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OPTIMAL CONTROL OF ASYNCHRONOUS DISCRETE-TIME SYSTEMS

by

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ABSTRACT

This paper is devoted to establishing necessary conditions for the optimal controls in a class of fixed duration asynchronous sampled-data processes. These conditions apply both to the amplitude of the controls and the sampling instants. The plants of the systems under consideration are assumed to be described by nonlinear differential equations which do not necessarily lead to nonlinear difference equations in a synchronous sampled-data mode of operation. The techniques used are analogous to the ones used originally in establishing the Pontryagin Maximum Principle and the results are similar to the extent that one gets a requirement of a local maximum or stationarity for a Hamiltonian-like functional, while the Transversality Conditions remain the same.

1. INTRODUCTION

The original developments of optimal control theory were stimulated by the realization that the optimal control for a number of continuous time problems would be bang-bang. The reason for this was that a relay action controller has two advantages: it is simple and it is most efficient in controlling large quantities of power. However, it is now

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known that for many important problems, the optimal controls are not bang-bang but complicated time functions.

When the processes are of fixed duration and the continuous time optimal controls are not bang-bang, it is possible to regain the practical advantages of relay control by using an asynchronous sampled-date mode of operation. In this mode of operation, the control is piecewise constant with the discontinuities occuring nonperiodically.

All previous work concerning the optimal control of discrete time systems has been concerned with systems described by difference equations or with sampled-data systems in which the sampling instants are fixed. (See Refs. 2, 3, 4, 5.)

This paper is devoted to establishing necessary conditions for the optimal controls in a class of fixed duration asynchronous sampled-data processes. These conditions apply both to the amplitude of the controls and the switching instants. The plants of the systems under consideration are assumed to be described by nonlinear differential equations which do not necessarily lead to nonlinear difference equations in a synchronous sampled-data mode of operation. The techniques used are analogous to the ones used originally in establishing the Pontryagin Maximum Principle and the results are similar to the extent that one gets a requirement of a local maximum or stationarity for a Hamiltonian like functional, while the transversality conditions remain the same. Computational aspects of this problem are dealt with in a separate paper to be published soon.

2. FORMULATION OF THE OPTIMAL CONTROL PROBLEM

System Equations: Consider a system which satisfies the differential equations

$$\frac{d}{dt}\ \underline{x}(t) \triangleq \dot{\underline{x}}(t) = \underline{f}\left[\underline{x}(t), \underline{u}(t)\right] \tag{1}$$

where

$$\underline{x} = \text{col}\,(x_1, \ldots, x_n) \in E^n \tag{2}$$

is the state,

$$\underline{u} = \text{col}\,(u_1, \ldots, u_r) \in U \subset E^r \tag{3}$$

is the control and $\underline{f} = \text{col}\,(f_1, \ldots, f_n)$. (4)

U is assumed to be defined so that if $\underline{u}' \in U$ then there exists an $\epsilon > 0$ and a $\delta\underline{u}$ such that $\underline{u}' + \epsilon\,\delta\underline{u} \in U$. It is also assumed that if $\underline{u}' + \epsilon^1 \delta\underline{u}^1 \in U$, $\underline{u}' + \epsilon^2 \delta\underline{u}^2 \in U$, then $\underline{u}' + \lambda\epsilon^1\delta\underline{u}^1 + (1-\lambda)\,\epsilon^2\delta\underline{u}^2 \in U$, $0 \leq \lambda \leq 1$. It is also assumed that $f_i \in C'$ on $E^n \times U$ $i = 1, 2, \ldots, n$.

Assume that the time interval over which the system is to operate is given as $[t_0, t_K]$. Assume that there are K-1 sampling times in this interval (i.e., $t_0 < t_1 \leq t_2 \leq \cdots \leq t_{K-1} < t_K$). Define the sampling sequence

$$\left[t_{1,\,K-1}\right] \triangleq \left[t_1, t_2, \ldots, t_{K-1}\right]. \tag{5}$$

Define the control sequence

$$\left[\underline{u}(0,\ K-1)\right] \triangleq \left[\underline{u}(0), \ldots, \underline{u}(K-1)\right]. \tag{6}$$

The sampling sequence, (4) and the control sequence (5) taken together define the control to be applied to the system over the time interval $[t_0, t_K]$ in the following manner

$$\underline{u}(t) = \begin{cases} \underline{u}(0) & t_0 \leq t \leq t_1 \\ \underline{u}(1) & t_1 < t \leq t_2 \\ \cdot & \\ \cdot & \\ \cdot & \\ \underline{u}(K-1) & t_{K-1} < t \leq t_K \end{cases} \quad . \tag{7}$$

Define the sequence of the values of the state at the times $t_0, t_1, \ldots, t_K$ corresponding to the control $u(t)$ defined by (7) as

$$\left[\underline{x}(t_{0,K})\right] \triangleq \left[\underline{x}(t_0), \underline{x}(t_1), \ldots, \underline{x}(t_K)\right] \quad . \tag{8}$$

The relation between $\underline{x}(t_k)$, $\underline{x}(t_{k-1})$, and $\underline{u}(k-1)$ is

$$\underline{x}(t_k) = \underline{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \underline{f}\left[\underline{x}(t), \underline{u}(k-1)\right]dt \qquad k = 1, 2, \ldots, K. \tag{9}$$

<u>Initial Conditions</u>: It will be assumed that the initial time, t_0, and the initial state, $\underline{x}(t_0)$, are specified

<u>Terminal Conditions</u>:

<u>Time</u>: The terminal time will be assumed to be fixed at t_K.

<u>State</u>: Two types of terminal state constraints, S, will be considered.

i) Assume that $S \epsilon E^n$ is closed, convex set. This definition allows

S to be a point in E^n, a subset of E^n, or the whole of E^n. If S is of the second type (i.e., a closed, convex subset of E^n having more than one point), it will be assumed that it has no sharp edges (i.e., at each point on its boundary surface a unique tangent plane exists).

ii) Assume that S is an $(n - \ell)$-dimensional manifold described by the ℓ equations

$$S = \left[\underline{x} \mid g_i(\underline{x}) = 0 \quad i = 1, 2, \ldots, \ell < n\right] . \tag{10}$$

In this case, S need not be convex, it will be assumed however, that the g_i have continuous partial derivatives with respect to the x_i and that $\text{grad}_{\underline{x}}\, g_i \neq 0$ for any $\underline{x} \in S$, $i = 1, \ldots, \ell$.

Cost Function: Let the cost of a transition from the state $\underline{x}(t_{k-1})$ to the state $\underline{x}(t_k)$ caused by the control $\underline{u}(k-1)$ be given by $\int_{t_{k-1}}^{t_k} f_0\left[\underline{x}(t), \underline{u}(k-1)\right] dt$, where $f_0 \epsilon C'$ on $E^n x\, \mathcal{U}$. Let $x_0(t_k)$ be the cost of operating the system from time t_0 to time t_k. Then $x_0(t_k)$ is the solution of the differential equation

$$\dot{x}_0(t) = f_0\left[\underline{x}(t), \underline{u}(t)\right] \tag{11}$$

with $x_0(t_0) = 0$. Also, the relation between $x_0(t_k)$, $x_0(t_{k-1})$, $\underline{x}(t_{k-1})$ and $\underline{u}(k-1)$ is

$$x_0(t_k) = x_0(t_{k-1}) + \int_{t_{k-1}}^{t_k} f_0\left[\underline{x}(t), \underline{u}(k-1)\right] dt. \tag{12}$$

Extended System Equations: Now, for convenience, the system equations will be extended to include the cost variable by defining the $(n + 1)$-dimensional vectors

$$\underline{\tilde{x}} \triangleq \mathrm{col}\,(x_0, \underline{x}) \tag{13}$$

$$\underline{\tilde{f}} \triangleq \mathrm{col}\,(f_0, \underline{f}). \tag{14}$$

From (9) and (12), the equations of the extended system are found to be

$$\underline{\tilde{x}}(t_k) = \underline{\tilde{x}}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \underline{\tilde{f}}\left[\underline{x}(t),\ \underline{u}(k-1)\right] dt \tag{15}$$

for $k = 1, 2, \ldots, K$, with $\underline{\tilde{x}}(t_0) = \mathrm{col}\,\left[0,\ \underline{x}(t_0)\right]$. Then, given $\left[\underline{u}(0,\ K-1)\right]$ and $\left[t_{1,\ K-1}\right]$, $\left[\underline{\tilde{x}}(t_{0,K})\right]$ can be computed and $x_0(t_K)$ will be the cost incurred in operating the system over the time interval $\left[t_0, t_K\right]$.

Problem Statement: The optimal control problem for the systems under consideration can be stated as follows.

(P): Given the initial time t_0 the final time t_K, the initial state $\underline{\tilde{x}}(t_0)$ and the terminal state constraints S for the system described by (15), find the control sequence $\left[\underline{u}(0, K-1)\right]$, $\underline{u}(k) \epsilon\, \mathcal{U}$, $k = 0, 1, \ldots, K-1$, and the sampling sequence $\left[t_{1, K-1}\right]$, $t_0 < t_1 \leq \cdots \leq t_{K-1} < t_K$, which transfers the state from the given initial point, $x(t_0)$, at t_0, to a point $\underline{x}(t_K) \epsilon\, S$ at t_K, such that $x_0(t_K)$ is minimized.

Definition: The control sequence $\left[\underline{u}^*(0, K-1)\right]$ and the sampling sequence $\left[t^*_{1,\ K-1}\right]$ which minimize $x_0(t_K)$ and satisfy the boundary conditions of (P) will be called the optimal control sequence and the optimal sampling sequence, respectively, for (P) and the corresponding trajectory $\left[\underline{\tilde{x}}^*(t_{0,K})\right]$ will be called the optimal trajectory.

3. NECESSARY CONDITIONS FOR AN OPTIMAL SOLUTION

The Adjoint System: Let $\tilde{\underline{p}}$ be an $(n + 1)$-dimensional vector satisfying the differential equation

$$\dot{\tilde{\underline{p}}}(t) = - \left\{ \frac{\partial \tilde{\underline{f}} [\underline{x}(t),\ \underline{u}(t)]}{\partial \tilde{\underline{x}}} \right\}^T \tilde{\underline{p}}(t) \tag{16}$$

where A^T is the transpose of A. The vector $\tilde{\underline{p}}(t)$ will be called the adjoint variable.

Since this system of equations is homogeneous, all that is needed to generate the trajectory of the adjoint system is the knowledge of $\tilde{\underline{p}}(t_K)$. The determination of this vector will be a major consideration of this work.

Notice that $\dfrac{\partial f_i [\underline{x}(t),\ \underline{u}(t)]}{\partial x_0} = 0, \ i = 0, 1, 2, \ldots, n.$

Consequently, $\dot{p}_0 = 0.$

In other words p_0 is constant for all t.

The Hamiltonian: Let the "Hamiltonian," H, be defined by

$$H\left[\tilde{\underline{p}}(t_k),\ \underline{x}(t_{k-1}),\ \underline{u}(k-1)\right] = \langle \tilde{\underline{p}}(t_k),\ \int_{t_{k-1}}^{t_k} \tilde{\underline{f}}\left[\underline{x}(\tau),\ \underline{u}(k-1)\right] d\tau \rangle. \tag{17}$$

Now the conditions necessary for a control sequence $\left[\underline{u}^*(0,\ K-1)\right]$ and a sampling sequence $\left[t^*_{1,K-1}\right]$ to be optimal are stated in Theorem 1.

Theorem 1. If $\left[\underline{u}^*(0,\ K-1)\right]$ is an optimal control sequence, $\left[t^*_{1,\ K-1}\right]$ an optimal sampling sequence, and $\left[\tilde{\underline{x}}^*(t_{0,K})\right]$ the corresponding optimal trajectory for (P), then there exists a function $\tilde{\underline{p}}^*(t)$, $t_0 \leq t \leq t_K$ satisfying (16) such that

i) $H\left[\tilde{\underline{p}}^*(t^*_k),\ \underline{x}^*(t^*_{k-1}),\ \underline{u}^*(k-1)\right] = \langle \tilde{\underline{p}}^*(t^*_k),\ \int_{t^*_{k-1}}^{t^*_k} \tilde{\underline{f}}\left[\underline{x}^*(\tau),\ \underline{u}^*(k-1)\right] d\tau \rangle$

is a local maximum or stationary with respect to $\underline{u}^*(k-1) \epsilon U$ for each k, $1 \leq k \leq K$.

ii) $< \tilde{\underline{p}}^*(t_k^*), \tilde{\underline{f}}\left[\underline{x}^*(t_k^*), \underline{u}^*(k)\right] > = < \tilde{\underline{p}}^*(t_k^*), \tilde{\underline{f}}\left[\underline{x}^*(t_k^*), \underline{u}^*(k-1)\right] >$

$k = 1, \ldots, K-1$

iii) $p_0^*(t_K) \leq 0$.

Discussion: This theorem is basic and holds regardless of the form of the terminal constraint set S.

The problem (P) is a two point boundary value problem. There are $(2n + 2 + rK + K - 1)$ unknowns in the problem; the values of $\left[\tilde{\underline{x}}(t_{0,K})\right]$, $\left[\tilde{\underline{p}}(t_{0,K})\right]$, $\left[\underline{u}(0, K-1)\right]$ and $\left[t_{1, K-1}\right]$. The given initial conditions on the state, $\tilde{\underline{x}}(t_0)$, eliminate $(n + 1)$ unknowns. Condition (i) of Theorem 1 eliminates (rK) unknowns and condition (ii) eliminates $(K - 1)$ unknowns. There remain $(n + 1)$ unknowns. Knowledge of the value of $\tilde{\underline{p}}(t_K)$ will eliminate these unknowns. The value of $\tilde{\underline{p}}(t_K)$ will be found by means of the Transversality Conditions which do depend upon the form of the terminal constraint set, S.

Theorem 1 will be proven by examining each type of terminal contraint in turn and establishing the Transversality Conditions for each case. These Transversality Conditions are developed in Theorems 2-5. It will be shown that for each type of terminal constraint, the conditions of Theorem 1 are necessary.

The basic technique to be used will be to assume that the optimal control and trajectory are known. The control will then be perturbed so as to affect the trajectory only slightly. The necessary conditions which the optimal control must satisfy will then arise from the realization that any admissible perturbed control which satisfies the terminal constraints must not result in a lower cost.

The first item, then, to be considered is the effect upon the trajectory of small perturbations in the control. Since the value of $\underline{\tilde{x}}(t_0)$ is given, no perturbations of its value need be considered. Only perturbations in each control vector $\underline{u}(k)$, $i = 0, \ldots, K-1$, and each sampling time t_i, $i = 1, \ldots, K-1$, must be considered.. It is at this point that the basic difference between this discrete time problem and the similar one for continuous time problems occurs. It is required that any perturbation must i) be such that the perturbed control is admissible, and ii) affect the trajectory only slightly. In the continuous time problem, the control is only assumed to be measurable. Consequently, the perturbed control can vary from the original control by large amounts, provided the length of time, over which the perturbations are large, is small. This allows one to search out all of the control space at each time and to therefore require that the Hamiltonian be an absolute maximum at each instant of time.

In the discrete time problem, however, the only perturbations which have a small effect on the trajectory are small perturbations. Consequently, only local conditions can be obtained.

<u>Variation of the Control, Sampling Times, and the Trajectory:</u> Assume that the optimal control sequence $\left[\underline{u}^*(0, K-1)\right]$ and the optimal sampling sequence $\left[t^*_{1, K-1}\right]$ with the corresponding optimal trajectory $\left[\underline{\tilde{x}}^*(t^*_{0, K})\right]$ exist and are known. Suppose that a perturbed control $\left[\underline{u}(0, K-1)\right]$, and a perturbed sampling sequence $\left[t_{1,,K-1}\right]$ are defined as,

$$\underline{u}(k) = \underline{u}^*(k) + \epsilon\delta\underline{u}(k) \quad k = 0, \ldots, K-1 \tag{18}$$

$$t_k = t_k^* + \epsilon\delta t_k \qquad k = 1, \ldots, K-1 \tag{19}$$

where $\epsilon > 0$ is a real number, independent of k and small. $\delta\underline{u}(k)$ is an r-dimensional vector It must be chosen so that $\underline{u}(k) \epsilon U$ for some $\epsilon > 0$, $i = 0, \ldots, K-1$.

δt_k is chosen in the following manner:

1) If $t_k^* \neq t_j^*$ $j = 1, 2, \ldots, K-1$, $j \neq k$, then δt_k is arbitrary
2) If $t_k^* = t_j^*$ $j = 1, 2, \ldots, K-1$, choose $\delta t_k = \delta t_j$.

Then for ϵ sufficiently small, $t_k^* + \epsilon\,\delta t_k \,\epsilon\, (t_0, t_K)$, $k = 1, 2, \ldots, K-1$. Furthermore, ϵ will be chosen small enough that if $t_k^* < t_j^*$, $t_k^* + \epsilon\,\delta t_k \leq t_j^* + \epsilon\,\delta t_j$, $j, k = 1, 2, \ldots, K-1$.

These perturbed sequences are applied to the system and result in a perturbed trajectory $\left[\tilde{\underline{x}}(t_{0,K})\right]$. Define

$$\delta\tilde{\underline{x}}(t_k) \triangleq \tilde{\underline{x}}(t_k) - \tilde{\underline{x}}^*(t_k^*), \quad k = 0, 1, \ldots, K. \tag{20}$$

Then,

$$\delta\tilde{\underline{x}}(t_K) = \epsilon\tilde{\underline{y}}(t_K) + \underline{\mathcal{O}}(\epsilon) \tag{21}$$

where

$$\tilde{\underline{y}}(t_K) = \sum_{i=0}^{K-1} \Phi(t_K, t_{i+1}^*) \int_{t_i^*}^{t_{i+1}^*} \Phi(t_{i+1}^*, \tau) \frac{\partial f\left[\underline{x}^*(\tau), \underline{u}^*(i)\right]}{\partial \underline{u}} d\tau\delta\underline{u}(i)$$

$$+ \sum_{i=1}^{K-1} \Phi(t_K, t_i)\left\{\tilde{\underline{f}}\left[\underline{x}^*(t_i^*), \underline{u}^*(i-1)\right] - \tilde{\underline{f}}\left[\underline{x}^*(t_i^*), \underline{u}^*(i)\right]\right\}\delta t_i \tag{22}$$

and $\Phi(t, \tau)$ is the state transition matrix for the linear variational equations associated with (1) for the optimal trajectory. Let $\underline{\tilde{y}}(t_K) \epsilon E^{n+1}_{t_K}$, where $E^{n+1}_{t_K}$ is obtained from E^{n+1} by translating the origin of E^{n+1} to $\underline{\tilde{x}}^*(t_K)$.

Define the set

$$\mathcal{K}_{t_K} \triangleq \left[\underline{\tilde{y}}(t_K) \mid \underline{y}(t_K) \text{ satisfies } (22)\right] . \tag{23}$$

$\mathcal{K}_{t_K}$ is a convex cone with its vertex at $\underline{\tilde{x}}^*(t_K)$ and will be called the "cone of attainability" due to a similar definition by Pontryagin. Notice that $\underline{\tilde{x}}^*(t_K) \epsilon \mathcal{K}_{t_K}$ and is the vertex of the cone since for $\delta\underline{u}(k) = 0$, $k = 0, 1, \ldots, K - 1$ and $\delta t_k = 0$, $k = 1, 2, \ldots, K - 1$, $\delta\underline{\tilde{x}}(t_K) \equiv 0$ which corresponds to $\underline{\tilde{x}}(t_K) = \underline{\tilde{x}}^*(t_K)$. Also notice that for $t \epsilon [t_0, t_K]$, $\underline{\tilde{y}}(t)$ will satisfy the linear variational differential equations of (1) about the optimal trajectory,

$$\dot{\underline{\tilde{y}}}(t) = \frac{\partial \underline{\tilde{f}}\left[\underline{x}^*(t), \underline{u}^*(t)\right]}{\partial \underline{\tilde{x}}} \underline{\tilde{y}}(t) + \frac{\partial \underline{\tilde{f}}\left[\underline{x}^*(t), \underline{u}^*(t)\right]}{\partial \underline{u}} \delta\underline{u}(t) . \tag{24}$$

Case 1: Right End Constrained to Lie on a Smooth Surface: Let the constraint set S be an $(n - \ell)$-dimensional manifold described by

$$S = \left[\underline{x} \mid g_j(\underline{x}) = 0, \quad j = 1, 2, \ldots, \ell < n\right] . \tag{25}$$

Since each g_j has continuous first partial derivatives and since $(\partial g_j / \partial \underline{x}) \neq 0$, $i = 1, \ldots, \ell$ for $\underline{x} \epsilon S$, there is an $(n - \ell)$-dimensional plane, T, tangent to S at each $\underline{x} \epsilon S$ described by

$$T = \{\underline{x}' \mid < \underline{x}' - \underline{x}, \left[(\partial g_j(\underline{x})/\partial \underline{x}\right] > = 0, \ j = 1, 2, \ldots, \ell\} . \tag{26}$$

Now construct the $(n + 1 - \ell)$-dimensional cylinder defined by

$$S^1 = \left[\underset{\sim}{\tilde{x}} \mid \underset{\sim}{\tilde{x}} = \operatorname{col}(x_0, \underline{x}),\ \underline{x} \in S\right] \tag{27}$$

(see Fig. 1). It has been assumed that $\underset{\sim}{\tilde{x}}^*(t_K) \in S^1$. S^1 will have an $(n + 1 - \ell)$-dimensional tanjent plane at $\underline{x}^*(t_K)$ described by

$$T^1 = \left[\underset{\sim}{\tilde{x}} \mid \underset{\sim}{\tilde{x}} = \operatorname{col}(x_0, \underline{x}),\ \underline{x} \in T^*\right] \tag{28}$$

where T^* is the tangent plane to S at $\underline{x}^*(t_K)$. Clearly, the projection of T^1 onto E^n is T^*.

Construct the hyperplane C passing through $\underset{\sim}{\tilde{x}}^*(t_K)$ perpendicular to the x_0 axis.

$$C = \left[\underset{\sim}{\tilde{x}} \mid x_0 = x_0^*(t_K)\right] \quad . \tag{29}$$

C will cut T^1 into two semi-infinite planes

$$T^{1+} = \left[\underset{\sim}{\tilde{x}} \mid \underset{\sim}{\tilde{x}} \in T^1,\ x_0 \geq x_0^*(t_K)\right] \tag{30}$$

$$T^{1-} = \left[\underset{\sim}{\tilde{x}} \mid \underset{\sim}{\tilde{x}} \in T^1,\ x_0 \leq x_0^*(t_K)\right] \tag{31}$$

with the common boundary

$$T^{1^o} = C \cap T^1 \quad , \tag{32}$$

C will also cut the cylinder S^1 into two semi-infinite cylinders

$$S^{1+} = \left[\underset{\sim}{\tilde{x}} \mid \underset{\sim}{\tilde{x}} \in S^1,\ x_0 \geq x_0^*(t_K)\right] \tag{33}$$

$$S^{1-} = \left[\underset{\sim}{\tilde{x}} \mid \underset{\sim}{\tilde{x}} \in S^1,\ x_0 \leq x_0^*(t_K)\right] \tag{34}$$

with the common boundary

$$S^{1^o} = C \cap S^1 \quad . \tag{35}$$

Let $\underline{\xi} = \text{col}(\xi_1, \xi_2, \ldots \xi_n)$ be an arbitrary n-dimensional vector lying in T^*. Let $\underline{p}(t_K)$ be a vector consisting of the last n-components of $\tilde{\underline{p}}(t_K)$

$$\underline{p}(t_K) \triangleq \text{col}\left[p_1(t_K), \ldots, p_n(t_K)\right] . \tag{36}$$

Then for this case, the following theorem holds;

Theorem 2: Consider the problem (P) when the constraint set S is an $(n - \ell)$-dimensional smooth manifold defined by (25). Then, necessary conditions that $\left[\underline{u}^*(0, K-1)\right]$ be an optimal control sequence and $\left[t^*_{1, K-1}\right]$ be an optimal sampling sequence are

i) The conditions stated in Theorem 1,

ii) $< \underline{p}^*(t_K), \underline{\xi} > = 0$.

Proof: Since $\left[\underline{u}^*(0, K-1)\right]$ is an optimal control sequence and $\left[t^*_{1, K-1}\right]$ an optimal sampling sequence, it is necessary that any admissible perturbed control sequence and sampling sequence, whose corresponding trajectory satisfies the terminal conditions, not give a lower cost. For this requirement to be fulfilled, it is necessary that there exist a hyperplane separating $\mathcal{K}_{t_K}$ and T^{1-}. This is shown by establishing Lemma 1.

Lemma 1: Let $\left[\tilde{\underline{x}}(t_{0,K})\right]$ be the trajectory starting from $\tilde{\underline{x}}(t_0)$, and corresponding to the control sequence $\left[\underline{u}(0, K-1)\right]$ and the sampling sequence $\left[t_{1, K-1}\right]$. Let G be a $g \leq n$-dimensional smooth manifold with an edge, G_e, in E^{n+1}. Let L be the half-plane tangent to G at $\tilde{\underline{x}}(t_K)$.

If the cones, $\mathcal{K}_{t_K}$ and L, having a common vertex at $\tilde{\underline{x}}(t_K)$, are not separated, then there exists a control sequence $\left[\underline{u}'(0, K-1)\right]$

and a sampling sequence $[t'_{1, K-1}]$ with a corresponding trajectory $[\tilde{\underline{x}}'(t'_{0,K})]$, starting from $\tilde{\underline{x}}(t_0)$ such that $\tilde{\underline{x}}'(t_K) \in G$ but $\tilde{\underline{x}}'(t_K) \notin G_e$.

Proof of Lemma 1: The proof of this lemma is identical with the one given for Lemma 10 in Ref. 1.

Let us apply this lemma to the proof of Theorem 2. It follows from Lemma 1 that if the cones $\mathcal{K}_{t_K}$ and T^{1-}, having the common vertex $\tilde{\underline{x}}^*(t_K)$, are not separated, then there exists a control sequence $[\underline{u}'(0, K-1)]$ and a sampling sequence $[t'_{1,K-1}]$ with a corresponding trajectory $[\tilde{\underline{x}}'(t_{0,K})]$ such that $\tilde{\underline{x}}'(t_K)$ lies in S^{1-} but not on the edge of S^{1-} and consequently will satisfy the constraints and have a lower cost.

Therefore, for $[\underline{u}^*(0, K-1)]$, $[t^*_{1,K-1}]$ and $[\underline{x}^*(t^*_{0,K})]$ to be optimal, it is necessary that there exist a hyperplane, call it A, separating $\mathcal{K}_{t_K}$ and T^{1-}. Let the (n + 1)-dimensional vector $\tilde{\underline{a}} = \text{col}(a_0, a_1, \ldots, a_{n+1})$ be the normal to A. Choose the direction of $\tilde{\underline{a}}$ so that

$$<\tilde{\underline{x}} - \tilde{\underline{x}}^*(t_K), \tilde{\underline{a}}> \leq 0 \text{ if } \tilde{\underline{x}} - \tilde{\underline{x}}^*(t_K) \in \mathcal{K}_{t_K} \tag{37}$$

then

$$<\tilde{\underline{x}} - \tilde{\underline{x}}^*(t_K), \tilde{\underline{a}}> \geq 0 \text{ if } \tilde{\underline{x}} - \tilde{\underline{x}}^*(t_K) \in T^{1-}. \tag{38}$$

Clearly, the hyperplane A contains $\tilde{\underline{x}}^*(t_K)$ and T^{1^o}. Let $\underline{\xi} = \text{col}(\xi_1, \ldots, \xi_n)$ be any vector in T^*. Then $\tilde{\underline{\xi}} = \text{col}(0, \underline{\xi})$ will be parallel to T^{1^o}. Since $T^{1^o} \subset A$, $<\tilde{\underline{a}}, \tilde{\underline{\xi}}> = 0$. But $\xi_0 = 0$. Therefore

$$<\tilde{\underline{a}}, \tilde{\underline{\xi}}> = \sum_{i=1}^{n} a_i \xi_k = 0. \tag{39}$$

Since x_0 does not appear in the constraint relation (25), any point lying on the vector emanating from $\underline{\tilde{x}}^*(t_K)$ and pointing in the direction $\underline{\eta} = \text{col}\,(-1, 0, 0, \ldots 0)$ belongs to T^{1-}. Consequently, from the way $\underline{\tilde{a}}$ was chosen, it follows that

$$< \underline{\tilde{a}},\ \underline{\tilde{\eta}} > \ \geq\ 0.$$

But $< \underline{\tilde{a}},\ \underline{\tilde{\eta}} > = -a_0 \geq 0$

and therefore

$$a_0 \leq 0. \tag{40}$$

Since $\underline{\tilde{y}}(t_K) \in \mathcal{K}\, t_K$, it follows from the necessary condition (37) that

$$< \underline{\tilde{a}},\ \underline{\tilde{y}}(t_K) > \ \leq\ 0. \tag{41}$$

Let

$$\underline{\tilde{p}}^*(t_K) \triangleq \underline{\tilde{a}}\ . \tag{42}$$

Then the necessary condition (41) becomes

$$< \underline{\tilde{p}}^*(t_K),\ \underline{\tilde{y}}(t_K) > \ \leq 0\ . \tag{43}$$

Now choose a special perturbation in the optimal control. Assume that the optimal control sequence or the optimal sampling sequence is perturbed at only one value; i. e.,

either

$$\underline{u}(t) = \begin{cases} \underline{u}^*(0) & t_0 \leq t \leq t_1^* \\ \vdots & \vdots \\ \underline{u}^*(\nu - 2) & t_{\nu-2}^* < t \leq t_{\nu-1}^* \\ \underline{u}^*(\nu - 1) + \epsilon\delta\underline{u}(\nu - 1) & t_{\nu-1}^* < t \leq t_\nu^* \\ \underline{u}^*(\nu) & t_\nu^* < t \leq t_{\nu+1}^* \\ \vdots & \vdots \\ \underline{u}^*(K - 1) & t_{K-1}^* < t \leq t_K \end{cases} \tag{44}$$

or

$$\underline{u}(t) = \begin{cases} \underline{u}^*(0) & t_0 \le t \le t_1^* \\ \vdots & \vdots \\ \underline{u}^*(\nu - 2) & t_{\nu-2}^* < t \le t_{\nu-1}^* \\ \underline{u}^*(\nu - 1) & t_{\nu-1}^* < t \le t_\nu^* + \epsilon\delta t_\nu \\ \underline{u}^*(\nu) & t_\nu^* + \epsilon\,\delta t_\nu < t \le t_{\nu+1}^* \\ \vdots & \vdots \\ \underline{u}^*(K - 1) & t_{K-1}^* < t \le t_K \end{cases} \quad . \tag{45}$$

Then

$$\delta\underline{\tilde{x}}(t_k^*) = \underline{\tilde{y}}(t_k^*) = 0 \quad \text{for} \quad k = 0, 1, \ldots, \nu - 1 \ . \tag{46}$$

Since $\delta\underline{u}(t) = 0$, $t_\nu^* < t \le t_K$, it is seen from (16) and (24) that

$$\frac{d}{dt} < \underline{\tilde{p}}^*(t), \underline{\tilde{y}}(t) > = < \underline{\dot{\tilde{p}}}^*(t), \underline{\tilde{y}}(t) > + < \underline{\tilde{p}}^*(t), \underline{\dot{\tilde{y}}}(t) > = 0 \ . \tag{47}$$

The necessary condition (43) becomes

$$<\underline{\tilde{p}}^*(t_K), \underline{\tilde{y}}(t_K) > = < \underline{\tilde{p}}^*(t_\nu^*), \underline{\tilde{y}}(t_\nu^*) > \le 0 \ . \tag{48}$$

If the optimal control sequence was perturbed as in (44),

$$\underline{\tilde{y}}(t_\nu^*) = \left\{ \int_{t_{\nu-1}^*}^{t_\nu^*} \Phi(t_\nu^*, \tau) \frac{\partial \underline{\tilde{f}}\left[\underline{x}^*(\tau), \underline{u}^*(\nu - 1)\right]}{\partial \underline{u}} d\tau \right\} \delta\underline{u}(\nu - 1) \ . \tag{49}$$

Then (48) becomes

$$< \underline{\tilde{p}}^*(t_\nu^*), \left\{ \int_{t_{\nu-1}^*}^{t_\nu^*} \Phi(t_\nu^*, \tau) \frac{\partial \underline{\tilde{f}}\left[\underline{x}(\tau), \underline{u}^*(\nu - 1)\right]}{\partial \underline{u}} d\tau \right\} \delta\underline{u}(\nu - 1) >$$

$$= < \nabla_{\underline{u}} H\left[\underline{\tilde{p}}^*(t_\nu^*), \underline{x}^*(t_{\nu-1}^*), \underline{u}^*(\nu - 1)\right], \delta\underline{u}(\nu - 1) > \le 0 \ . \tag{50}$$

Equivalently, $H\left[\underline{\tilde{p}}^*(t_\nu^*), \underline{x}^*(t_{\nu-1}^*), \underline{u}^*(\nu - 1)\right]$ must be a local maximum or stationary with respect to $\underline{u}^*(\nu - 1) \epsilon\, U$, $\nu = 1, 2, \ldots, K$, and condition (i) of Theorem 1 is proven for this terminal constraint.

If the optimal control were perturbed as in (45),

$$\underline{\tilde{y}}(t_\nu^*) = \left\{\underline{f}\left[\underline{x}^*(t_\nu^*), \underline{u}^*(\nu - 1)\right] - \underline{\tilde{f}}\left[\underline{x}^*(t_\nu^*), \underline{u}^*(\nu)\right]\right\}\delta t_\nu \quad . \tag{51}$$

Then from (48),

$$< \underline{\tilde{p}}^*(t_\nu^*), \left\{\underline{f}\left[\underline{x}^*(t_\nu^*), \underline{u}^*(\nu - 1)\right] - \underline{\tilde{f}}\left[\underline{x}^*(t_\nu^*), \underline{u}^*(\nu)\right]\right\}\delta t_\nu > \leq 0 \quad . \tag{52}$$

But δt_ν can be positive or negative hence (52) is satisfied only when

$$< \underline{\tilde{p}}^*(t_\nu^*), \underline{\tilde{f}}\left[\underline{x}^*(t_\nu^*), \underline{u}^*(\nu - 1)\right] > = < \underline{\tilde{p}}^*(t_\nu^*), \underline{\tilde{f}}\left[\underline{x}^*(t_\nu^*), \underline{u}^*(\nu)\right] >$$

$$\nu = 1, 2, \ldots, K - 1. \tag{53}$$

Condition (ii) of Theorem 1 is proven. Recall that $\underline{\tilde{p}}^*(t_K) = \underline{\tilde{a}}$, $a_0 \leq 0$ and $\sum_{i=1}^n a_i \xi_i = 0$. Consequently, Condition (ii) of Theorem 1 and Condition (ii) of Theorem II, are proven. This completes the proof of Theorem 2.

Case 2: Right End Constrained to Lie at a Point: Next, consider the problem when S is a point in E^n. For this case, the following Theorem holds.

Theorem 3: Consider the problem (P) when the constraint set S is a point in E^n. Then, necessary conditions for $\left[\underline{u}^*(0, K - 1)\right]$ and $\left[t^*_{1, K-1}\right]$ to be an optimal control sequence and an optimal sampling sequence, respectively, are the conditions stated in Theorem 1.

Proof: (See Fig. 2.) Since S is a point, S^1 is a line perpendicular to E^n(i.e., parallel to the x_0 axis) and passing through S. S^{1-} is the semi-infinite line consisting of those points in S^1 which are in or below C. Clearly, $T^{1-} = S^{1-}$. It follows from Lemma 1 that $\mathcal{K}_{t_K}$ and S^{1-} must be separated by a hyperplane A, with its normal $\underline{\tilde{a}}$. As in Theorem 2, one may choose $a_0 \leq 0$. However, since T^{1^o} is a point, no transversality conditions are imposed on $\underline{\tilde{a}}$. Proceeding as in Theorem 2, it is found by letting $\underline{\tilde{p}}^*(t_K) = \underline{\tilde{a}}$ that conditions (i) (ii) and (iii) of Theorem 1 are necessary for $[\underline{u}^*(0, K-1)]$ to be an optimal control sequence and $[t^*_{1,K-1}]$ to be an optimal sampling sequence. This completes the proof of Theorem 3.

Case 3: Free Right End: Consider the problem where S is the whole space E^n. For these terminal conditions the following theorem holds.

Theorem 4: Consider the problem (P) when the constraint set $S = E^n$. Then, necessary conditions for $[\underline{u}^*(0, K-1)]$ and $[t^*_{1,K-1}]$ to be an optimal control sequence and an optimal sampling sequence, respectively are,

i) The conditions stated in Theorem 1

ii) $p^*_i(t_K) = 0 \quad i = 1, 2, \ldots, n.$

Proof: (See Fig. 3.) Clearly, S^{1-} will be the closed half-space in E^{n+1} consisting of all points lying below or in C. Let T^{1^o} be an arbitrary line in C passing through $\underline{\tilde{x}}^*(t_K)$ and let T^{1-} be the semi-infinite hyperplane consisting of those points lying in T^{1^o} or directly below it. It follows from Lemma 1 that T^{1-} and $\mathcal{K}_{t_K}$ must be separated by some hyperplane A with its normal $\underline{\tilde{a}}$. It follows from Theorem 2 that $\underline{\tilde{a}}$ must be perpendicular to T^{1^o}. But T^{1^o} has arbitrary direction in C, consequently, $\underset{\sim}{\tilde{a}}$ must be orthogonal to any vector in C, i. e., to C itself. It can only have the value

$$\underline{\tilde{a}} = \text{col}\,(-1, 0, 0, \ldots, 0) \tag{54}$$

and therefore A coincides with C. Proceeding as in Theorem 2 and letting $\underline{\tilde{p}}^{*}(t_K) = \underline{\tilde{a}}$, the conditions of Theorem 1 are shown to be necessary and in addition the Transversality Conditions (ii) of Theorem 4 are shown to be necessary. This completes the proof of Theorem 4.

<u>Case 4</u>: <u>Right End Constrained to Lie in an n-Dimensional Subset of E^n</u>: The problems where S is a point in E^n, a manifold of dimension $n - \ell < n$ and the whole of E^n have been considered. The only problem left is that where S is an n-dimensional proper subset of E^n. Assume that S is closed and convex.

S^{1-} will be a semi-infinite cylinder consisting of all those points in S^1 which lie in or below C. Two possibilities can occur. (See Fig. 5.)

<u>Case A:</u> $\underline{\tilde{x}}^{*}(t_K)$ may lie on the surface of S^1. It will also lie in C. Let T^1 be the tangent plane to S^1 at $\underline{\tilde{x}}^{*}(t_K)$. T^1 will be cut in half by C. Let T^{1+} and T^{1-} be the upper and lower halves as before with the common boundary $T^{1^o} = C \cap T^1$. Let $\underline{\tilde{h}}$ be the normal to T^1 at $\underline{\tilde{x}}^{*}(t_K)$ which points away from S^1

$$\underline{\tilde{h}} = \text{col}\,(0, h_1, \ldots, h_n)\ . \tag{55}$$

Let $\underline{\tilde{c}}$ be the upward pointing normal to C, $\underline{\tilde{c}} = \text{col}\,(1, 0, \ldots, 0)$. (56)

<u>Case B</u>: $\underline{\tilde{x}}^{*}(t_K)$ may lie interior to S^1. It will still lie in C. In this case a tangent plane cannot be defined and $\underline{\tilde{h}}$ will be defined

as the zero vector

$$\underline{\tilde{h}} = \text{col}\,(0, 0, \ldots, 0) \tag{57}$$

$\underline{\tilde{c}}$ will be as in case A.

Then in either case the following theorem holds.

Theorem 5: Consider the problem (P) when the constraint set S is a closed, convex, n-dimensional proper subset of E^n. Then, necessary conditions for $[\underline{u}^*(0, K-1)]$ and $(t^*_{1, K-1})$ to be an optimal control sequence and an optimal sampling sequence, respectively, are,

i) The conditions stated in Theorem 1,

ii) $\underline{\tilde{p}}^*(t_K) = \lambda\,\underline{\tilde{h}} + \mu\,\underline{\tilde{c}}$ where λ, μ are nonpositive constants.

Proof: (See Fig. 4.) Consider Case B. This situation is identical with that of the free right end and Theorem 4 holds. Since $\underline{\tilde{h}} = 0$ in this case

$$\underline{\tilde{p}}^*(t_K) = \mu\underline{\tilde{c}} = \text{col}\,(\mu, 0, 0, \ldots, 0) \quad . \tag{58}$$

But from Theorem 4 it is seen that $\mu \leq 0$. Therefore Theorem 5 is true for Case B.

Consider Case A. Let L be a half-hyperplane having T^{1^o} as an edge and intersecting S^{1-}. Then it follows from Lemma 1 that $\mathcal{K}_{t_K}$ must be separated from L. Therefore, $\mathcal{K}_{t_K}$ must be separated from the closed quarter-space, Q, bounded from above by C and from the side by T^{1-}. Let A be the hyperplane which separates $\mathcal{K}_{t_K}$ from Q and let $\underline{\tilde{a}}$ be the normal to A which points into Q. Then $-\underline{\tilde{a}}$ will lie in the space spanned by $\underline{\tilde{h}}$ and $\underline{\tilde{c}}$ and will lie between them. Therefore, $\underline{\tilde{a}}$ can be written as a negative linear combination of $\underline{\tilde{h}}$ and $\underline{\tilde{c}}$.

$$\underline{\tilde{a}} = \lambda \underline{\tilde{h}} + \mu \underline{\tilde{c}} \qquad \lambda, \ \mu \leq 0. \tag{59}$$

Then, proceeding as in Theorem 2 and letting $\underline{\tilde{p}}^*(t_K) = \underline{\tilde{a}}$, the conditions of Theorem 1 as well as the Transversality Conditions (ii) of Theorem 5 are shown to be necessary.

The conditions of Theorem 1 have been shown to be necessary for each terminal constraint under consideration. Therefore, the proof of Theorem 1 is completed.

4. CONCLUSIONS

Although all the results in this paper were developed for sampled-data systems in which one could choose the sampling instants, most of them also remain valid when the sampling sequence $(t_{1, K-1})$ is fixed. For this case, condition (ii) of Theorem 1 no longer applies but conditions (i) and (iii) of Theorem 1 and the Transversality Conditions stated in Theorems 2 to 5 are still necessary.

There is one particular problem in which some of the restrictions on the control constraint set may be relaxed. For the free right-end case it is only necessary that for any $\underline{u}' \epsilon U$, there exist some ϵ and some $\delta \underline{u}$ such that $\underline{u}' + \epsilon \delta \underline{u} \epsilon U$; i.e., it is not necessary that the set $(\delta \underline{u} \mid \underline{u}' + \epsilon \delta \underline{u} \epsilon U$ for some $\epsilon)$ be convex. The reason for this is that S^{1-} for this case is a whole half space in E^{n+1} and the separating hyperplane is uniquely defined. Consequently, the proofs no longer depend on the convexity of $\mathcal{K}_{t_K}$.

For all cases considered here, it is possible for the control constraint set to change with the time step k $[$i.e., $U = U(k)]$, provided that each of the $U(k)$, $k = 0, 1, \ldots, K - 1$, is an admissible constraint set.

It is hoped that the results of this paper will be useful in developing optimal control systems which are more readily engineered than those operating in a continuous time mode.

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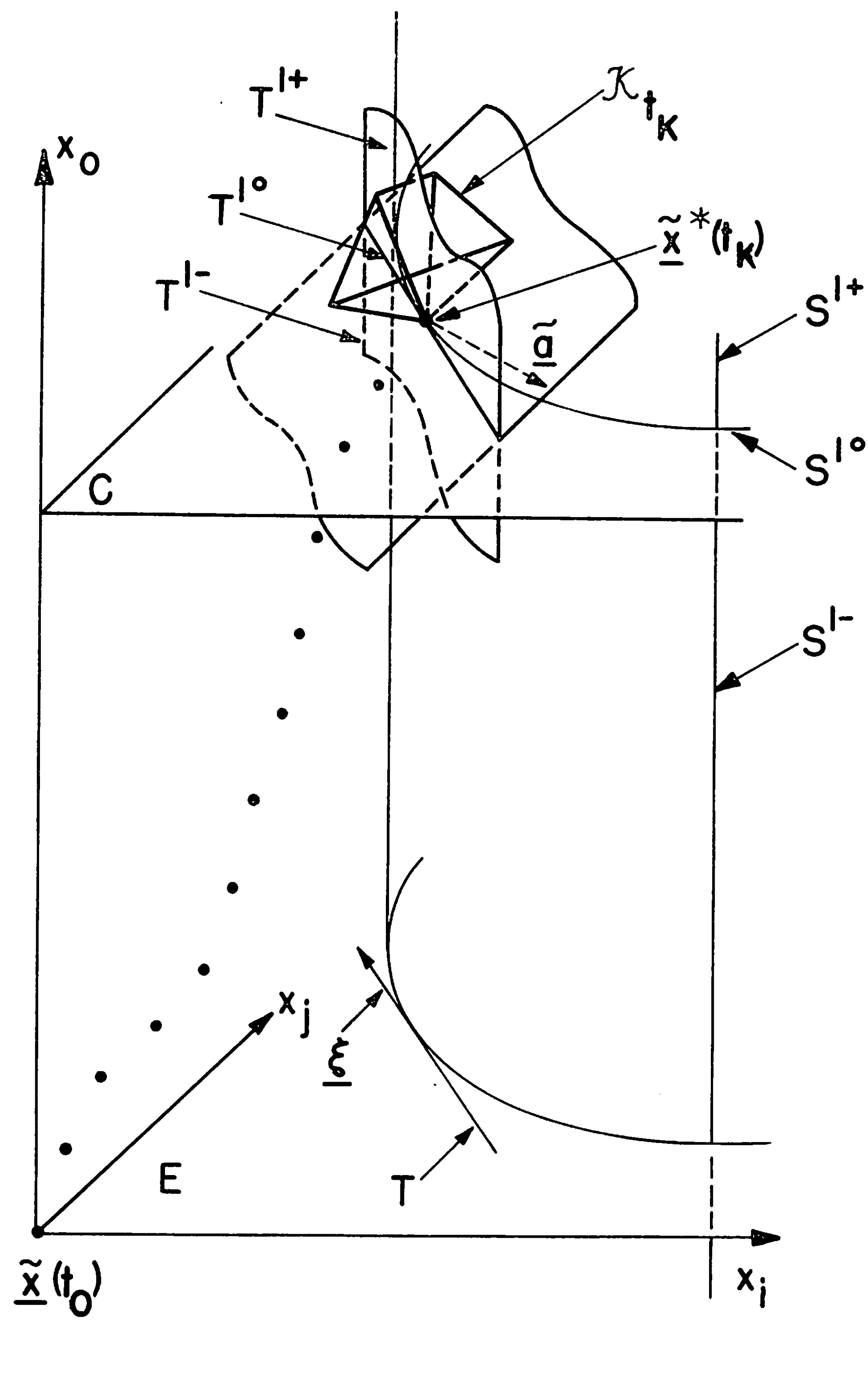

Fig. 1. Illustration for Theorem 2.

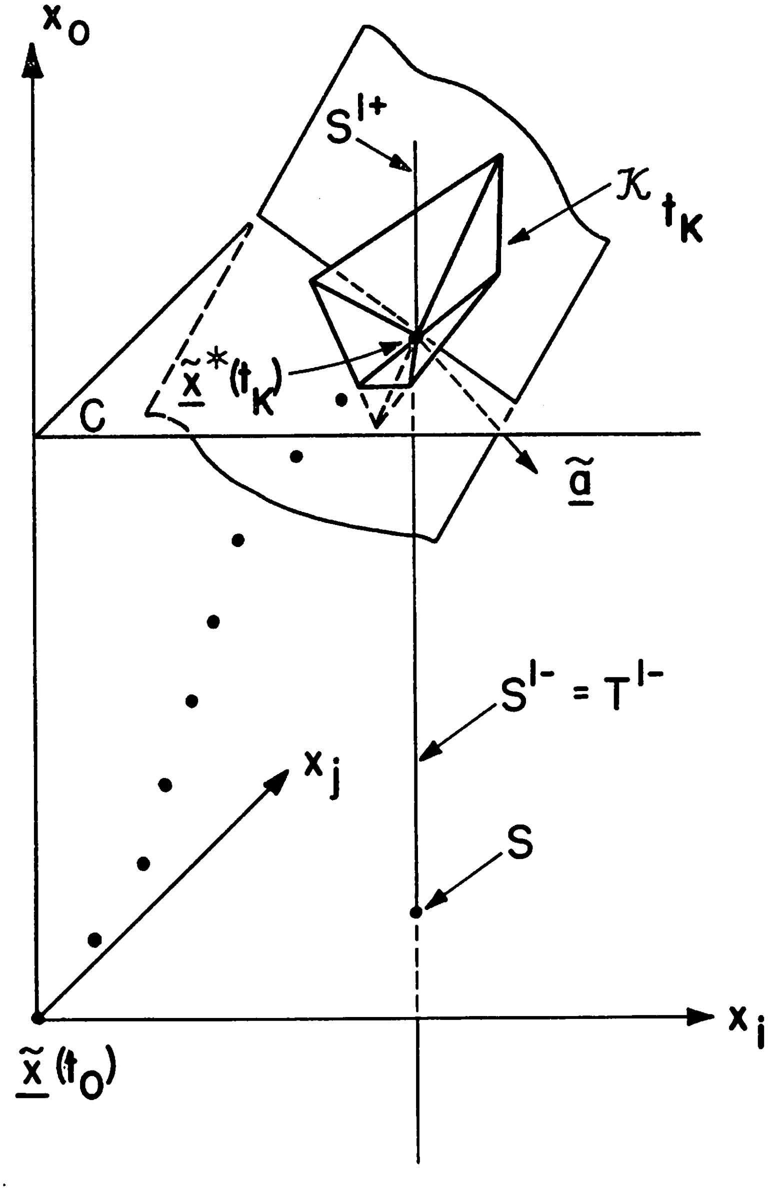

Fig. 2. Illustration for Proof of Theorem 3.

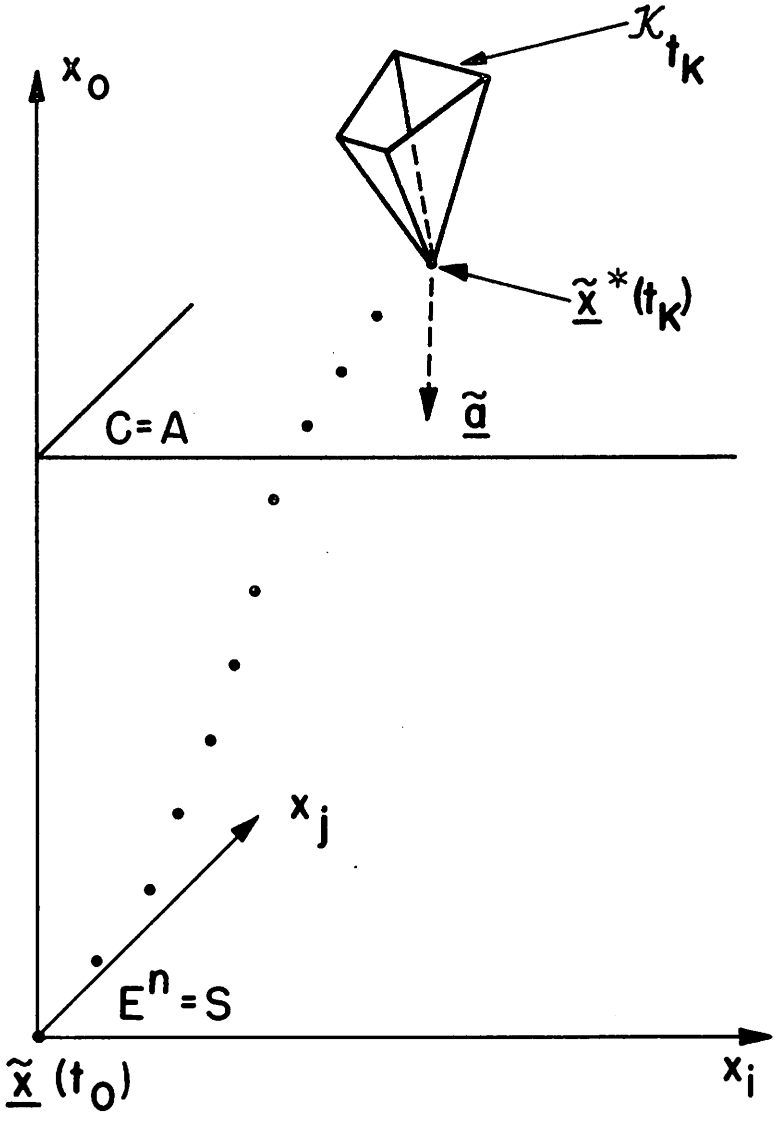

Fig. 3. Illustration for the Proof of Theorem 4.

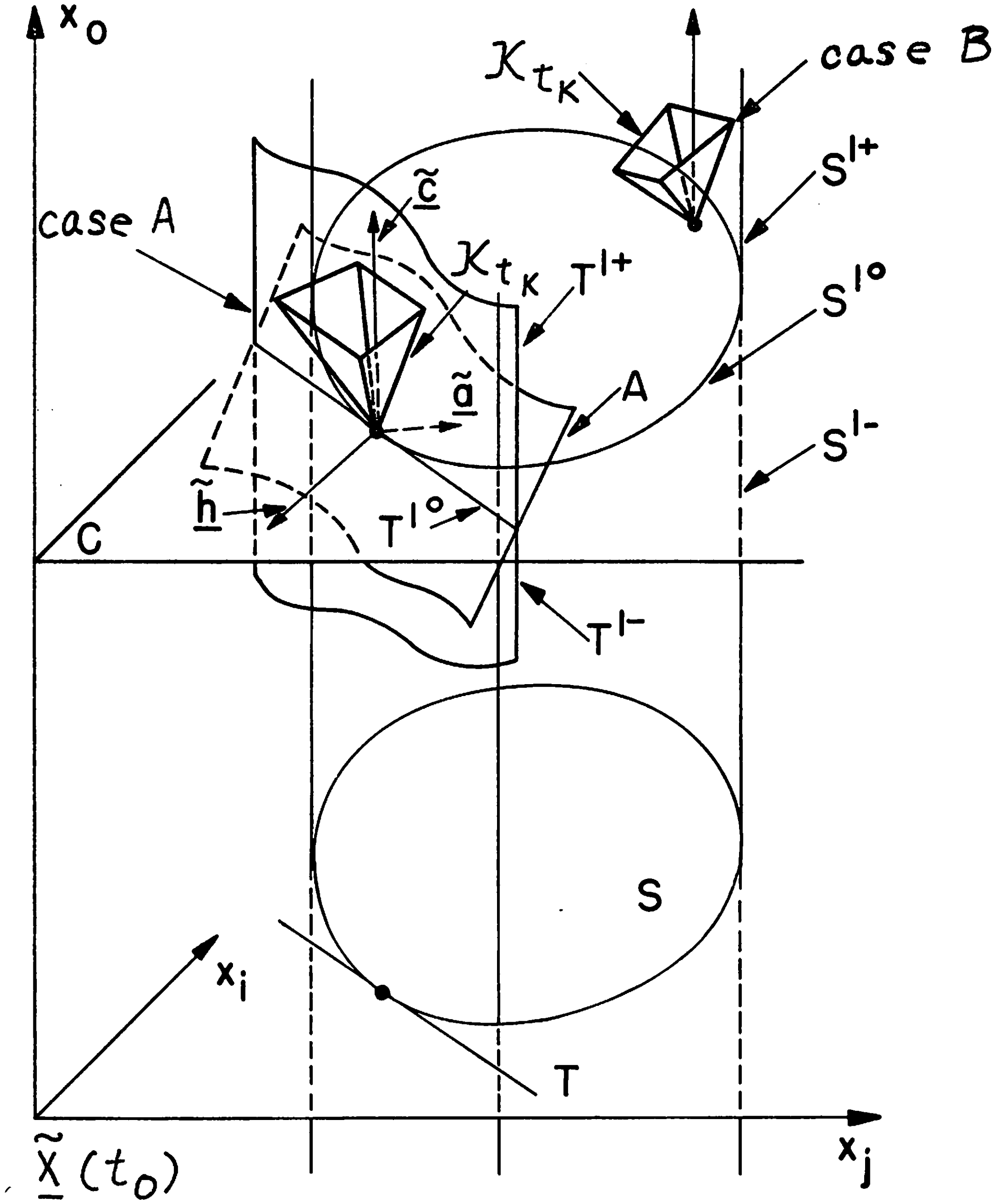

Fig. 4. Illustration for the Proof of Theorem 5.