to the inequality

$$\sum_{j=2}^{n} d_{1j} P_j \geq d_1 \tag{19}$$

while $\underline{d}_1' \cdot \underline{P}_1^{\,0} \geq d_1 + |d_{11}|$ is equivalent to inequality (18). But (19) is a weaker constraint than (18), and since $\mathcal{D}_j^{\,0} \subset \mathcal{D}_j^{\,1}$, we can replace (19) by (18) without effecting the set of solutions in the solution space. Hence $\mathcal{D}' = \mathcal{D}$. Now by construction, the number of negative elements in $\mathcal{D}'$ is one less than that in $\mathcal{D}$. By repeating this process (i.e., by forming $\mathcal{D}''$, $\mathcal{D}'''$, etc.) a finite number of times, we finally obtain $\underline{H}$ and $\underline{h}$. We have $h_{ij} = d_{ij}$ if $d_{ij} \geq 0$ and $h_{ij} = 0$ if $d_{ij} < 0$, and $h_i = d_i + \sum_j |d_{ij}|$ where the sum is taken over only those j for which $d_{ij} < 0$.

Note that if $h_i \leq 0$, then the i-th inequality may be deleted since it is redundant, since for all $\underline{P}$, $\underline{h}_i \cdot \underline{P} \geq 0$. Hence all P.R. systems δ can be put into the form where for all i and j, $d_{ij} \geq 0$ and $d_i > 0$.

D. PROPER REPLACEMENT REDUCTION (SYNTHESIS)

In this section, various procedures for finding P.R. systems will be presented. If δ is the P.R. system for α, it would be desirable if $||\delta||$ were minimal. If $||\delta||_{min} = 1$, then finding δ is the same as finding the 1-realization of $\mathcal{C}_\alpha$.

By employing I.L.P., it is possible to determine the δ system such that $||\delta||$ is minimal. (Obviously if δ exists, then it can be written with all integral coefficients.) However, the resulting system required to be solved is much larger than the original system, and hence this procedure will not be considered here.

We now present a very simple way for determining a δ system from $\mathcal{F}_\alpha'$. This synthesis procedure is based on the following theorem.

Theorem 12: If under the evaluation $P_1 = P_2 = \ldots = P_s = 0$ and $\underbrace{P_{s+1} = P_{s+2} = \ldots = P_t}_{t-s \text{ terms}} = 1$, the Boolean expression $\mathcal{P}$ is reduced to $\mathcal{P}'$, which is a threshold function having a 1-realization

$$\sum_{j=t+1}^{n} d_{ij}P_j \geq d_i \qquad (20)$$

then the inequality which expresses this condition is

$$d_i[P_1 + \ldots + P_s] + \sum_{j=t+1}^{n} d_{ij}P_j \geq d_i. \qquad (21)$$

Proof: Consider the inequality

$$d_i[P_1+P_2+\ldots+P_s+(1-P_{s+1})+(1-P_{s+2})+\ldots+(1-P_t)] + \sum_{j=t+1}^{n} d_{ij}P_j \geq d_i. \qquad (22)$$

Note that the term $[\ldots]$ in (22) is zero only for the evaluation of the variables stated in the theorem. For this evaluation (22) reduces to (20). For all other evaluations of the variables P_1 through P_t, the term $[\ldots] \geq 1$, and hence (22) is satisfied regardless of the value of the variables P_{t+1} through P_n. But by Theorem 11, inequality (22) may be replaced by (21).

Synthesis Algorithm:

If $P_i = 0$, then $\mathcal{P}' = [\mathcal{P}| P_i = 0]$, else $\mathcal{P}'' = [\mathcal{P}| P_i = 1]$. If either $\mathcal{P}'$ or $\mathcal{P}''$ are threshold functions, then Theorem 1 can be employed. If $\mathcal{P}'$ or $\mathcal{P}''$ are not threshold, then the procedure is repeated on each of the nonthreshold functions in turn. The evaluation $\mathcal{P}|P_i$ is chosen so that P_i is a variable appearing in $\mathcal{P}$.

Example: Let

$$\underline{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\mathcal{P}_\alpha = (P_1P_2 \vee P_3P_4)P_5 \vee P_1P_3P_4$ which is not a threshold function. If $P_5 = 0$, then $[\mathcal{P}_\alpha | P_5 = 0] = P_1P_3P_4$ which is a threshold function with a 1-realization $P_1 + P_3 + P_4 \geq 3$. Otherwise $[\mathcal{P}_\alpha | P_5 = 1] = P_1P_2 \vee P_3P_4$ which is not a threshold function. If $P_1 = 0$, then $[\mathcal{P}_\alpha | P_5 = 1, P_1 = 0] = P_3P_4$ which is a threshold function with a 1-realization $P_3 + P_4 \geq 2$. Otherwise $P_1 = 1$ and $[\mathcal{P}_\alpha | P_5 = 1, P_1 = 1] = P_2 \vee P_3P_4$ which is a threshold function with a 1-realization $2P_2 + P_3 + P_4 \geq 2$. Therefore, from Theorem 12, we have the δ system

$$\begin{aligned} P_1 \qquad + P_3 + P_4 + 3P_5 &\geq 3 \\ 2P_1 \qquad + P_3 + P_4 \qquad &\geq 2 \\ 2P_2 + P_3 + P_4 \qquad &\geq 2. \end{aligned}$$

For the system $\underline{A}\,\underline{P} \geq \underline{1}$, with $\underline{A}$ defined above, we can combine rows 1 and 2, 3 and 4, and 5 and 6 according to Theorem 4, and obtain the following δ system,

$$\begin{aligned} P_1 + 2P_3 \qquad + P_5 &\geq 2 \\ P_2 \qquad + 2P_4 + P_5 &\geq 2 \\ 2P_1 \qquad + P_4 + P_5 &\geq 2. \end{aligned}$$

One procedure for determining a P.R. system δ for α is to select two inequalities from α and let them be an α' system. If δ' is a P.R. system for α', then by replacing α' by δ' in α we have δ. The process can be repeated until no δ' can be found. This procedure is based on the concept of pair-wise proper replacement reduction, and is implemented by using a universal table of pair-wise P.R.'s. The table consists of pairs of constraints and their P.R., which is a single constraint. By employing this precomputed table, the reduction of the number of constraints in a system reduces to a table look-up or search process. The P.R. may be determined by either graphical, analytic, or intuitive means.

The analytic procedure for finding the δ system is quite straightforward. Let the α system be $\sum_{j=1}^{n} a_{ij}P_j \geq b_i$ for i=1, 2, ..., m and the δ system be $\sum_{j=1}^{n} d_i P_j \geq d$. Then for all $\underline{P} \epsilon \mathcal{A}$ we have $\underline{d} \cdot \underline{P} \geq d$, and for all $\underline{P} \epsilon \overline{\mathcal{A}}$ we have $\underline{d} \cdot \underline{P} < d$. We have 2^n inequalities and only $n+1$ unknowns, namely $d_1, \ldots d_n$, d. There are therefore a great number of redundant inequalities. The minimal set of inequalities required in order to solve for the unknowns can be found from the "worse case" conditions; that is, from those cases where the number of P_j's equal to one in $\underline{P}$ is minimal, and yet the inequality is satisfied, and from those cases where the number of P_j's equal to one in $\underline{P}$ is maximal, and yet the inequality is not satisfied. This information is determined by inspecting the reduced normal forms of $\mathcal{P}_\alpha$ and $\overline{\mathcal{P}}_\alpha$.* In fact, each clause in $\left\{\begin{matrix}\overline{\mathcal{P}}_\alpha \\ \mathcal{P}_\alpha\end{matrix}\right\}$ corresponds to an $\left\{\begin{matrix} < \\ \geq \end{matrix}\right\}$ inequality. The $\underline{P}$ corresponding

*This procedure is analogous to the one given by Winder[4](p. 75) for finding the minimal set of inequalities for testing whether a function is threshold or not.

to the clause $w_i' = \bigwedge_{j=1}^{n} a_{ij}' P_j$ in $\mathcal{P}_\alpha$ is $\underline{P}^i = (a_{i1}', \ldots, a_{in}')$, and the $\underline{P}$ corresponding to the clause $\overline{w}_k = \bigwedge_{j=1}^{n} a_{kj} \overline{P}_j$ in $\overline{\mathcal{P}}_\alpha$ is $\underline{P}^k = (1 - a_{k1}, \ldots, 1 - a_{kn})$, where $a_{ij} \in \{0, 1\}$. These results follow from the fact that each clause is a prime implicant, and hence contains a minimal number of literals.

Example: Find the P.R. system for

$$\begin{aligned} P_1 + P_2 \quad\;\; &\geq 1 \\ P_2 + P_3 &\geq 1 \\ P_1 \quad\;\; + P_3 &\geq 1. \end{aligned}$$

Now $\mathcal{P}_\alpha = P_1P_2 \vee P_2P_3 \vee P_1P_3$ and $\overline{\mathcal{P}}_\alpha = \overline{P}_1\overline{P}_2 \vee \overline{P}_2\overline{P}_3 \vee \overline{P}_1\overline{P}_3$. Therefore the worse case test conditions in $\mathcal{Q}$ are (1, 1, 0), (0, 1, 1) and (1, 0, 1), and in $\overline{\mathcal{Q}}$ they are (0, 0, 1), (1, 0, 0), (0, 1, 0). If δ is of the form

$$d_1P_1 + d_2P_2 + d_3P_3 \geq d,$$

then we have

$$\begin{aligned} d_1 + d_2 \qquad\quad &\geq d \\ d_2 + d_3 \qquad &\geq d \\ d_1 \quad + d_3 \qquad &\geq d \\ d_1 \qquad\qquad + \epsilon_1 &\leq d \\ d_2 \qquad + \epsilon_2 &\leq d \\ d_3 + \epsilon_3 &\leq d \end{aligned}$$

where $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$. Let $\epsilon = \min(\epsilon_1, \epsilon_2, \epsilon_3)$, and scale the problem so that $\epsilon = 1$. Hence we get the system shown in the table below.

	d_1	d_2	d_3	d		
(1)	1	1	0	-1	≥ 0	
(2)	0	1	1	-1	≥ 0	
(3)	1	0	1	-1	≥ 0	
(4)	-1	0	0	1	≥ 1	
(5)	0	-1	0	1	≥ 1	
(6)	0	0	-1	1	≥ 1	
(7) = (1) + (4)	0	1	0	0	≥ 1	
(8) = (2)	0	1	1	-1	≥ 0	
(9) = (3) + (4)	0	0	1	0	≥ 1	
(10) = (5)	0	-1	0	1	≥ 1	
(11) = (6)	0	0	-1	1	≥ 1	
(12) = (7) + (10)		0	0	1	≥ 2	
(13) = (8) + (10)		0	1	0	≥ 1	
(14) = (9)		0	1	0	≥ 1	same as (13)
(15) = (11)		0	-1	1	≥ 1	
(16) = (12)			0	1	≥ 2	
(17) = (13) + (15)			0	1	≥ 2	same as (16)

We have $d \geq 2$. Try $d = 2$. Substituting into (4), (5) and (6) we have $-d_i \geq 1$, i.e., $d_i \leq 1$. Let $d_i = 1$ for $i = 1, 2, 3$, then since (1), (2) and (3) are satisfied we have that the P.R. is $P_1 + P_2 + P_3 \geq 2$.

The table of P.R.'s is given in terms of pairs of constraints having a P.R. consisting of a single constraint. The replacement indicator notation $N(\lambda, \beta) \rightarrow \gamma$ states that there are $N(N < n)$ variables in the two constraints being replaced by a P.R. such that the coefficient of all N of these variables in the first constraint is λ, the coefficient of these variables in the second constraint is β, and γ is the value of the coefficient of these variables in the P.R. If $\lambda = \beta = 0$, then $\gamma = 0$ and hence we do not mention such variables in the table. The notation $R(\lambda, \beta) \longrightarrow \gamma$ refers to the constants on the right hand side of the constraints. A P.R. can be completely described by a set of such replacement indicators. Note that if λ and β in each replacement indicator of a set describing a P.R. are interchanged, the result is still a P.R. relation.

As an example of this notation, the P.R.

$$\left.\begin{array}{l} P_1 + P_3 + P_5 + P_7 \geq 1 \\ P_5 + P_7 + P_8 \geq 1 \end{array}\right\} \overset{\text{P.R.}}{\Longrightarrow} P_1 + P_3 + 3P_5 + 3P_7 + 2P_8 \geq 3$$

would be indicated as

$$2(1, 0) \longrightarrow 1, \quad 2(1, 1) \longrightarrow 3, \quad 1(0, 1) \longrightarrow 2, \quad R(1, 1) \longrightarrow 3.$$

The set of replacement indicators defining a P.R. can easily be coded, i.e., reduced to a number. With these numbers listed in numerical order, and under appropriate headings, it is possible to take two constraints from the system being solved, code them, and search the table to see if a P.R. exists. If no P.R. is found, another pair is selected. If a P.R. is found, it replaces the two constraints, and the process continues. The process ends when no further P.R.'s can be found. Unfortunately, the final system of constraints is not unique, and another set of pairings of constraints may produce a final system of fewer constraints. However, the integer solution set is the same as the original system. Table 1 lists a few classes of P.R.'s.

Example:

$$\left.\begin{array}{ll} r_1: & P_1 \quad +P_3 \geq 1 \\ r_2: & \quad P_2+P_3 \geq 1 \end{array}\right\} \xRightarrow[\text{P.R.}]{\substack{\text{Case 1} \\ i=1}} P_1+P_2+2P_3 \geq 2$$

$$\left.\begin{array}{ll} r_3: & P_1+P_2 \quad +P_4 \geq 1 \\ r_4: & \quad P_4+P_5 \geq 1 \end{array}\right\} \xRightarrow[\text{P.R.}]{\substack{\text{Case 1} \\ i=2}} P_1+P_2+2P_3+3P_4 \geq 3$$

$$\left.\begin{array}{l} P_1+P_2+2P_3 \geq 2 \\ P_1+P_2+2P_3+3P_4 \geq 3 \end{array}\right\} \xRightarrow[\text{P.R.}]{\substack{\text{Case 10} \\ i=1,\ j=2}} P_1+P_2+2P_3+P_4 \geq 3$$

$$\left.\begin{array}{ll} r_5: & P_2 \quad +P_4+P_5 \geq 1 \\ & P_1+P_2+2P_3+P_4 \geq 3 \end{array}\right\} \xRightarrow[\text{P.R.}]{\text{Case 4}} 3P_1+4P_2+6P_3+4P_4+P_5 \geq 10$$

The system of inequalities r_i, $i=1, 2, \ldots, 5$ can be replaced by the inequality $3P_1 + 4P_2 + 6P_3 + 4P_4 + P_5 \geq 10$. The application of this pair-wise P.R.'s procedure has been found to be quite successful in reducing the number of constraints in a system.

TABLE I

Partial Table of Proper Replacements (all integer)

R(1, 1)

1. $i(1, 0) \longrightarrow 1$, $1(0, 1) \longrightarrow i$, $j(1, 1) \longrightarrow i + 1$, $R(1, 1) \longrightarrow i + 1$

All other inequalities of the form R(1, 1) do not have a P.R.

R(2, 1)

This case consists of all inequality pairs of the form

$$2P_{i_1} + P_{i_2} + P_{i_3} \geq 2 \qquad \text{and} \quad P_{j_1} + P_{j_2} + \ldots + P_{j_u} \geq 1.$$

2. $1(2,1) \longrightarrow 2(k+1)$, $2(1,0) \longrightarrow k$, $k(0,1) \longrightarrow 2$, $R(2,1) \longrightarrow 2(k+1)$ where if k is even, the values of the constants in the P.R. may be divided by 2.

3. $2(1,0) \longrightarrow 2k+1$, $2(1,1) \longrightarrow k+1$, $k(0,1) \longrightarrow 1$, $R(2,1) \longrightarrow 2k+2$. All other pairs do not have a P.R. These pairs are:

a) $1(2,0)$, $1(1,0)$, $1(1,1)$, $i(0,1)$, $R(2,1)$ for $i \geq 1$,

b) $1(2,0)$, $2(1,0)$, $i(0,1)$, $R(2,1)$ for $i \geq 2$.

<u>R(3, 1)</u>

4. $1(0,1) \longrightarrow 1$, $2(1,1) \longrightarrow 4$, $1(1,0) \longrightarrow 3$, $1(2,0) \longrightarrow 6$, $R(3,1) \longrightarrow 10$.

<u>R(2, 2)</u>

We first give the results for all pairs of the form

$$2P_{i_1} + P_{i_2} + P_{i_3} \geq 2 \quad \text{and} \quad 2P_{j_1} + P_{j_2} + P_{j_3} \geq 2.$$

5. $1(2,0) \longrightarrow 2$, $1(1,2) \longrightarrow 3$, $1(1,1) \longrightarrow 2$, $1(0,1) \longrightarrow 1$, $R(2,2) \longrightarrow 5$.

6. $1(2,2) \longrightarrow 3$, $1(1,0) \longrightarrow 1$, $1(1,1) \longrightarrow 1$, $1(0,1) \longrightarrow 1$, $R(2,2) \longrightarrow 3$.

7. $1(2,1) \longrightarrow 1$, $1(1,2) \longrightarrow 1$, $1(1,1) \longrightarrow 1$, $R(2,2) \longrightarrow 2$.

8. $1(2,1) \longrightarrow 3$, $1(1,0) \longrightarrow 1$, $1(1,1) \longrightarrow 2$, $1(0,2) \longrightarrow 2$, $R(2,2) \longrightarrow 5$.

9. $1(2,2) \longrightarrow 4$, $2(1,0) \longrightarrow 1$, $2(0,1) \longrightarrow 1$, $R(2,2) \longrightarrow 4$.

All other pairs do not have a P.R. These pairs are:

a) $1(2,0)$, $1(1,2)$, $1(1,0)$, $2(0,1)$, $R(2,2)$

b) $1(2,0)$, $2(1,0)$, $1(0,2)$, $2(0,1)$, $R(2,2)$

c) $1(2,0)$, $1(1,0)$, $1(1,1)$, $1(0,1)$, $1(0,2)$, $R(2,2)$

d) $1(2,0)$, $2(1,1)$, $1(0,2)$, $R(2,2)$.

TABLE 1 (Continued)

The previous cases can be generalized as follows:

for case 5, replace $l(1, 1) \to 2$ by $i(1, 1) \to 2$.
" " 6, " $l(2, 2) \to 3$ " $i(2, 2) \to 3$ and
" $l(1, 1) \to 1$ " $j(1, 1) \to 1$.
" " 7, " $l(1, 1) \to 1$ " $i(1, 1) \to 1$.
" " 8, " $l(1, 1) \to 2$ " $i(1, 1) \to 2.5$.*
" " 9, " $l(2, 2) \to 4$ " $i(1, 1) \to 4$.

$\underline{R(3, 2)}$
10. $i(2, 2) \to 2$, $l(3, 0) \to 1$, $j(1, 1) \to 1$, $R(3, 2) \to 3$
etc.

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* Nonintegral form.