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A NOTE ON THE ABSOLUTE STABILITY OF A CLASS OF NONLINEAR SAMPLED-DATA SYSTEMS

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E. I. Jury and B. W. Lee

This research was supported by the Air Force Office of Scientific Research Contract AF-AFOSR 292-63.

January 10, 1964

TABLE OF CONTENTS

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E. I. Jury† and B. W. Lee††

INTRODUCTION

In a previous paper,[1] a stability theorem for a class of nonlinear sampled-data feedback systems has been presented. This theorem provided a sufficient criterion for absolute stability which involves the plotting of the frequency response characteristics of the linear plant part of the system. This criterion yielded a gain restriction comparable to those obtainable from Lyapunov's direct method using the general form of Lyapunov's function of the Lur'e type.

The purpose of this note is to present certain generalizations of the preceding stability theorem[1] which would give a certain significance to the various inequalities that are obtainable. From this generalization we can obtain sufficient conditions for stability for certain specific sub-classes of systems, which involves certain types of nonlinearities. An illustrative example of sampled-data systems with dead-zone is studied to illustrate the application of the material developed in this note. Finally, a discussion on the possible future work on this problem concludes the material of this note.

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*This research was supported by the Air Force Office of Scientific Research Contract AF-AFOSR 292-63.

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RECAPITULATION

The conditions for absolute stability of a class of sampled-data systems characterized by a stable linear part and a memoryless non-linear element is considered. A system of this class has the typical configuration shown in Fig. 1, wherein the linear part is stable and has a Laplace Transform representation given by:

$$G(s) = G_1(s) + \gamma/s. \tag{1}$$

$G_1(s)$ in (1) is of m-th order, and has no singularity in the right half of the s-plane or on the real frequency axis. Moreover, it is assumed that

$$\lim_{s \to \infty} G_1(s) = 0. \tag{2}$$

The z-Transform of (1) is given by:

$$G^*(z) = G_1^*(z) + \gamma \frac{z}{z-1}. \tag{3}$$

The memoryless nonlinear gain function is assumed to be integratable and, moreover, satisfies the following conditions:

$$\begin{aligned} &\text{(i)} \quad \varphi(0) = 0 \\ &\text{(ii)} \quad \infty > K > \frac{\varphi(\sigma)}{\sigma} > 0, \ \forall \sigma \neq 0. \end{aligned} \tag{4}$$

NOMENCLATURE

All sampled-data systems satisfying the above description for a specific $K > 0$ will be denoted by $\Gamma(K)$. Subclasses of $\Gamma(K)$ generated by imposing various additional restrictions on the derivative of $\varphi(\sigma)$ will be denoted by $\Gamma(K, K_1, K_2)$, where K_1, K_2 denote respectively the lower and upper bounds of a closed interval on the real line in which the derivative of $\varphi(\sigma)$ is restricted. As an example $\Gamma(K, -K', K')$ denotes a subclass of systems belonging to $\Gamma(K)$, i.e., systems satisfying conditions (1) to (4), and the additional restriction

$$\left|\frac{d\varphi}{d\sigma}\right| \leq K'. \tag{5}$$

Note that this is identical to the subclass Γ_1 defined in Ref. 1. As another example, $\Gamma(K, -\infty, K')$ denotes a subclass of systems included in $\Gamma(K)$ and satisfies in addition the condition

$$-\infty \leq \frac{d\varphi}{d\sigma} \leq K'. \tag{6}$$

In what follows, many of the computational steps parallel those developed in detail in Ref. 1. Results demonstrated in Ref. 1 will be assumed and used extensively. To facilitate this procedure, the same nomenclature as used in Ref. 1 will be re-employed whenever possible. In general, corresponding symbols, unless explicitly redefined here, will carry the same definitions as in Ref. 1.

AN ABSOLUTE STABILITY THEOREM FOR $\Gamma(K)$

Theorem 1: A system belonging to a subclass of $\Gamma(K)$ is absolutely stable with respect to the null solution if a real q, non-negative numbers a_1, a_2, a_3, a_4, and auxiliary functions $\psi_{N,q}(n)$ and $V_{N,q}(n)$, exist such that

(i) $\psi^*_{N,q}(z)$, $V^*_{N,q}(z)$ exist $\forall N \geq 0$, $\forall q$ real;

(ii) $$\sum_{n=0}^{\infty} \psi_{N,q}(n)\, V_{N,q}(n)$$

$$\geq q \int_{\sigma(0)}^{\sigma(N)} \varphi d\sigma - a_4 |\widetilde{\xi(N)}|^2 - (a_1|\widetilde{\xi(N)}| + a_2)|\widetilde{X(0)}| - a_3|\xi_0|,$$

$\forall N < 0;$

(iii) $\operatorname{Re}\left\{G^*(z) + 1/K - \psi^*_{N,q}(z)\, V^*_{N,q}(z) \,/\, |\varphi^*_N(z)|^2\right\}$

$$\geq 0, \ \forall |z| = 1;$$

where

$$|\widetilde{\xi(N)}| \triangleq \sup_{0 \leq \zeta \leq N} |\xi(\zeta)|, \text{ and } |\widetilde{X(0)}| \triangleq \max_{i,j=1,\ldots,m} (|x_i(0)|, |x_i(0)x_j(0)|).$$

It is of interest to note that $\int_{\sigma(0)}^{\sigma(N)} \varphi d\sigma$ represent a certain area as shown in Fig. 2.

Proof: The auxiliary functions $\varphi_N(n)$, $\sigma_N(n)$, and $\xi_N(n)$ as defined by (16), (17), and (18) of Ref. 1 will be assumed, whereas the auxiliary function $\lambda_N(n)$ is redefined as follows:

$$\lambda_N(n) \triangleq \sigma_N(n) + A\left[\xi_N(n) - \xi_N(n-1)\right] - \varphi_N(n)/K - \sum_{i=1}^{m} f_i(n)x_i(0) + A\,\xi_0\,\delta_{n0}, \tag{7}$$

where A is an arbitrary positive number. The z-transform of (7) is given by:

$$\lambda^*_N(z) = -\,G^*(z)\,\varphi^*_N(z) - \varphi^*_N(z)/K + A\left(\frac{z+1}{z-1}\right)\varphi^*_N(z). \tag{8}$$

To begin the proof, a suitable Popov function of N is defined by:

$$\rho(n) \triangleq \sum_{n=0}^{\infty}\left[\lambda_N(n)\,\varphi_N(n) + \psi_{N,q}(n)\,V_{N,q}(n)\right], \ \forall N \geq 0. \tag{9}$$

By applying the Lyapunov-Parseval theorem to (9), and following the computational procedure employed in Ref. 1, it can be demonstrated easily that condition (iii) of the theorem implies

$$\rho(N) \leq 0, \quad \forall N > 0. \tag{10}$$

Putting appropriate time-domain quantities into (9), and using (10), condition (ii) of the theorem, and results demonstrated in Ref. 1, inequality (9) leads to the inequality

$$\rho(N) \geq \rho_1(N) + q\left[F(N) - F(0)\right] + A\,\xi^2(N) - a_4|\widetilde{\xi(N)}|^2 - (C_1|\widetilde{\xi(N)}| + C_2)|\widetilde{X(0)}| - a_3|\xi_0| \leq 0 \tag{11}$$

where

$$\rho_1(N) \triangleq \sum_{n=0}^{N} \varphi(n)\sigma(n)\left[1 - \frac{\varphi(n)}{K\sigma(n)}\right] > 0, \tag{12}$$

$$F(N) \triangleq \int_0^{\sigma(N)} \varphi\, d\sigma \geq 0, \tag{13}$$

$$C_1 \triangleq a_1 + \beta'_1 + \beta'_2 \geq 0, \tag{14}$$

$$C_2 \triangleq a_2 + (\beta'_1 + \beta'_2)\,|\xi_0| \geq 0, \tag{15}$$

for $q \geq 0$, inequality (11) yields

$$(A - a_4)\,|\widetilde{\xi(N)}|^2 - (C_1|\widetilde{\xi(N)}| + C_2)\,|\widetilde{X(0)}| - a_3|\xi_0| - qF(0) \leq 0. \tag{16}$$

Since A may be an arbitrary positive number, the coefficient of the first term of (16) can be chosen positive. Then (16) is of the same type of inequality as (45) of Ref. 1. Using exactly the same argument as employed in Ref. 1, it can be concluded that

$$|\widetilde{\xi(N)}| \leq H\left(|\xi_0|, \ |\widetilde{X(0)}|, |\sigma(0)|\right), \tag{17}$$

where H is a well defined positive function of the initial state variables $|\xi_0|$, $|\widetilde{X(0)}|$, and $|\sigma(0)|$. More important, this function is specifically independent of N, which is necessary in order to conclude that the system is stable in the Lyapunov sense, i.e., given $\epsilon > 0$, $\delta > 0$ exist such that

$$|\widetilde{\xi(N)}| \leq H < \epsilon, \qquad |C(n)| = |\sigma(n)| < \epsilon, \tag{18}$$

whenever

$$|\widetilde{X(0)}| < \delta, \qquad |\xi_0| < \delta, \text{ and } |\sigma(0)| < \delta. \tag{19}$$

Next, suppose $q < 0$; then $-qF(0)$ is positive and may be discarded from (11). The term $q\,F(N)$ is negative, however, and must be retained. From (4), it follows that

$$q\,F(N) \geq -\frac{|q|\,K}{2}\,\sigma^2(N) \geq -\frac{K\,|q|}{2}\,|\sigma(N)|^2. \tag{20}$$

Utilizing results from Appendixes I and II of Ref. 1, it can be demonstrated that

$$|\sigma(N)|^2 \leq d_1\,|\widetilde{\xi(N)}|^2 + d_2|\widetilde{\xi(N)}| \cdot |\widetilde{X(0)}| + d_3|\widetilde{X(0)}|^2, \tag{21}$$

where d_1, d_2, d_3 are positive numbers independent of N. Putting (20) and (21) into (11) yields, for the $q < 0$ case, the inequality:

$$\left(A - \frac{|q|\ Kd_1}{2}\right) |\widetilde{\xi(N)}|^2 - \left[\left(C_1 + \frac{K|q|\ d_2}{2}\right) |\widetilde{\xi(N)}| + \left(C_2 + \frac{|q|\ Kd_3}{2}\right) |\widetilde{X(0)}|\right] |\widetilde{X(0)}| - a_3 |\xi_0| \leq 0. \tag{22}$$

Again, since A may be an arbitrary positive number, the first coefficient in (22) can be made positive, provided K is finite. This is assured for systems in $\Gamma(K)$ by condition (4). Hence, inequality (22) is of the same type as (16) and results in the same type of bound on $|\widetilde{\xi(N)}|$ as specified by (17), (18), and (19).

To show that the system is also asymptotically stable, it is necessary only to obtain from (17) and (11) the inequality

$$\rho_1(N) = \sum_{n=0}^{N} \varphi(n)\sigma(n) \left[1 - \frac{\varphi(n)}{K\sigma(n)}\right] \leq \tilde{f}\left(|\xi_0|,\ \ |\widetilde{X(0)}|,\ \ |\sigma(0)|\right), \tag{23}$$

where $\tilde{f}$ is also a well defined positive function of the initial state variables and specifically independent of N. This is sufficient to obtain the results

$$\lim_{n \to \infty} \sigma(n) = \lim_{n \to \infty} \varphi(n) = \lim_{n \to \infty} \xi(n) = 0, \tag{24}$$

which imply asymptotic stability. This completes the proof of Theorem 1.

Corollary 1: A system belonging to $\Gamma(\infty)$ is absolutely stable if Theorem 1 is satisfied and q is non-negative.

SPECIFIC SUBCLASSES OF $\Gamma(K)$

Theorem 1 is a general result and can be used to obtain theorems relating to the absolute stability of systems belonging to specific subclasses of $\Gamma(K)$. As the first case in point, Theorem 1 will be used to prove a slightly restated version of the theorem presented in Ref. 1 for the subclass $\Gamma(K, -K', K')$. It should be noted that the restated version is slightly more general in that q need no longer be non-negative but can be any finite real number. The significance of this generalization and its meaning will be discussed in a separate section.

Theorem 2: A system of subclass $\Gamma(K, -K', K')$ is absolutely stable with respect to the null solution if a real number q (positive or negative) exist such that

$$\text{Re}\; G^*(z)\left[1 + q(z - 1)\right] + 1/K - \left(|q|\; K'/2\right)\; |(z - 1)\, G^*(z)|^2 \geq 0, \quad (25)$$

is satisfied for all $|z| = 1$.

Proof: For all systems of subclass $\Gamma(K, -K', K')$, the nonlinear gain function $\varphi(\sigma)$ satisfies conditions (4) and (5). These conditions imply that function $\varphi(\sigma)$ satisfies, between sampling instants n - 1 and n, the following inequality:

$$q\, \varphi(n - 1) \nabla\sigma(n) + \frac{|q|\, K'}{2} \left(\nabla\sigma(n)\right)^2 \geq q \int_{\sigma(n-1)}^{\sigma(n)} \varphi(n)\, d\sigma(n), \quad (26)$$

the validity of this assertion is illustrated in Fig. 2, where both the cases $q \geq 0$ and $q < 0$ are considered. For suitable auxiliary functions $\psi_{N,q}(n)$ and $V_{N,q}(n)$, (26) suggests the following definitions:

$$\psi_{N,q}(n) \triangleq q\, \varphi_N(n - 1) + \frac{K'|q|}{2} \left(\nabla\sigma(n)\right) - \frac{K'|q|}{2} \left(\sum_{i=1}^{m} \nabla f_i(n)\, X_i(0) - \gamma\, \xi_0\, \delta_{n0}\right), \quad (27)$$

$$V_{N,q}(n) \triangleq \nabla \sigma_N(n) - \left(\sum_{i=1}^{m} \nabla f_i(n) \, X_i(0) - \gamma \, \xi_0 \delta_{n0} \right). \tag{28}$$

The z-transforms of (27), (28) are given by

$$\psi^*_{N,q}(z) = q \, z^{-1} \, \varphi^*_N(z) - \frac{K'|q|}{2} \left(\frac{z-1}{z} \right) G^*(z) \, \varphi^*_N(z), \tag{29}$$

and

$$V^*_{N,q}(z) = - \left(\frac{z-1}{z} \right) G^*(z) \varphi^*_N(z). \tag{30}$$

This shows that condition (i) of Theorem 1 is satisfied.

As for condition (ii),

$$\sum_{n=0}^{\infty} \psi_{N,q}(n) \, V_{N,q}(n) \geq q \int_{\sigma(0)}^{\sigma(N)} \varphi d\sigma + q \, \varphi(N) \nabla \sigma_N(N+1)$$

$$- \left(b_1 + b_2 \, |\widetilde{\xi(N)}| \right) |\widetilde{X(0)}| - b_3 |\xi_0| , \tag{31}$$

where inequalities (43) and (44) of Ref. 1 have been used. By direct computation, it can be shown that

$$|\nabla \sigma_N(N+1)| \leq m \, h_3 |\widetilde{X(0)}| + \left(h_4 + h_6 / (1 - e^{-h_0}) \right) |\widetilde{\xi(N)}| , \tag{32}$$

and

$$|\varphi(N)| = |\xi(N) - \xi(N-1)| \leq 2 |\widetilde{\xi(N)}| . \tag{33}$$

Hence,

$$q \, \varphi(N) \nabla \sigma_N(N+1)$$

$$\geq -|q| \left\{ 2 \left[h_4 + h_6 / (1 - e^{-h_0}) \right] |\widetilde{\xi(N)}|^2 + 2mh_3 |\widetilde{\xi(N)}| \cdot |\widetilde{X(0)}| \right\} . \tag{34}$$

Combining these results yields

$$\sum_{n=0}^{\infty} \psi_{N,q}(n)\, V_{N,q}(n)$$

$$\geq \int_{\sigma(0)}^{\sigma(N)} \varphi\, d\sigma - a_4 |\widetilde{\xi(N)}|^2 - \left(a_1 |\widetilde{\xi(N)}| + a_2\right) |\widetilde{X(0)}| - a_3 |\xi_0| , \qquad (35)$$

where

$$a_4 = 2\,|q| \left[h_4 + h_6/(1 - e^{-h_0}) \right],$$

$$a_3 = b_3,$$

$$a_2 = b_1,$$

$$a_1 = b_2 + 2\, m h_3 .$$

This shows that condition (ii) of Theorem 1 is also satisfied. As for condition (iii), (29) and (30) can be used to verify that this condition reduces to criterion (25), which completes the proof Theorem 2.

Theorem 3: A system of subclass $\Gamma(K, -\infty, K')$ is absolutely stable with respect to the null solution if a non-negative q exist such that criterion (25) is satisfied for all $|z| = 1$.

The proof of this theorem is identical to that of Theorem 2, with the exception that inequality (26) and hence (35) hold only if $q \geq 0$. This assertion is illustrated in Fig. 3 and accounts for the restriction on q.

Theorem 4:* A system of subclass $\Gamma(K, -K', \infty)$ is absolutely stable

*A special case of this theorem in which $K' = 0$ has been suggested by Ya. Z. Tsypkin.

with respect to the null solution if a non-negative q[†] exist for which

$$\text{Re } G^*(z)\left(1 + q\frac{z-1}{z}\right) + 1/K - \frac{K'q}{2}\left|(z-1)\, G^*(z)\right|^2 \geq 0, \; \forall |z| = 1. \tag{36}$$

<u>Proof:</u> For any system of $\Gamma(K, -K', +\infty)$, the nonlinear gain function $\varphi(\sigma)$ satisfies (4) and

$$-K' \leq \frac{d\varphi}{d\sigma} \leq +\infty. \tag{37}$$

Then for $q \geq 0$

$$q\,\varphi(n)\,\nabla\sigma(n) + \frac{K'q}{2}\left(\nabla\sigma(n)\right)^2 \geq q\int_{\sigma(n-1)}^{\sigma(n)} \varphi(n)\, d\,\sigma(n), \tag{38}$$

the validity of which is illustrated in Fig. 4. This suggests taking $V_{N,q}(n)$ as defined by (28) and a new $\psi_{N,q}(n)$ defined by

$$\psi_{N,q}(n) \triangleq q\,\varphi_N(n) + \frac{K'q}{2}\nabla\sigma_N(n) - \frac{K'q}{2}\left(\sum_{i=1}^{m}\nabla f_i(n)\, X_i(0) - \gamma\xi_0\,\delta_{n0}\right). \tag{39}$$

The z-transform of (39) is readily determined to be

$$\psi^*_{N,q}(z) = q\,\varphi^*_N(z) - \frac{K'q}{2}\left(\frac{z-1}{z}\right) G^*(z)\;\varphi^*(z), \tag{40}$$

which implies that condition (i) of Theorem 1 is satisfied. Since

$$-\overline{\psi^*_{N,q}(z)}\; V^*_{N,q}(z) / \left|\varphi^*_N(z)\right|^2 = q\left(\frac{z-1}{z}\right) G^*(z) - \frac{K'q}{2}\left|(z-1)\, G^*(z)\right|^2, \tag{41}$$

[†]This theorem is true for both positive or negative q if the subclass is $\gamma(K, -K', K')$.

condition (iii) of Theorem 1 reduces to criterion (36) for the case in point.

From (38), one obtains

$$\sum_{n=0}^{\infty} \psi_{N,q}(n)\, V_{N,q}(n)$$

$$\geq \int_{\sigma(0)}^{\sigma(N)} \varphi\, d\sigma - \left(b_1 + b_2 \left|\widetilde{\xi(N)}\right|\right) \left|\widetilde{X(0)}\right| - b_3 |\xi_0| , \quad \forall N \geq 0, \forall q \geq 0 \tag{42}$$

hence, condition (ii) is also satisfied. This proves Theorem 4.

SIGNIFICANCE OF NEGATIVE q

For $\Gamma(K, -K', K')$, Theorem 2 allows the possibility for q to be negative. By taking the limiting case as $T \to 0$, in the manner as prescribed in Ref. 1, a corresponding generalization of Popov's theorem[2] applicable to a class of continuous systems is also valid.

To investigate the significance and applicability of criterion (25) when q is negative, suppose for $z = \exp(j\bar{w})$,

$$G(z) = X(\bar{w}) + j\, Y(\bar{w}), \qquad 0 \leq \bar{w} < 2\pi. \tag{43}$$

Letting $q = 0$ in (25), the criterion for absolute stability reduces to

$$\operatorname{Re} G^{*}(z) = X(\bar{w}) \geq -1/K_0, \tag{44}$$

where K_0 may be viewed as a first estimate of K. Inequality (44) is recognized as Tsypkin's criterion[3] for $\Gamma(K)$, and this was shown by examples in Ref. 1 to be generally conservative when applied to $\Gamma(K, -K', K')$. Further, suppose that for some $z_0 = \exp(j\bar{w}_0)$, $0 \leq \bar{w}_0 \leq \pi$,

$$X(\bar{w}_0) = \min_{|z| = 1} \operatorname{Re} G^*(z). \quad (45)$$

Without loss of generality, $X(\bar{w}_0)$ may be assumed negative, for otherwise (44) would have yielded the result $K = K_0 = \infty$. Then, from (44),

$$X(\bar{w}_0) = -1/K_0. \quad (46)$$

To realize a less conservative estimate for K than implied by (46), it is clear from an inspection of criterion (25) that this is possible only if

$$\operatorname{Re} q(z_0 - 1)\, G^*(z_0) - \frac{|q|\, K_0}{2} \left| (z_0 - 1)\, G^*(z_0) \right|^2 > 0. \quad (47)$$

For the case $q < 0$, a necessary and sufficient condition for (47) to hold is:

$$\text{(i)} \quad 0 \leq (1/2K_0)(\tan\theta_0 - \sqrt{\tan^2\theta_0 - 8}) < Y(\bar{w}_0) < (1/2K_0)(\tan\theta_0 + \sqrt{\tan^2\theta_0 - 8}), \quad \theta_0 = 90^\circ - \bar{w}_0/2; \quad (48)$$

$$\text{(ii)} \quad 0 \leq \bar{w}_0 < 38.94^\circ.$$

Condition (48) implies that $q < 0$ needs to be considered only if a point on the unit circle within the first 38.94 degree sector of the z-plane is mapped by $G^*(z)$ into the critical point $[x(\bar{w}_0),\ Y(\bar{w}_0)]$, and the critical point is within the sector, specified by (i) of (48), in the second quadrant of the $G^*(z)$-plane. This is illustrated in Fig. 5. For most systems of $\Gamma(K)$ condition (48) can not be satisfied. Indeed, attempts to construct an example satisfying (48) thus far have not been successful. This, of course, does not imply that examples may not exist, but it is an indication that the applicability of negative q may be limited to rather special cases.

For the limiting case as $T \longrightarrow 0$, i. e., for continuous system satisfying the requirements of Popov's theorem[2, 4, 5] the limiting condition corresponding to (48) is simply

$$Y(w_0) > 0, \tag{48a}$$

where

$$G(jw) \triangleq X(w) + j\, Y(w), \quad \forall \text{ real } w, \tag{48b}$$

and

$$X(w_0) = \min_{0 \le w \le \infty} \text{Re } G(jw). \tag{48c}$$

This condition indicates that for continuous sytems, the case $q < 0$ will be applicable only if the point $[X(w_0), Y(w_0)]$ on the frequency locus of the linear part of the system is located in the second quadrant of the G-plane. This is illustrated in Fig. 6.

It is known that for systems whose linear part has low pass characteristic and contains only real poles and zeros, condition (48a) can not be satisfied, and therefore, for these systems the case $q < 0$ is of no practical significance. However, an example in which (48a) is satisfied is a system with pure time delay and a under-damped second-order resonant dynamic.

A SYSTEM WITH DEADBAND NONLINEARITY

A number of examples illustrating the applications of either Theorem 2 or Theorem 3 was considered in Ref. 1. To illustrate the application of Theorem 4, a sampled-data system with deadband nonlinearity will be considered. Specifically, consider a sample-data system of $\Gamma(K, 0, +\infty)$ with a deadband type of nonlinear element described by:

$$\varphi(\sigma) = \begin{cases} -1, & \text{for } -\infty \le \sigma < -\alpha, \\ 0, & \text{for } -\alpha \le \sigma \le +\alpha, \\ +1, & \text{for } +\alpha < \sigma \le +\infty, \end{cases} \tag{49}$$

and a linear part whose z-transform is given by:

$$G^{*}(z) = \frac{1}{z-1} - \frac{1-e^{-1}}{z-e^{-1}} . \tag{50}$$

The problem is to relate the deadband width α to condition for asymptotic stability.

For this example, $K' = 0$, and criterion (36) reduces to

$$\text{Re } G^{*}(z)\,(1 + q\frac{z-1}{z}) + 1/K \geq 0, \;\forall |z| = 1, \; q \geq 0. \tag{51}$$

By letting $q = 0$, Tsypkin's criterion yields

$$K \leq 2/3, \text{ and } \alpha \geq 3/2, \tag{52}$$

for absolute and hence asymptotic stability. For $q \geq 0$, it can be shown by direct computation that

$$\underset{q \geq 0}{\text{Max}} \; \underset{|z|=1}{\text{Min}} \; \text{Re } G^{*}(z)\,(1 + q\frac{z-1}{z}) \cong -1. \tag{53}$$

Hence Theorem 4 insures asymptotic stability for

$$K \leq 1 \text{ and } \alpha \geq 1. \tag{54}$$

For comparison, the necessary and sufficient conditions for asymptotic stability, obtained by phase-plane construction, are:

$$K < 2 \quad \text{and } \alpha > 0.5. \tag{55}$$

It may appear that result (54) is still conservative compared to (55). An important difference which must be made clear is that result (55) insures complete stability for the nonlinearity defined by (49) only, whereas, result (54) insures complete stability for all forms of nonlinear functions in $\Gamma(K, 0, \infty)$, which includes, among others, (49) as a specific example. This is the important difference between a system which is completely stable and one which is absolutely stable.

CONCLUSION

In this note a certain inequality which is applicable to stability test of nonlinear discrete systems has been demonstrated. The importance of this theorem lies in the fact that various inequalities for subclasses of systems can be readily obtained. Furthermore, the significance of negative q in the new inequality as well as in Popov's theorem has been discussed.

It is indicated that generally, for physical systems, the applicability of negative q may be limited to rather special cases. This point, however, requires further exploration. By restricting q to positive values, the range of application of the theorem proposed in Ref. 1 can then be extended to other types of nonlinearities.

Further research in extending the proposed inequalities to cover a certain practical type of nonlinearities is being presently attempted. Also, the problem of obtaining a criterion for absolute stability, which contains both the necessary and sufficient condition will be one of the challenging topics for further investigations.

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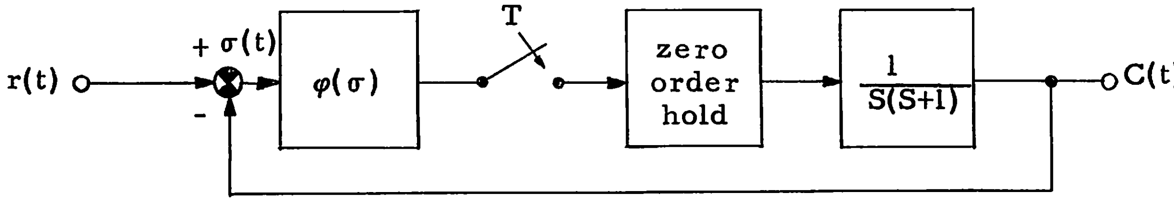

Fig. 1

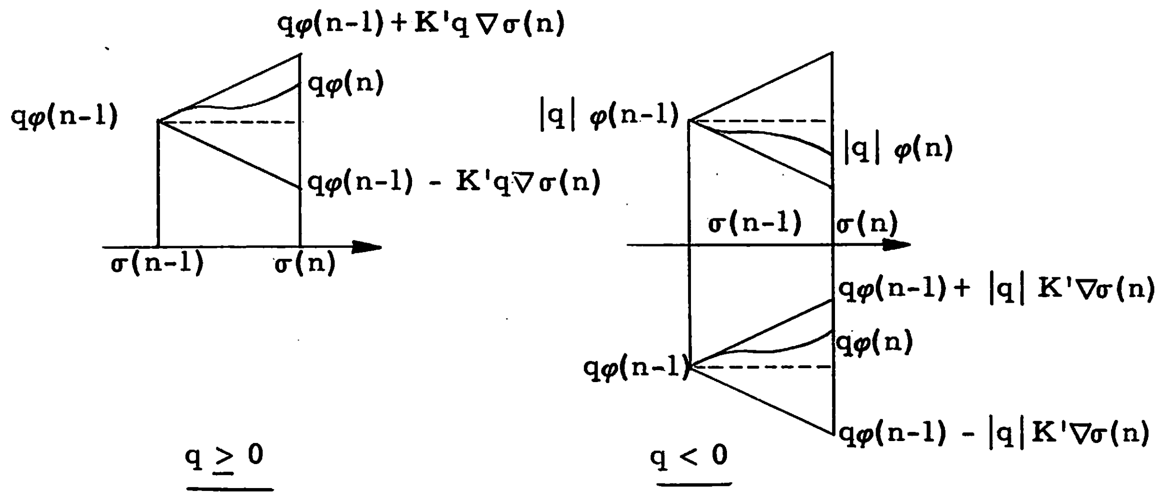

Fig. 2. Area inequality - Theorem 2.

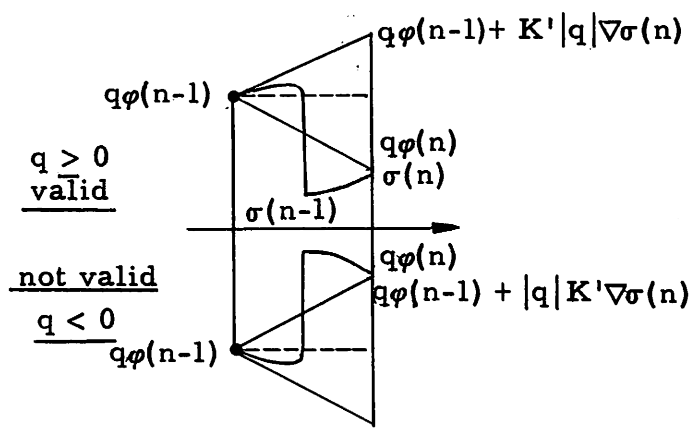

Fig. 3. Area inequality - Theorem 3.

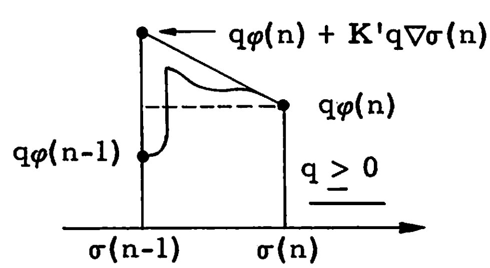

Fig. 4. Area inequality - Theorem 4.

Ⓐ $Y = -\frac{X}{2}(\tan\theta_0 - \sqrt{\tan^2\theta_0 - 8})$

Ⓑ $Y = -\frac{X}{2}(\tan\theta_0 + \sqrt{\tan^2\theta_0 - 8})$

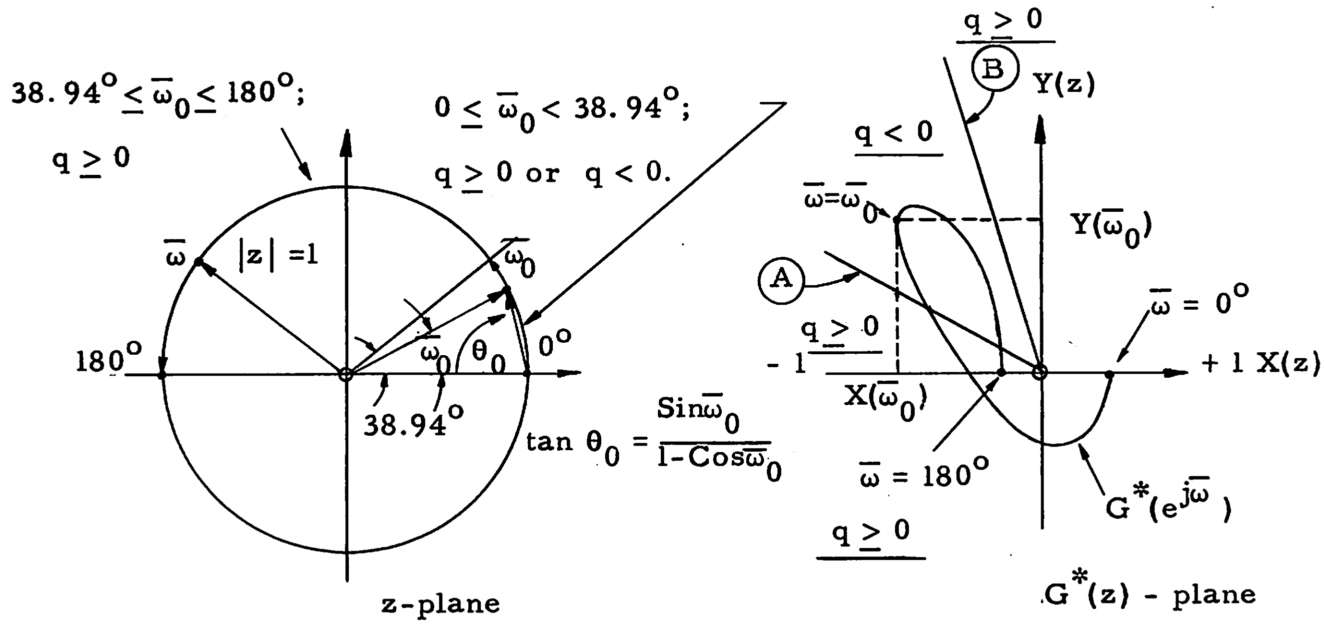

Fig. 5. Conditions for applicability of negative q - S.D.S. case.

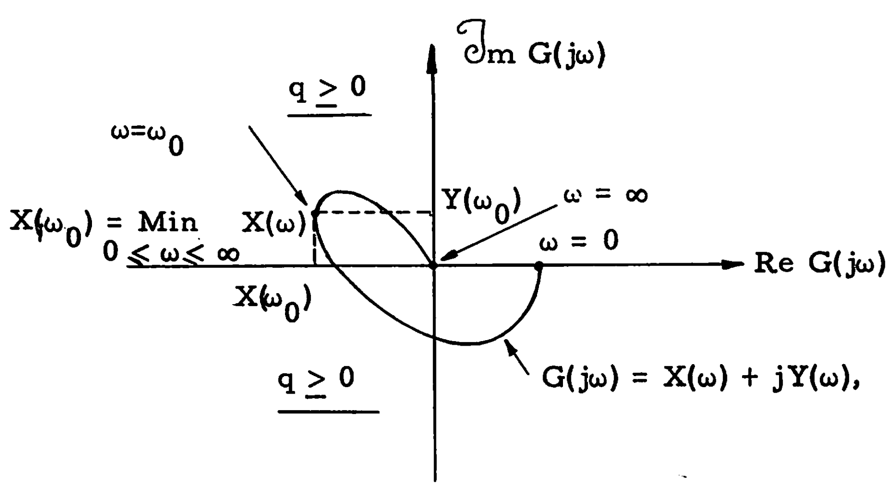

Fig. 6. Condition for applicability of negative q - extended Popov's theorem.