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A UNIFIED THEORY OF SYNTHESIS WITH UNSYMMETRIC
FOUR-ELEMENT LOSSLESS LATTICES

by

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ABSTRACT

A complete theory is given for the use of unsymmetric lossless lattices with four elements in the cascade synthesis of transfer functions and driving-point functions. The major results are a new lattice equivalent for a Darlington-C section under specified constraints on the driving-point impedance and a new, simple condition of applicability for the Miyata Lattice. For both lattices explicit formulas for the element values are developed in terms of the given impedance.

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A UNIFIED THEORY OF SYNTHESIS WITH UNSYMMETRIC FOUR-ELEMENT LOSSLESS LATTICES*

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1. Introduction

It has recently been shown that the Foster and Cauer realizations of lossless positive real functions can be replaced by cascades of unsymmetric four-element lattices¹ and Miyata² has shown that under certain conditions a cascade of a Brune section and a Darlington section can be converted to the unsymmetric four-element lossless lattice of Fig. 1(b). These results naturally lead us to ask two questions:

1. Are there any other unsymmetric four-element lossless lattices which can be used to replace sections of more complicated driving point or transfer-function realizations?

2. Is there a direct method of obtaining the Miyata lattice without first realizing the equivalent Brune and Darlington sections?

The limitation to lattices of only four elements in question 1 is a practically meaningful one, since it guarantees that the new realizations are simple and hence economical. Question 2 is important since it deals with the practical applicability of the lattices.

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In this paper complete answers are given to both questions 1 and 2, and as a consequence the two major results are:

1. It is proved that the lattice in Fig. 1(c), which we will call N_3 , is equivalent to the network in the dotted box of Fig. 3 under certain given restrictions. A direct realization method is given in terms of the lattice.

2. A direct realization scheme is given for the Miyata lattice in Fig. 1(b), to be called N_2 , and a condition of applicability is given in terms of the driving-point impedance.

The approach used is to classify all four element lattices into three nontrivial sets such that the lattices in a given set can be realized by the same method. Conditions are determined under which the lattices in each set can be used in a realization by determining the equivalents of these lattices in terms of standard realizations.

Except for the trivial situation in which the lattice contains only one type of element, the four-element lattices can be divided into three sets and their duals as illustrated by typical lattices in Fig. 1; Type I, used by H. B. Lee; Type II, used by F. Miyata; and Type III, which we introduce, consisting of three elements of one kind and one of the other. The single element (say a capacitor) can either be put in one of the two series arms or one of the two shunt arms of the lattice. The remaining arms then consist of the other kind of element. The two cases for the single element in the series arm are the same so far as the synthesis is concerned. The cases for the single element in the shunt arm, however, are different. We thus have three different kinds of lattices in Type III. In the paper a new method is given for the networks in Type II and the representative network of Type III lattice, N_3 , is considered. The synthesis of other networks in Type III is discussed briefly at the end of section 2.

2. Synthesis Using a Type III Lattice

Let us first consider the circumstances under which a Type III lattice can be used. Considering the lattice as a two-port network, we see it always has a transmission zero at $s = \sigma_0, \sigma_0$ real. This means,

we may try to use the lattice to replace a Darlington-C section and thus eliminate an ideal transformer under certain restrictions in the process of synthesis. In addition, since in N_3 , there is one direct inductance path connecting the input terminals, the driving point impedance to be synthesized by the lattice must have a zero at $s = 0$ (corresponding to L'_c in Fig. 2). The lattice may be used to try to replace a section of the form shown in Fig. 2, where the ideal transformer is introduced to equate the degrees of freedom of N_3 and the network in Fig. 2. L_1 is introduced to give $Z(s)$ a pole at ∞ .

We now proceed to show that the two port of Fig. 2 is equivalent to N_3 under a set of restrictions to be developed.

The transmission matrix of the lattice N_3 is:

$$T_L = \frac{1}{L_3 L_4 (s^2 - \frac{L_1}{C_1 L_3 L_4})} \begin{bmatrix} (L_1 + L_4)(L_3 s^2 + \frac{1}{C_1}) & L_1 L_3 L_4 s^3 + (L_1 L_4 + L_1 L_3 + L_3 L_4) \frac{s}{C_1} \\ (L_1 + L_3 + L_4) s + \frac{1}{C_1 s} & (L_1 + L_3)(L_4 s^2 + \frac{1}{C_1}) \end{bmatrix} \quad (1)$$

The transmission matrix of the network in Fig. 2 is given by:

$$T_C = \frac{1}{MCs^2 + 1} \begin{bmatrix} \frac{1}{\rho} [(L'_a + M)Cs^2 + 1] & \rho [(L'_a + L_b) s + C (L'_a L_b + ML_b + L'_a M)s^3] \\ \frac{1}{\rho} [Cs + \frac{1}{sL_c} + \frac{(L'_a + M)C}{L_c} s] & \rho [\frac{C(L'_a L_b + ML'_a + ML_b)}{L_c} s^2 + \frac{L'_a + L_b}{L_c} + (L_b + M)Cs^2 + 1] \end{bmatrix} \quad (2)$$

†As will be shown, the shunt inductance cannot be arbitrary. Thus, in the course of derivation, we find restrictions on L_c . We then try to obtain L_c , the required inductance, from L'_c , obtained from $Z(s)$, in the actual calculation.

Equations (4)-(12) can be reduced to six independent equations which can be solved for C_1, L_1, L_3, L_4, ρ and L_1^a as in (13)-(18).

$$(12) \quad \frac{L_1 + L_3}{L_1 + L_4} \cdot \frac{C_1}{1} = \frac{L_3^3 L_4^4}{L_1^a + L_3^b + L_4^c} \rho$$

$$(11) \quad \frac{L_3}{L_1 + L_3} = \frac{ML^c}{\rho(L_1^a L_3^b + ML^c)} + \frac{M}{\rho(L_3^b + M)}$$

$$(10) \quad \frac{L_3}{L_1} + \frac{L_4}{L_1} + 1 = \frac{C_1}{1} \cdot \frac{MC}{\rho(L_1^a + L_3^b)}$$

$$(9) \quad L_1 = \frac{M}{\rho(L_1^a L_3^b + ML^c) + L_1^a M}$$

$$(8) \quad \frac{C_1 L_3^3 L_4^4}{1} = \frac{L_3^c \rho MC}{1}$$

$$(7) \quad \frac{L_3}{L_1} + \frac{L_4}{L_1} + \frac{\rho M}{L_1^a + M} = \frac{L_3^3 L_4^4}{L_1} + \frac{ML^c \rho}{L_1^a + M}$$

$$(6) \quad \frac{L_1 + L_4}{1} = \frac{L_3^3 L_4^4 C_1}{\rho MC}$$

$$(5) \quad \frac{L_1 + L_4}{L_1 + L_4} = \frac{L_4^4}{1} \cdot \frac{\rho M}{L_1^a + M}$$

$$(4) \quad -L_1 = \frac{C_1 L_3^3 L_4^4}{1} = \frac{MC}{1}$$

(3) $T_L = T_C$ and equate coefficients of like powers of s , we have equations (4)-(12).

If we let

$$C_1 = C \quad (13)$$

$$L_4 = \frac{L_c(L_b + M)}{L_b + L_c} \quad (14)$$

$$L_1 = \frac{L_c(L_c - M)}{L_b + L_c} \quad (15)$$

$$L_3 = M \frac{M - L_c}{L_b + M} \quad (16)$$

and

$$\rho = \frac{M - L_c}{L_c + L_b} \quad (17)$$

$$L'_a = (-M) \frac{L_b + L_c}{L_b + M} \quad (18)$$

In the equivalent Darlington realization $M < 0$, $L_b + M > 0$ and thus C_1 , L_4 , L_1 , L_3 and L'_a are all positive.

Since L_a , L_b , C and M are found from the Darlington-C section, they are all known. We thus can solve for C_1 , L_4 , L_1 , L_3 and ρ from equations (13)-(18) once " L_c " is determined.

As shown in Fig. 3, $L'_a = L_a + L_i$. From equation (18)

$$L'_a = \frac{(-M)L_b}{L_b + M} + \frac{(-M)L_c}{L_b + M} = L_a + \frac{(-M)L_c}{L_b + M} \quad (19)$$

where L_a is the inductance required for the Darlington-C section. We have

$$L_i = \frac{(-M)L_c}{L_b + M} \quad (20)$$

or

$$\frac{L_i}{L_c} = \frac{(-M)}{L_b + M} \quad (21)$$

This equation gives us the constraint on the driving point function to be synthesized by Type III lattice. As in equation (21), the ratio of L_i to L_c is required to be a fixed value. Thus to synthesize the driving point function by N_3 , we can first obtain the network of Fig. 2. It is unlikely that the ratio of L_i to L'_c obtained directly from the driving point function will be equal to the value as required by equation (21). We have the two following different situations

$$(1) \text{ If } L_i/L'_c > \frac{-M}{L_b + M} \quad (22)$$

the quantity on the left side can be made smaller by splitting L'_c into two larger inductors L_c and L''_c in parallel while L_c is chosen to fulfill (21) with equality.

$$(2) \text{ If } L_i/L'_c < \frac{-M}{L_b + M} \quad (23)$$

the method does not apply. (It can be shown that the condition is the same if L_i is first partially removed to change all the element values, or if L_i (or L_c) is moved to the other side of the Darlington-C section and a new set of relations are derived.)

The conditions on a positive real driving point function for the realizability with a section N_3 terminated in a positive real function can be summarized as follows:

$$\text{I(a)} \quad \text{Re}Z(\sigma_0) = 0 \text{ with } \sigma_0 \text{ real}$$

II(a) $Z(s)$ must have a zero at the origin and a pole at infinity corresponding to L'_c and L_i as in Fig. 2.

$$\text{III(a)} \quad L_i/L'_c > \frac{-M}{L_b + M}$$

Finally the calculations can be made directly in terms of the driving point impedance if we use Youla's results³ to obtain M , C , and L_b directly in terms of $Z_2(s)$, the impedance after removing L_i and L'_c . (22) then becomes:

$$\frac{L_i}{L'_c} > \frac{\frac{Z_2(\sigma_o)}{\sigma_o} + Z'_2(\sigma_o)}{\frac{Z_2(\sigma_o)}{\sigma_o} - Z'_2(\sigma_o)} \quad (24)$$

We also have,

$$M = \frac{-2}{\frac{Z_2(\sigma_o)}{\sigma_o} - Z'_2(\sigma_o)} \quad (25)$$

$$C = \frac{\frac{2}{\sigma_o^2}}{\frac{Z_2(\sigma_o)}{\sigma_o} - Z'_2(\sigma_o)} \quad (26)$$

$$L_b = \frac{\frac{Z_2(\sigma_o)}{\sigma_o}}{\left(\frac{Z_2(\sigma_o)}{\sigma_o}\right)^2 - (Z'_2(\sigma_o))^2} \quad (27)$$

The systematic method of synthesis is given below:

- (1) remove L'_c and L_i from $Z(s)$ to obtain $Z_2(s)$.
- (2) test the applicability of the method by (24) and calculate L_c defined by

$$L_c = L_i \frac{\frac{Z_2(\sigma_o)}{\sigma_o} - Z'_2(\sigma_o)}{\frac{Z_2(\sigma_o)}{\sigma_o} + Z'_2(\sigma_o)} \quad (28)$$

- (3) use (25)-(28) and (13)-(17) to compute $L_1, C_1, L_3, L_4,$ and ρ .
- (4) from $Z_2(s)$, using equation (10) of Youla³, obtain $Z^*(s)$, the impedance before the transformer in Fig. 3, and get $W(s)$ from

$$W(s) = \rho^2 Z^*(s) \quad (29)$$

We can claim now that conditions I(a) - III(a) on a positive real function are necessary and sufficient for the removal of N_3 . They are necessary because if I(a) or II(a) is not satisfied, transmission matrices T_L of (1) and T_C of (2) will not have the same form; if III(a) is not satisfied, L''_C in Fig. 3 will become negative. They are also sufficient for if they are all satisfied, the lattice can be pulled out by steps (1)-(4) as we have shown.

The dual lattice of N_3 which contains three capacitors and one inductor can be considered similarly, except L_i and L_C here have to be changed to C_i and C_C .

The synthesis of other kinds of type III lattices, as mentioned in the Introduction, can be treated in exactly the same fashion. Their equivalent networks have the same form as in Fig. 3 and hence the transmission matrices T_C are the same as equation (2). But the transmission matrices of the lattices will be different from N_3 , as will be the formulas for the element values.

Example 1: Let $Z(s) = \frac{2s^4 + 3s^3 + 7s^2 + 6s}{5s^3 + 6s^2 + 4s + 3}$ having a transmission zero

$\sigma_0 = 1$. Removing two inductors as in Fig. 4 we have

$$Z_2(s) = \frac{\frac{3}{4}s^2 + \frac{27}{4}s + 6}{4s^2 + 4.5s + \frac{1}{2}}$$

$$Z_2(\sigma_0) = Z_2(1) = \frac{13.5}{9}$$

$$Z'_2(\sigma_0) = -\frac{7}{6},$$

and

$$\frac{\frac{Z_2(\sigma_0)}{\sigma_0} + Z_2'(\sigma_0)}{\frac{Z_2(\sigma_0)}{\sigma_0} - Z_2'(\sigma_0)} = \frac{1}{8}$$

Since $\frac{L_i}{L_c} = \frac{1}{4} > \frac{1}{8}$ the method applies. L_c is separated into two inductors such that $L_c = 4$.

Using equations (25)-(27), we have

$$M = -\frac{4}{3}$$

$$C = \frac{3}{4}$$

$$L_b = 12$$

Substituting in (13)-(16) yields

$$C_1 = \frac{3}{4}$$

$$L_1 = \frac{4}{3}$$

$$L_4 = \frac{8}{3}$$

$$L_3 = \frac{2}{3}$$

and ρ is found by (17)

$$\rho = -\frac{1}{3}$$

$$Z^*(s) = \frac{32}{3} \text{ and } W(s) = \frac{1}{9} \cdot \frac{32}{3} = \frac{32}{27}$$

The result is shown in Fig. 5. The equivalent Darlington realization is in Fig. 6.

3. Synthesis Using a Type II Lattice

As mentioned in the Introduction, Miyata has derived necessary and sufficient conditions for the utilization of N_2 in a driving point synthesis. However, these conditions depend upon I_1 and I_2 , the values of turn ratios in Fig. 8, which are not expressed directly in terms of the given impedance. Thus, the network in Fig. 8 must first be found and then converted to the lattice. We will derive necessary and sufficient conditions directly in terms of the given impedance by finding conditions in terms of I_1 and I_1' , two quantities to be defined directly in terms of $Z(s)$.

The type II lattice in Fig. 1(b), denoted by N_2 , has transmission zeros at $s_o = \pm j\omega_o, \pm \omega_o$ where

$$\omega_o = \frac{1}{4\sqrt{L_1 L_2 C_1 C_2}} \quad (30)$$

Suppose N_2 can be removed directly from a pertinent driving point function $Z(s)$; then at the transmission zero, $s = s_o, Z(s_o)$ is simply the input impedance of N_2 with the output terminals open circuited as in Fig. 7. Thus at $s = \omega_o$

$$\begin{aligned} Z(\omega_o) &= \frac{\omega_o^4 L_1 L_2 C_1 C_2 + \omega_o^2 (L_1 C_2 + L_2 C_1) + 1}{\omega_o^3 (L_1 + L_2) C_1 C_2 + \omega_o (C_1 + C_2)} \\ &= \frac{2 + \omega_o^2 (L_1 C_2 + L_2 C_1)}{\omega_o^3 (L_1 + L_2) C_1 C_2 + \omega_o (C_1 + C_2)} \end{aligned} \quad (31)$$

At $s = j\omega_o$

$$jX(\omega_o) = Z(j\omega_o) = j \frac{2 - \omega_o^2 (L_1 C_2 + L_2 C_1)}{\omega_o^3 C_1 C_2 (L_1 + L_2) - \omega_o (C_1 + C_2)} \quad (32)$$

We now have three independent equations (30), (31), and (32) that $Z(s)$ must satisfy. We thus need one more independent relation for $Z(s)$ to satisfy so we can determine exactly under what conditions the lattice can be removed and so we have four equations to solve for L_1 , L_2 , C_1 and C_2 .

In order to derive the fourth relation, consider a driving point function $Z(s)$ which is realized at the input terminals of a two-port terminated in $Z^*(s)$ then

$$Z(s) = \frac{AZ^*(s) + B}{CZ^*(s) + D} \quad (33)$$

where A, B, C, D represent the cascade parameters of the two port.

At a transmission zero s_0 of the two port,

$$Z(s_0) = \frac{B(s_0)}{D(s_0)} \quad (34)$$

If the transmission zero is of double order, we have

$$Z'(s) \Big|_{s=s_0} = \left(\frac{B(s)}{D(s)} \right)' \Big|_{s=s_0} \quad \dagger\dagger \quad (35)$$

Having equation (35), we may go ahead to prove the fourth relation.

It is clear that any transmission zeros on the $j\omega$ axis must be of double order; thus (35) can be applied. It assures us that $\frac{dx(\omega)}{d\omega}$ when $\omega = \omega_0$ is equal to the derivative of the right side of (32) for $\omega = \omega_0$ or,

$$X'(\omega_0) = \frac{(\omega_0^2 C_1 C_2 (L_1 + L_2) + (C_1 + C_2))(\omega_0^2 (L_1 C_2 + L_2 C_1) - 2)}{(\omega_0^3 C_1 C_2 (L_1 + L_2) - \omega_0 (C_1 + C_2))^2}$$

\dagger The transmission zeros are defined to be the roots of the equation $AD - BC = 0$.

$\dagger\dagger$ A complete proof can be found, for example, in E. S. Kuh, E. E. 241 Class notes, Ch. IV, University of California, Berkeley.

$$X'(\omega_o) = - \frac{X(\omega_o)}{\omega_o} \frac{\omega_o^2 C_1 C_2 (L_1 + L_2) + (C_1 + C_2)}{\omega_o^2 C_1 C_2 (L_1 + L_2) - (C_1 + C_2)} \quad (36)$$

The turns ratio of the Brune cycle removed from $Z(s)$ as derived by Youla³ is

$$I_1 = \frac{X'(\omega_o) - \frac{X(\omega_o)}{\omega_o}}{X'(\omega_o) + \frac{X(\omega_o)}{\omega_o}} \quad (37)$$

Substituting (36) into (37) we get

$$I_1 = \omega_o^2 \frac{C_1 C_2 (L_1 + L_2)}{(C_1 + C_2)} \quad (38)$$

The transmission zeros on the real σ -axis can always be constructed by augmenting the driving point function with the factor $(s + \sigma)$ or $(s + \sigma)^2$ for a double transmission zero.

By (35), we can say once more that $\frac{dZ(\omega_o)}{d\omega_o}$ is equal to the derivative of the right side of equation (31) or

$$Z'(\omega_o) = - \frac{Z(\omega_o)}{\omega_o} \frac{\omega_o^2 C_1 C_2 (L_1 + L_2) - (C_1 + C_2)}{\omega_o^2 C_1 C_2 (L_1 + L_2) + (C_1 + C_2)} \quad (39)$$

If we calculate the turnsratio of the Darlington-C section removed from $Z(s)$, we have

$$I_1' = \frac{\frac{Z(\omega_o)}{\omega_o} - Z'(\omega_o)}{\frac{Z(\omega_o)}{\omega_o} + Z'(\omega_o)} \quad (40)$$

Substituting equation (39) into (40),

$$I_1' = \frac{\omega_0^2 C_1 C_2 (L_1 + L_2)}{(C_1 + C_2)} \quad (41)$$

Comparing (38) and (41), we immediately see that

$$I_1 = I_1' \quad (42)$$

or in another form

$$\frac{\frac{Z(\omega_0)}{\omega_0} - Z'(\omega_0)}{\frac{Z(\omega_0)}{\omega_0} + Z'(\omega_0)} = \frac{X'(\omega_0) - \frac{X(\omega_0)}{\omega_0}}{X'(\omega_0) + \frac{X(\omega_0)}{\omega_0}} \quad (43)$$

This equation can be simplified to

$$Z(\omega_0)X(\omega_0) = \omega_0^2 Z'(\omega_0)X'(\omega_0)$$

Note that I_1 and I_1' are turns ratios of sections both removed directly from $Z(s)$. It is this fact which enables us to improve Miyata's conditions.

We will next prove the following theorem:

Theorem. Lattice N_2 can be used as a section in realizable driving point synthesis of $Z(s)$ if and only if

$$\text{I(b)} \quad \text{Re}Z(s) = 0 \text{ at } s = \pm j\omega_0, \omega_0 \text{ real}$$

II(b) Condition (44) is satisfied.

The proof consists of two parts, and we proceed as follows.

Miyata has shown that the lattice is equivalent to one Brune cycle and one Darlington-C section cascaded together (Fig. 8).

As s goes to infinity, the capacitors in N_2 become a short-circuit path. This amounts to a direct but reversed connection between terminals 1-1' and 2-2'. To let the cascade sections have the same behavior, we must have

$$I_1 \cdot I_2 = 1 \quad (45)$$

where I_1 and I_2 are absolute values of the turns ratios of the Brune cycle and the Darlington-C section respectively.

The first part of the proof shows that the condition for the lattice to be pulled out is equivalent to condition (45) which can be proved by writing two port parameters of the lattice and the cascade section and setting them equal. The detail of the proof is given by Miyata and will not be repeated here.

The second part, which is given below, is to prove that condition (44) is exactly equivalent to (45).

Equation (45) can be written in another form

$$\frac{X'(\omega_o) - \frac{X(\omega_o)}{\omega_o}}{X'(\omega_o) + \frac{X(\omega_o)}{\omega_o}} \cdot \frac{\frac{Z_1(\omega_o)}{\omega_o} - Z_1'(\omega_o)}{\frac{Z_1(\omega_o)}{\omega_o} + Z_1'(\omega_o)} = 1 \quad (46)$$

or

$$\frac{Z_1'(\omega_o)}{Z_1(\omega_o)} + \frac{X(\omega_o)}{\omega_o^2 X'(\omega_o)} = 0 \quad (47)$$

where Z_1 is the remaining impedance after the Brune cycle has been pulled out from the given impedance $Z(s)$, as in Fig. 8.

The relation between $Z(s)$ and $Z_1(s)$ can be represented in terms of L_p , L_s , M and C as follows:

$$Z_1(s) = \frac{(L_s C s^2 + 1)Z(s) - (L_p + L_s - 2M)s}{-Cs Z(s) + L_p C s^2 + 1}$$

We differentiate equation (48) and substitute $Z_1(s)$ and $Z_1'(s)$ into equation (47). The result can be simplified to

$$\frac{[\omega_o^2 Z_1'(\omega_o) X'(\omega_o) - Z(\omega_o) X(\omega_o)] [\omega_o^2 X'^2(\omega_o) - X^2(\omega_o)]}{\omega_o^2 X'(\omega_o) [X'(\omega_o) \omega_o - Z(\omega_o)] [Z(\omega_o) X'(\omega_o) \omega_o - X^2(\omega_o)]} = 0 \quad (49)$$

Equation (49) implies that at least one of the two numerator factors is zero. Suppose $(\omega_0^2 X'^2(\omega_0) - X^2(\omega_0)) = 0$ or $(\omega_0 X'(\omega_0) + X(\omega_0)) (\omega_0 X'(\omega_0) - X(\omega_0)) = 0$. $(\omega_0 X'(\omega_0) + X(\omega_0)) = 0$ corresponds to infinite turns ratio and $(\omega_0 X'(\omega_0) - X(\omega_0)) = 0$ corresponds to zero turns ratio of the Brune cycle. These are, of course, nonpractical. So the only possible equivalent form of (45) is

$$\omega_0^2 Z'(\omega_0) X'(\omega_0) - Z(\omega_0) X(\omega_0) = 0 \quad (50)$$

which is exactly the same as equation (44) and the proof is thus completed.

Suppose the given driving point function satisfies conditions I(b), II(b); we can solve for L_1 , L_2 , C_1 and C_2 in terms of ω_0 , $X(\omega_0)Z(\omega_0)X'(\omega_0)$ and $Z'(\omega_0)$. This is done by solving equations (30), (31), (32) and (39).

The result is

$$L_1 = \frac{r_1 - r_2}{b - dr_2} \quad (51)$$

$$L_2 = \frac{r_1 - r_2}{dr_1 - b} \quad (52)$$

$$C_1 = \frac{r_2(dr_1 - b)}{r_1 - r_2} \quad (53)$$

$$C_2 = \frac{r_1(b - dr_2)}{r_1 - r_2} \quad (54)$$

where

$$b = \frac{2(Z(\omega_0) + Z'(\omega_0)\omega_0)}{\omega_0(Z^2(\omega_0) - Z'(\omega_0)\omega_0 X(\omega_0))} \quad (55)$$

$$d = \frac{2\omega_0(Z(\omega_0) - Z'(\omega_0)\omega_0)}{(Z^2(\omega_0) - Z'(\omega_0)\omega_0 X(\omega_0))} \quad (56)$$

$$r_1, r_2 = \frac{Z(\omega_0) \pm \sqrt{Z'(\omega_0)\omega_0 X(\omega_0)}}{\omega_0^2(Z^2(\omega_0) - Z'(\omega_0)\omega_0 X(\omega_0))} \quad (57)$$

Thus if the given impedance requires a Brune cycle in its realization and if condition (44) is satisfied, the equivalent lattice can easily be found by simply substituting the values of ω_0 , $X(\omega_0)$, $Z(\omega_0)$ and $Z'(\omega_0)$ into the above equations.

Now the problem is that it is unlikely that a given driving point function will satisfy condition (44). To overcome this problem, we partially remove a pole or zero at the origin or infinity to adjust the coefficients of the function such that (44) is satisfied. That is, assume the driving point function to start with is $Z_0(s)$, which requires a Brune cycle in its realization but which does not satisfy (44). For the time being, assume $Z_0(s)$ has a pole at infinity, corresponding to a series inductor. Assume L henrys of inductance must be subtracted from $Z_0(s)$ in order to make the remained function $Z(s)$ satisfy condition (44).

We have

$$Z(\omega_0) = Z_0(\omega_0) - L\omega_0 \quad (58)$$

$$Z'(\omega_0) = Z'_0(\omega_0) - L \quad (59)$$

$$X(\omega_0) = X_0(\omega_0) - \omega_0 L \quad (60)$$

$$X'(\omega_0) = X'_0(\omega_0) - L \quad (61)$$

Substituting the values of $Z(\omega_0)$, $Z'(\omega_0)$, $X(\omega_0)$, $X'(\omega_0)$ into (44) and solving for L , we get

$$L = \frac{Z'_o(\omega_o)X'_o(\omega_o) - \frac{Z_o(\omega_o)X_o(\omega_o)}{\omega_o^2}}{X'_o(\omega_o) - \frac{X_o(\omega_o)}{\omega_o} - \frac{Z_o(\omega_o)}{\omega_o} + Z'_o(\omega_o)} \quad (62)$$

Suppose $Z_o(s)$ has a pole at the origin corresponding to a series capacitor C . Similarly, we have

$$Z(\omega_o) = Z_o(\omega_o) - \frac{1}{\omega_o C} \quad (63)$$

$$Z'(\omega_o) = Z'_o(\omega_o) + \frac{1}{\omega_o^2 C} \quad (64)$$

$$X(\omega_o) = X_o(\omega_o) + \frac{1}{\omega_o C} \quad (65)$$

$$X'(\omega_o) = X'_o(\omega_o) - \frac{1}{\omega_o^2 C} \quad (66)$$

Substituting these equations into (44), we get

$$C = \frac{\frac{Z_o(\omega_o)}{\omega_o} + Z'_o(\omega_o) - X'_o(\omega_o) - \frac{X_o(\omega_o)}{\omega_o}}{\omega_o^2 \left(Z'_o(\omega_o)X'_o(\omega_o) - \frac{X_o(\omega_o)Z_o(\omega_o)}{\omega_o^2} \right)} \quad (67)$$

For the shunt inductor or capacitor case, we may apply our method to $\frac{1}{Z(s)}$. After getting the network, we then find its dual network which is our answer.

The complete method of synthesis is then as follows:

- (1) Calculate ω_o , $Z_o(\omega_o)$, $Z'_o(\omega_o)$, $X_o(\omega_o)$ and $X'_o(\omega_o)$ from $Z_o(s)$ whose real part vanishes at $s = \pm j\omega_o$.

- (2) Calculate L or C from (62) or (67) and so Z(s) by $Z(s) = Z_0(s) - Ls$ or $Z(s) = Z_0(s) - \frac{1}{Cs}$.
- (3) Examine whether L or C is positive and whether they are smaller than the corresponding residues of $Z_0(s)$ at infinity and zero respectively.
- (4) Through equations (58)-(61) or (63)-(66), we compute $Z(\omega_0)$, $Z'(\omega_0)$, $X(\omega_0)$ and $X'(\omega_0)$ and solve for L_1 , L_2 , C_1 and C_2 from (51)-(57).
- (5) The remaining function $Z^*(s)$ is found by the equation

$$Z^*(s) = \frac{DZ(s) - B}{-CZ(s) + A} \quad (68)$$

where

$$A = (sL_1 + \frac{1}{sC_2})(sL_2 + \frac{1}{sC_1}) \quad (69)$$

$$B = \frac{L_1}{sC_1C_2} + \frac{L_1L_2}{C_2} s + \frac{L_1L_2}{C_1} s + \frac{L_2}{sC_1C_2} \quad (70)$$

$$C = s(L_1 + L_2) + \frac{1}{sC_1} + \frac{1}{sC_2} \quad (71)$$

$$D = (sL_1 + \frac{1}{sC_1})(sL_2 + \frac{1}{sC_2}) \quad (72)$$

A, B, C, D are the transmission parameters of the lattice.

Example 2. If

$$Z_0(s) = \frac{2s^3 + 5s^2 + 5s + 8}{s^3 + s^2 + 2s}$$

$\text{Re}Z_0(s) = 0$ at $s = j1$; $Z_0(s)$ has a pole at $s = 0$ corresponding to a series capacitor

and

$$Z_o(1) = 5$$

$$X_o(1) = -3$$

$$X'_o(1) = 7$$

$$Z'_o(1) = -3.5$$

Substituting into equation (67)

$$C = \frac{5 - 3.5 - 7 + 3}{-7 \times 3.5 + 15} = \frac{5}{19}$$

which is larger than the series capacitor of $Z_o(s) = \frac{1}{4}$ and the method applies.

$$Z(s) = Z_o(s) - \frac{1}{\frac{5}{19}s} = \frac{10s^3 + 6s^2 + 6s + 2}{5s^3 + 5s^2 + 10s}$$

From equations (63)-(66)

$$Z(\omega_o) = 5 - \frac{19}{5} = 1.2$$

$$Z'(\omega_o) = -3.5 + \frac{19}{5} = 0.3$$

$$X(\omega_o) = -3 + \frac{19}{5} = 0.8$$

$$X'(\omega_o) = 7 - \frac{19}{5} = 3.2$$

having these data, the element values of the lattice follows.

$$\text{First, we have } b = \frac{2(1.2 + 0.3)}{1.44 - 0.24} = 2.5$$

$$d = \frac{2(1.2 - 0.3)}{1.2} = 1.5$$

$$r_1, r_2 = \frac{(1.2 + 0.24)^2}{1.44 - 0.24} = 2.38, 0.42$$

$$L_1 = 1.05$$

$$L_2 = 1.82$$

$$C_1 = 0.23$$

$$C_2 = 2.27$$

To compute $Z^*(s)$, we use the transmission parameters of the lattice by equations (69)-(72).

$$A = (1.05s + \frac{1}{2.27s})(1.82s + \frac{1}{0.23s})$$

$$B = 9.12s + \frac{1}{s} 5.46$$

$$C = 2.87s + 4.78 \frac{1}{s}$$

$$D = (1.05s + \frac{1}{0.23s})(1.82s + \frac{1}{2.27s})$$

$$Z^* = \frac{DZ - B}{-CZ + A} = \frac{19s + 3.8}{9.55s} = 2 + \frac{1}{2.51s}$$

The realized network is shown in Fig. 9. The corresponding Brune realization is shown in Fig. 10.

4. Conclusion

Since the completion of this paper, the authors have learned that Yarlagadda and Tokad⁴ have independently considered the use of the lattice N_2 in synthesis. However, their conditions are more restrictive than ours since they require transmission zeros at $\bar{\tau} \omega_0$, $\bar{\tau} j\omega_0$ whereas we can achieve the zeros at $\bar{\tau} \omega_0$ by augmentation. Furthermore they do not give formulas for the element values directly in terms of the given impedance, as we do. Finally, the derivations are different and we believe ours give more intuitive insight.

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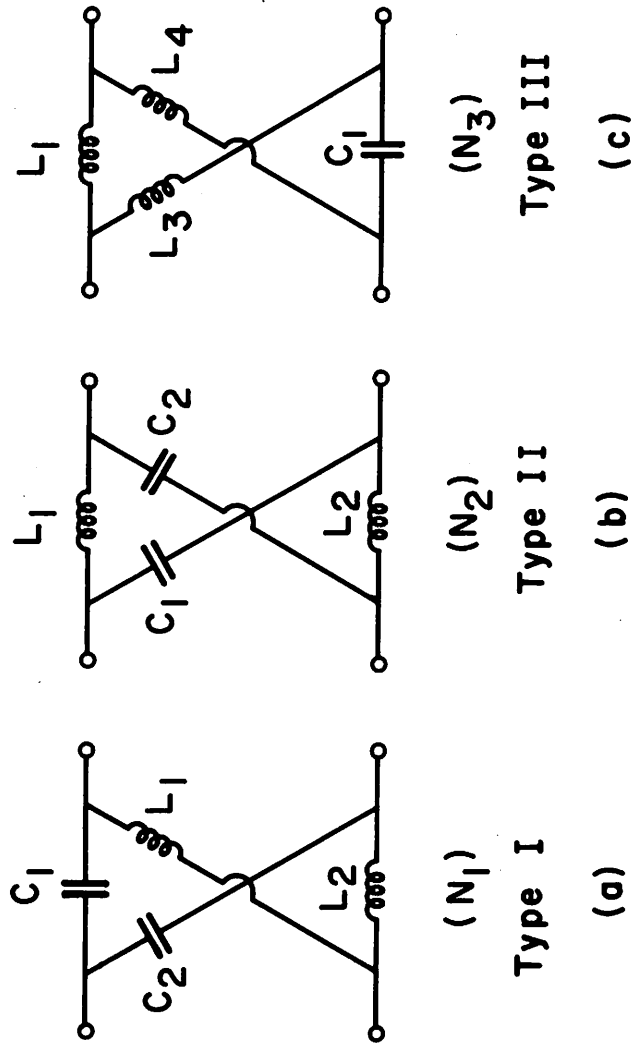


Fig. 1. Unsymmetric four-element lossless lattice networks.

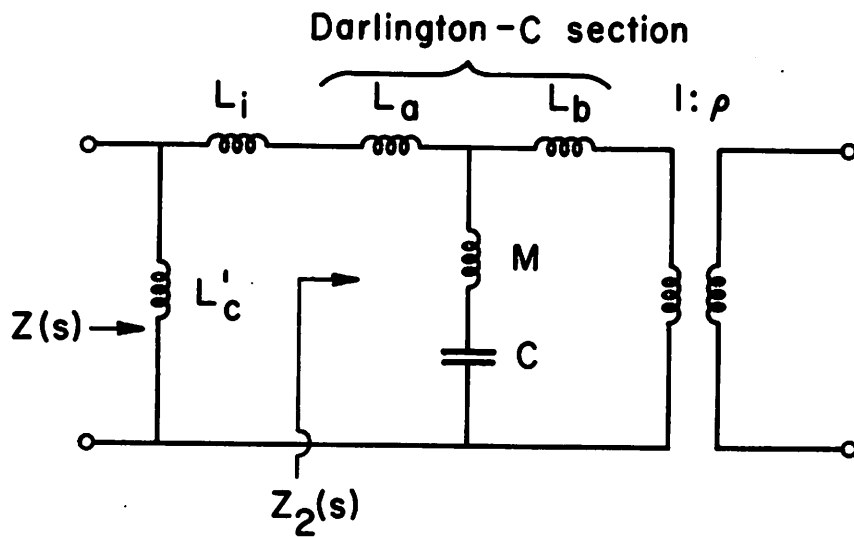


Fig. 2. Two-port network equivalent to N_3 .

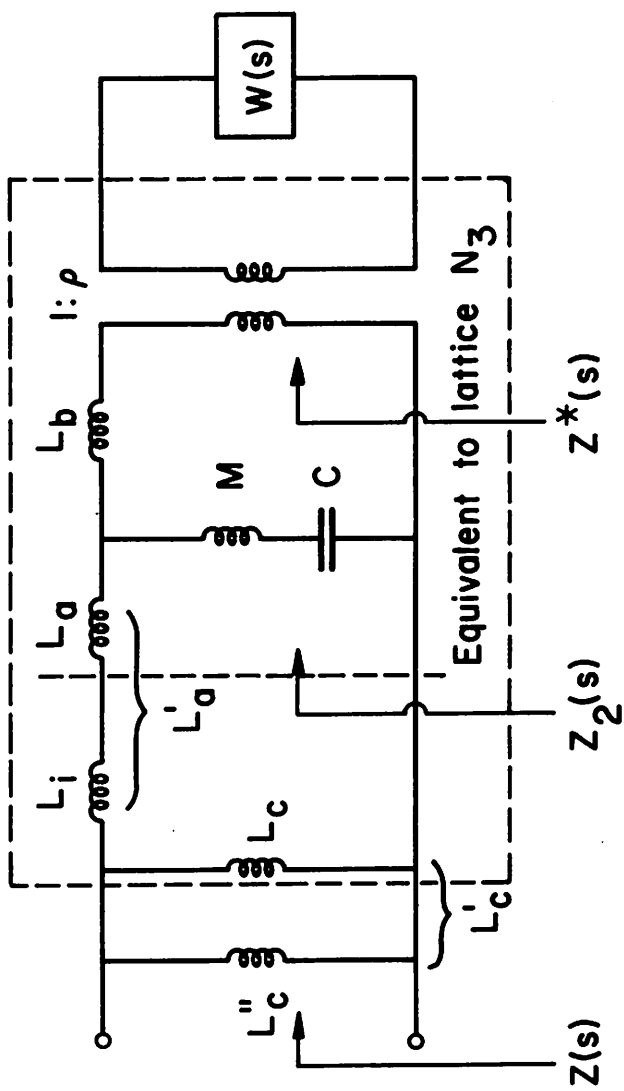


Fig. 3. Network used for removal of N_3 .

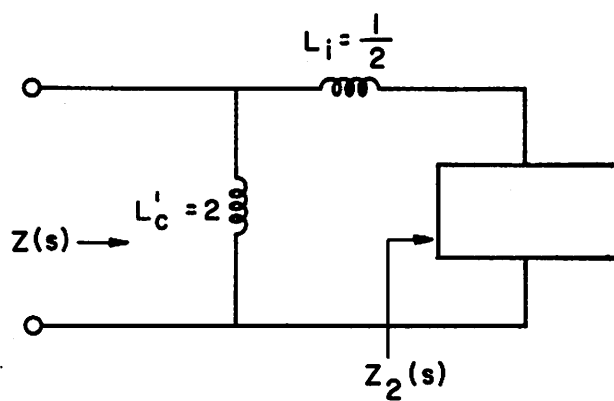


Fig. 4. Removal of L_i and L_c in Example 1.

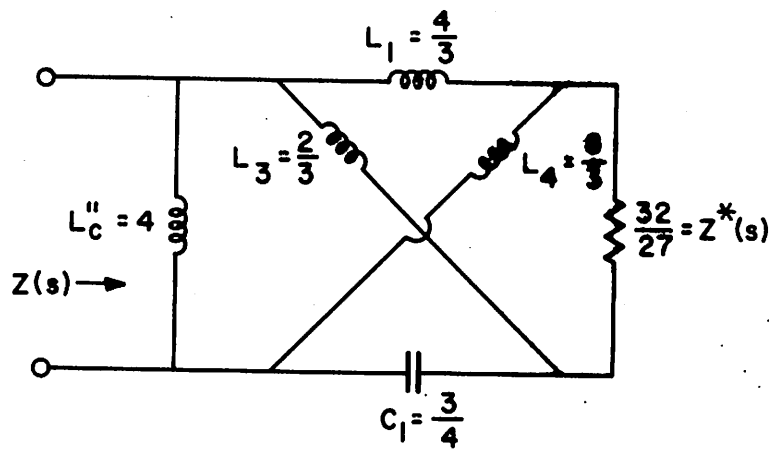


Fig. 5. Realization for Example 1.

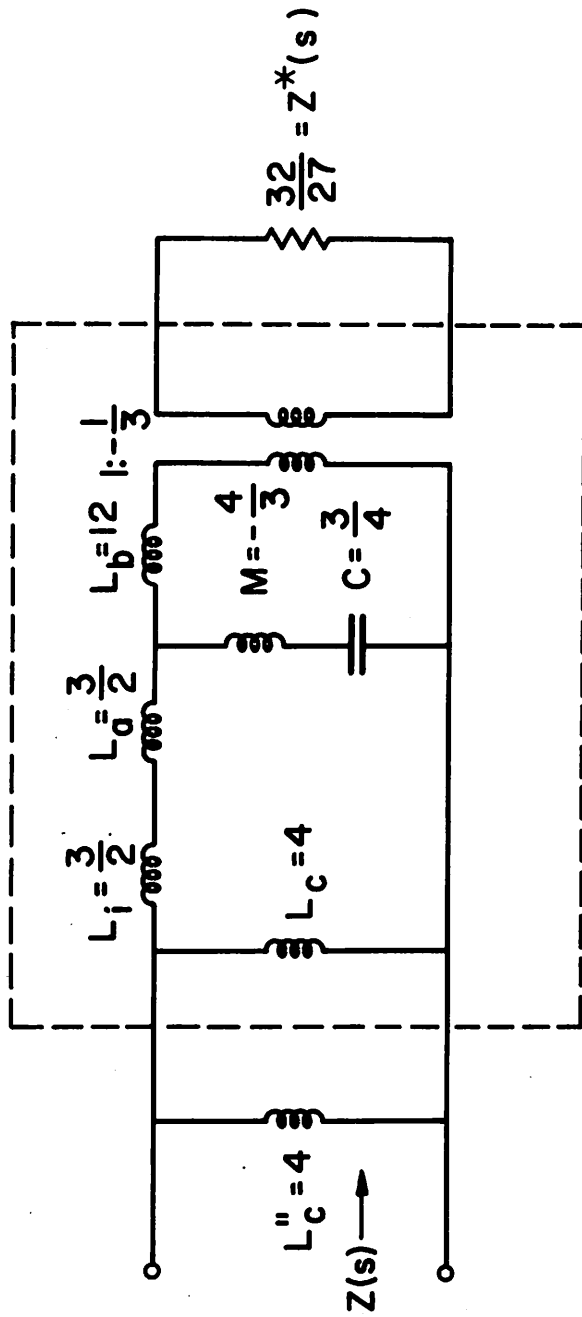


Fig. 6. Darlington-C section for Example 1.

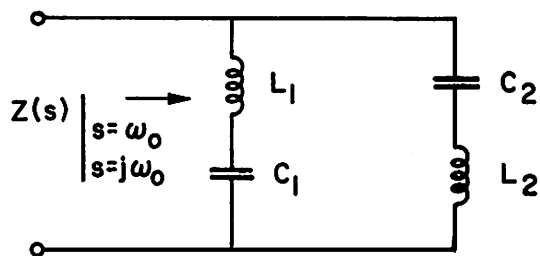


Fig. 7. Network used to calculate $Z(s)|_{s=\omega_0, j\omega_0}$ for N_2 .

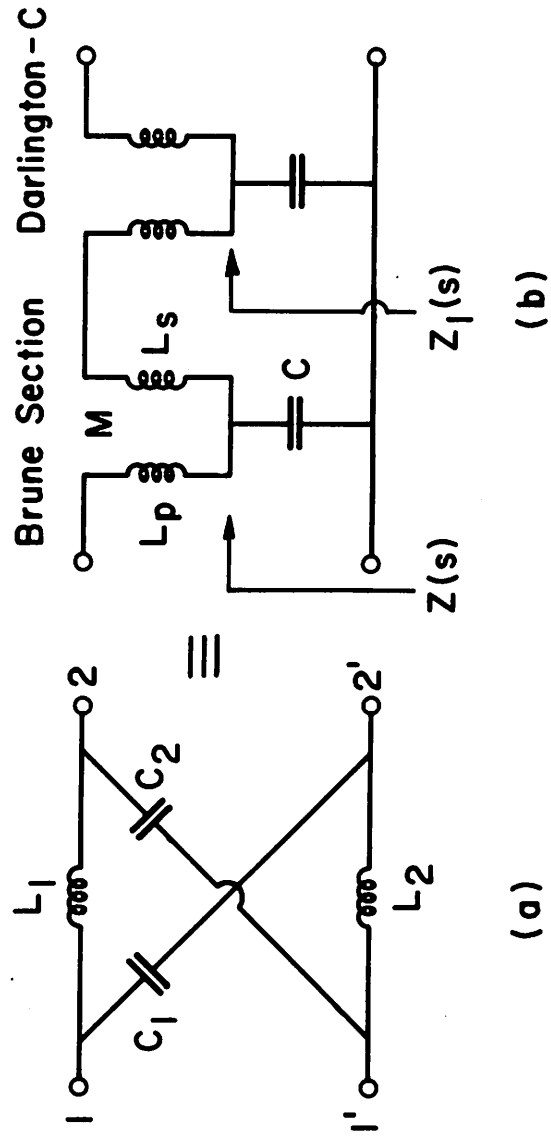


Fig. 8. N_2 and equivalent network.

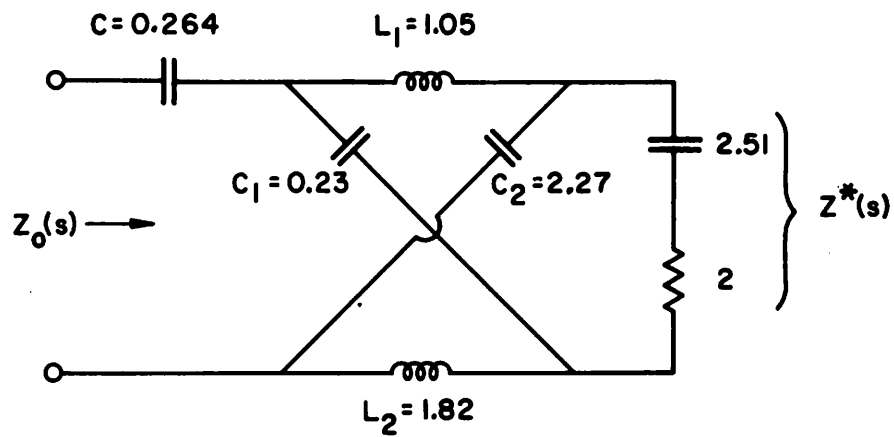


Fig. 9. Realization, using N_2 , for Example 2.

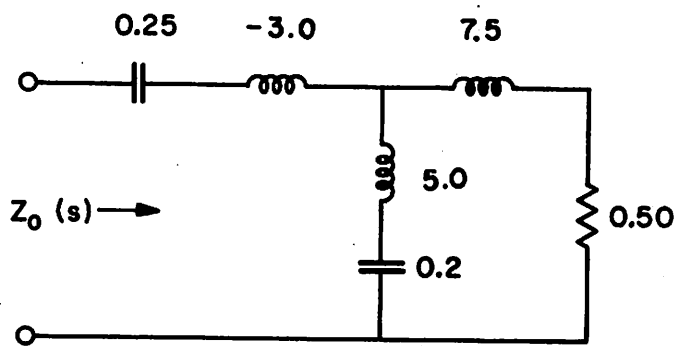


Fig. 10. Equivalent realization for Example 2.