TIME-VARYING NETWORKS—THE STATE VARIABLES, STABILITY AND ENERGY BOUNDS*

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Introduction

Linear time-varying circuits of special kind such as parametric converters and amplifiers have been studied to a great extent. Notable success in analysis and design has been obtained during the past five years. On the other hand theory and techniques for general time-varying networks have been developed much more slowly. Only recently some important results on methods of characterization and analysis, stability, and amplification began to come forth.

This paper deals with general time-varying networks which contain time-varying resistors, inductors and capacitors. The state variable description is used to analyze the circuit and to calculate the total energy stored. Sufficient conditions for stability are then derived in terms of the incidence submatrices and branch parameters.

Upper and lower energy bounding functions are formulated, from which sharp conditions for stability and instability for periodically varying circuits are obtained. These results are similar to the recent work of Darlington and Rohrer. However, the method is more general and the stability conditions are sharper.

The State Space Representation and Stability

Following the work of Bashkow and Bryant, we draw a special tree for a general circuit to include the maximum number of capacitances and the minimum number of inductances.

Let us designate v and j as the branch voltage and branch current matrices respectively. We partition these matrices to distinguish between tree branches (with subscripts 1) and links (with subscripts 2). Thus

$$v = (v_1, v_2)^t = (e, v_2)^t$$

$$j = (j_1, j_2)^t = (j_1, i)^t$$

where e represents the tree branch voltages and i the link currents. We next introduce the branch parameters and the voltage--current relations. C, G and Γ represent tree branch capacitances, conductances and reciprocal inductances. L, R and S represent link inductances, resistances and elastances.

$$e = (e_C, e_G, e_\Gamma)^t, \quad v_2 = (v_L, v_R, v_S)^t$$

$$j_1 = (j_C, j_G, j_\Gamma)^t, \quad i = (i_L, i_R, i_S)^t$$

$$\begin{bmatrix} j_C \\ i_S \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} e_C \\ v_S \end{bmatrix}$$

$$\begin{bmatrix} j_G \\ v_R \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} e_G \\ i_R \end{bmatrix}$$

$$\begin{bmatrix} e_\Gamma \\ v_L \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} j_\Gamma \\ i_L \end{bmatrix}$$

For the special tree picked, Kirchoff current law states

$$Aj = \begin{bmatrix} 1, D \end{bmatrix} \begin{bmatrix} j_1 \\ i \end{bmatrix} = 0$$

where A is the incidence matrix and D is partitioned into incidence submatrices as follows to separate the three kinds of branch parameters:

$$D = \begin{bmatrix} \alpha & \beta & \epsilon \\ \gamma & \delta & 0 \\ \mu & 0 & 0 \end{bmatrix}$$

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The zeros in D are due to the particular method of constructing the tree. After eliminating all variables except e_C and i_L from the above equations, we obtain the following important representation.

$$\begin{bmatrix} \mathcal{C} & 0 \\ 0 & \mathcal{L} \end{bmatrix} \begin{bmatrix} \dot{e}_C \\ i_L \end{bmatrix} =$$

$$\begin{bmatrix} -\beta P^{-1}\beta^t - \dot{\mathcal{C}} & -(\alpha - \beta P^{-1}\delta^t R_1 \gamma) \\ (\alpha^t - \gamma^t Q^{-1}\delta G_2 \beta^t) & -\gamma^t Q^{-1}\gamma - \dot{\mathcal{L}} \end{bmatrix} \begin{bmatrix} e_C \\ i_L \end{bmatrix}$$

where

$$R_1 = G_1^{-1}, \qquad G_2 = R_2^{-1}$$

$$P = R_2 + \delta^t R_1 \delta, \qquad Q = G_1 + \delta G_2 \delta^t$$

$$\mathcal{C} = C_1 + \epsilon C_2 \epsilon^t$$

$$\mathcal{L} = L_{22} - \mu^t L_{12} - L_{21}\mu + \mu^t L_{11}\mu$$

We now compute the energy stored in the circuit and its time derivative. Using the stability criterion of Liapunov, we can prove that

$$\beta P^{-1}\beta^t + \frac{\dot{\mathcal{C}}}{2} \geq 0$$

$$\gamma^t Q^{-1}\gamma + \frac{\dot{\mathcal{L}}}{2} \geq 0$$

are sufficient conditions for stability.

Energy Bounds

It can be shown that the energy stored is bounded by two functions as follows:

$$E(t_0)\exp\left\{-\int_{t_0}^{t} \lambda_\ell(\xi)d\xi\right\} \leq E(t)$$

and

$$E(t) \leq E(t_0)\exp\left\{-\int_{t_0}^{t} \lambda_u(\xi)d\xi\right\}$$

where $\lambda_u(t)$ and $\lambda_\ell(t)$ are the instantaneous minimum and maximum defined by

$$\left.\begin{aligned} \lambda_u(t) &= \min_t (\lambda_{1i}, \lambda_{2j}) \\ \lambda_\ell(t) &= \max_t (\lambda_{1i}, \lambda_{2j}) \end{aligned}\right\} \quad \begin{matrix} i \\ j \end{matrix}\Big\} = 1, 2, \ldots$$

λ_{1i} and λ_{2j} represent the zeros of

$$\det\left[\lambda_1 \frac{\mathcal{C}}{2} - (\beta P^{-1}\beta^t + \frac{\dot{\mathcal{C}}}{2})\right]$$

$$\det\left[\lambda_2 \frac{\mathcal{L}}{2} - (\gamma^t Q^{-1}\gamma + \frac{\dot{\mathcal{L}}}{2})\right].$$

It can be shown that a sufficient condition for stability of a periodicially varying network with period T is given by

$$\int_{t_0}^{t_0+T} \lambda_u(\xi)d\xi > 0.$$

A sufficient condition for instability is

$$\int_{t_0}^{t_0+T} \lambda_\ell(\xi)d\xi < 0.$$

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