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Electronics Research Laboratory
University of California
Berkeley, California

# A LIMITING CASE FOR ABSOLUTE STABILITY CRITERION

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V. M. Popov[2] had investigated the absolute stability of a class of continuous nonlinear control systems such as represented in Fig. 1, where the nonlinear function $\varphi(\sigma)$ is assumed to be Class A, i.e., a continuous function satisfies the conditions:

$$\varphi(0) = 0 \tag{1}$$

and

$$\varphi(\sigma)\,\sigma > 0 \quad \text{for} \quad \sigma \neq 0. \tag{2}$$

Popov's theorem states that a continuous system of this class is absolutely stable if a non-negative $q_c$ exist such that the inequality

$$\text{Re } G(jw)\left[1 + jwq_c\right] \geq 0 \tag{3}$$

is satisfied for all real $w$ in the interval $(-\infty, +\infty)$.

Despite the obvious difference that these systems are continuous rather than discrete, they nevertheless bear strong resemblances to

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†University of California, Department of Electrical Engineering, Berkeley, Calif.,

††Noller Control Systems, Inc., Richmond, California.

the sampled-data systems represented in Fig. 2. We shall establish that the continuous systems of Fig. 1 are the limits of corresponding sampled-data systems represented in Fig. 2 when $T \rightarrow 0$. Further, we shall establish that Popov's theorem may be obtained as a limiting case of the new stability theorem developed for Class $\Gamma_1$, and in the process, obtain an extension of Popov's theorem, applicable to a broader class of nonlinear functions.

As a starting point, we first consider the transform of the signal applied in Fig. 2 to the linear plant whose Laplace transform is $G(s)$. This is given by:

$$\mathcal{L}(\tilde{\sigma}) = \mathcal{Z}\left\{T\,\varphi\left(\sigma[nT]\right)\right\} = \sum_{n=0}^{\infty} T\,\varphi\left(\sigma[nT]\right) e^{-nTs} \tag{4}$$

where it is noted that

$$z = e^{Ts}. \tag{5}$$

By taking the limit of (4) as $T \longrightarrow 0$ in a way such that $nT \longrightarrow t$, we have the result

$$\lim_{T \to 0} \mathcal{L}\left(\tilde{\sigma}(t)\right) = \int_0^{\infty} \varphi\left(\sigma[t]\right) e^{-ts}\, dt = \mathcal{L}\left(\varphi(\sigma[t])\right). \tag{6}$$

Since the transform of the signal applied in Fig. 1 to the linear plant $G(s)$ is given by the last term on the right of (6), we have established the result that, in the limit as $T \longrightarrow 0$, identical input signals are applied to the linear plants (which is identical) in both Figs. 1 and 2, provided the inputs to the nonlinear element $\varphi(\sigma)$ are identical. The latter, however, must be true in the limit since the systems are closed loop systems and, in the absence of external inputs, the inputs to the nonlinear element are the outputs of the linear plants which are identical in the limit. These considerations lead to the result that in the limit as $T \to 0$ the responses and behaviors of both the continuous system represented in Fig. 1 and the sampled-data system represented in Fig. 2 are identical. This establishes the assertion that the continuous systems considered by Popov may be taken as the limits as $T \to 0$ of the corresponding sampled-data systems of class $\Gamma_1$.

Next, we consider the block in Fig. 2 containing the gain factor "T" as a part of a new linear plant whose z-transform is given by

$$G_d^*(z) = \mathcal{Z}\left[TG(s)\right] = \sum_{n=0}^{\infty} T\, g(nT) e^{-nTs}, \tag{7}$$

where $g(t)$ is the impulse response of $G(s)$. With this linear plant, the sampled-data system is of Class $\Gamma_1$ for each value of the sampling period $T$, provided the nonlinear element $\varphi$ satisfies the conditions

$$\varphi(0) = 0, \tag{8}$$

$$0 < \frac{\varphi(\sigma)}{\sigma} < K, \qquad \text{for } \sigma \neq 0, \tag{9}$$

$$\left|\frac{d\varphi(\sigma)}{d\sigma}\right| < K' . \tag{10}$$

We then can apply the new stability theorem[1] to obtain the result that the sampled-data system of Fig. 2 is absolutely stable for each $T$ if a non-negative $q_d(T)$ exist for each $T$ such that the inequality

$$\operatorname{Re} G_d^*\left(e^{jwT}\right)\left\{1 + T\, q_d(T)\left[\frac{e^{jwT}-1}{T}\right]\right\} + \frac{1}{K} - \frac{T^2 q_d(T)K'}{2}\left|\left[\frac{e^{jwT}-1}{T}\right] G_d^*\left(e^{jwT}\right)\right|^2 \geq 0 \tag{11}$$

is satisfied for all real $w$ in the interval $\left(-\frac{\pi}{T}, +\frac{\pi}{T}\right)$.

To obtain the stability criterion for the system in the limit as $T \to 0$, we shall consider the limit of each term on the left of inequality (11). First, we have from (7) that

$$\lim_{T \to 0} G_d^*(z) = \lim_{T \to 0} \sum_{n=0}^{\infty} g(nT)e^{-nTs}\, T = \int_0^{\infty} g(t)e^{-ts}dt = G(s), \tag{12}$$

and therefore

$$\lim_{T \to 0} G_d^*(e^{jwT}) = G(jw) \,. \tag{13}$$

Second, we have

$$\lim_{T \to 0} \left( \frac{e^{jwT} - 1}{T} \right) = jw. \tag{14}$$

Next, we can consider (11) as an inequality dependent on the continuous variable T. Then for each value of T, either a finite non-negative $q_d(T)$ exist or the system may not be absolutely stable. It follows therefore that in the limit as $T \to 0$ either

$$q_c = \lim_{T \to 0} T\, q_d(T) \tag{15}$$

exist as a finite non-negative quantity or the system may not be absolutely stable in the limit. For the case $q_c$ exist, we have

$$\lim_{T \to 0} \frac{T^2 q_d(T) K'}{2} = \lim_{T \to 0} \frac{T\, q_c K'}{2} = 0 \tag{16}$$

Combining the results (12) to (16), the inequality (11) in the limit as $T \to 0$ is equivalent to the following inequality;

$$\text{Re}\; G(jw)\left[1 + jwq_c\right] + \frac{1}{K} \geq 0 \tag{17}$$

for all real w in the interval $(-\infty, \infty)$. Since the response and behavior of the sampled-data system approaches that of the continuous system in the limit, they in the limit must be governed by an identical stability criterion. We therefore have established the following corollary to the new stability theorem[1] developed for sample-data systems belonging to class $\Gamma_1$:

Corollary: "A nonlinear sampled-data system of $\Gamma_1$ represented by Fig. 2 in the limit as $T \to 0$, or a continuous nonlinear system such as represented by Fig. 1, is absolutely stable with respect to its null solution for a nonlinear function $\varphi(\sigma)$ satisfying the conditions $\varphi(\sigma) = 0$ and

$$0 < \frac{\varphi(\sigma)}{\sigma} < K, \quad \text{for} \quad \sigma \neq 0,$$

if a non-negative $q_c$ exist such that the inequality

$$\text{Re} \; G(jw)\left[1 + jwq_c\right] + \frac{1}{K} \geq 0$$

is satisfied for all real w in the interval $(-\infty, +\infty)$." Note that this corollary places no restriction on K' (i.e., the slope of the function $\varphi$) except that its derivation implies the condition

$$\lim_{T \to 0} T\,K'_{max} = 0 \qquad (18)$$

should be satisfied.

The above corollary reduces to Popov's theorem[2] as a special case for class A functions. For these functions,

$$K = \infty \qquad (19)$$

and the stability criterion from the above corollary reduces to

$$\text{Re } G(jw)\left[1 + jwq_c\right] \geq 0,$$

which is identical to that of Popov's theorem. However, the above corollary extends Popov's theorem to a class of nonlinear function which are less restrictive than class A, namely, functions which are confined to given sectors of the first and third quadrants of the $\varphi$-$\sigma$ plane. For these cases, the above corollary yields significantly less conservative sufficient conditions for absolute stability.

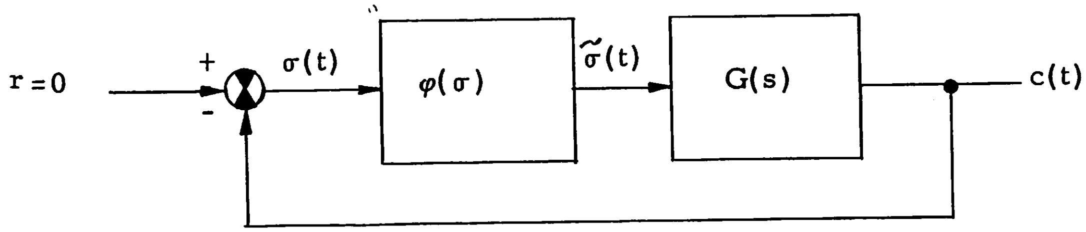


Fig. 1.

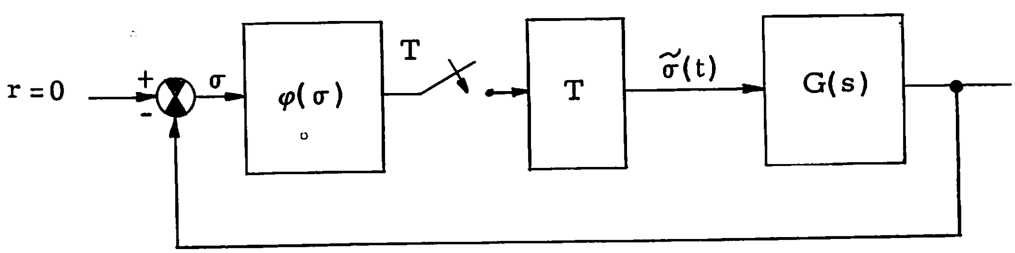


Fig. 2.

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