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STUDIES IN OPTIMAL SYSTEM THEORY*

by

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Consider the very simple problem of optimal control with fixed time and fixed endpoints of a scalar linear system in the continuous time deterministic case subject to a quadratic cost functional. The plant is

$$\dot{x} = ax + bu \tag{1}$$

where a, b are constants. The cost is

$$J = 1/2 \int_0^T u^2(t)\, dt \tag{2}$$

(this is sometimes called the minimum energy problem).

There is no constraint on u(t). The object is to transfer (1) from the given initial state $x(0) = x_o$ to the given final state $x(T) = X$ such that (2) is minimized.

Using the conventions of Ref. 1, the Hamiltonian is

$$H(x, p) = \max_u \; [p(ax + bu) - u^2/2\,] \, . \tag{3}$$

In this case the maximum may be found by setting the partial derivative of (3) equal to zero. This gives

$$u(t) = b\, p(t)\, . \tag{4}$$

Substituting (4) into (3) gives

$$H(x, p) = b^2 p^2/2 + pax\, . \tag{5}$$

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The Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2} b^2 \left(\frac{\partial S}{\partial x}\right)^2 + a x \frac{\partial S}{\partial x} = 0 . \qquad (6)$$

Here $S = S(x, X, t)$, where we choose the given final state X as a canonical constant of integration.[1] The solution to (6) is then

$$S(x, X, t) = C - \frac{a\left[x\, e^{\frac{a(T-t)}{2}} - X e^{\frac{a(t-T)}{2}}\right]^2}{2b^2 \sinh a(T - t)} . \qquad (7)$$

Here C is an arbitrary additive constant whose value is unimportant. One readily verifies by direct substitution that (7) satisfies (6). To complete the solution to the problem we have

$$p = \frac{\partial S(x, X, t)}{\partial x} . \qquad (8)$$

Using (7), (8), and (4) the optimal control is

$$u = \frac{a \operatorname{csch} a(t - T)}{b} \left[x\, e^{a(T-t)} - X\right] . \qquad (9)$$

In the special case when the desired final state is the origin, we set $X = 0$ and have

$$u = \frac{a}{b} \left[\coth a(t - T) - 1\right] x . \qquad (10)$$

Thus we have the familiar result that the control u may be realized from the current state x simply by multiplication by a time-varying gain. This completes the essential part of the solution of the optimal control problem.

We continue to explore the ramifications of this example, looking for possible embellishments. The trajectory $x(t)$ actually followed by the system (1) when driven by the control (9) may be found explicitly, as follows:

As explained in Ref. 1 the constant of integration canonically conjugate to X is given by

$$P = - \frac{\partial S(x, X, t)}{\partial X} . \tag{11}$$

Using (7), this is explicitly

$$P = \frac{a}{b^2} \operatorname{csch} a(T-t) \left[x - X e^{a(t-T)} \right]. \tag{12}$$

Now, in a particular problem the numerical value of the desired final state X will be part of the given data. However, the numerical value of the constant P will not be given. Rather, the other appropriate part of the given data is the numerical value of the initial state x_o. Since P is constant, Eq. (12) holds for all $0 \leq t \leq T$, so that we may insert $t = 0$ and $x = x_o$ into (12) to evaluate P. We need never actually even calculate P itself, though, because equating the right-hand sides of (12) for $t = 0$ and for current t provides what we want:

$$\frac{x - X e^{a(t-T)}}{\sinh a(T-t)} = \frac{x_o - X e^{-aT}}{\sinh (aT)} . \tag{13}$$

Solving (13) for $x(t)$ explicitly yields

$$x(t) = x_o \operatorname{csch} (aT) \sinh a(T-t) + X \left[\cosh a(T-t) - \coth (aT) \sinh a(T-t) \right] \tag{14}$$

We see that x(t) as given by (14) does indeed satisfy the boundary conditions $x(0) = x_o$, $x(T) = X$. The explicit form of the corresponding control u(t), as a function of time only, may be examined by substituting (14) into (9). This gives

$$u(t) = \frac{a \operatorname{csch}(aT)}{b} \left(X - x_o \epsilon^{aT}\right) \epsilon^{-at} = u(0) \epsilon^{-at} . \quad (15)$$

We note that, even though the "gain" in (9) or (10) becomes infinite at t = T, the actual control u(t) as an explicit function of time is bounded for all $0 \leq t \leq T$.

The physical significance of (15) is that if we drove the system (1) with the explicit time function (15), i.e., "open loop" control via a function generator, it would still necessarily execute the trajectory given by (14) and would still transfer from state x_o to state X such that J in (2) is minimized. In contrast, (10) gives a "feedback" or "closed-loop" method of realizing the same result.

Let us return now to (12), and explore a different idea. Solving (12) explicitly for x(t) gives

$$x(t) = X \epsilon^{a(t-T)} + \frac{b^2}{a} P \sinh a(t-T) . \quad (16)$$

Using (8), or alternatively now (4) and (9),

$$p(t) = \frac{a \operatorname{csch} a(t-T)}{b^2} \left[x(t) \epsilon^{-a(t-T)} - X\right] . \quad (17)$$

Substituting (16) into (17) reduces it to

$$p(t) = P \epsilon^{-a(t-T)} . \quad (18)$$

Thus, we do have p(T) = P, as required from the definition of P. Equations (16) and (18) together may be written in matrix form:

$$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} e^{a(t-T)} & (b^2/a) \sinh a(t-T) \\ 0 & e^{-a(t-T)} \end{bmatrix} \begin{bmatrix} X \\ P \end{bmatrix} . \qquad (19)$$

The determinant of this matrix is unity, which is consistent with the fact that this must be a contact transformation.[1]

Instead of using the Hamilton-Jacobi parital differential equation, when we reached Eq. (5) we could have employed Hamilton's canonical equations of motion:

$$\dot{x} = \frac{\partial H}{\partial p} ; \quad \dot{p} = - \frac{\partial H}{\partial x} . \qquad (20)$$

Using (5) in (20) and writing the result in matrix form, we have

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} a & b^2 \\ 0 & -a \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} . \qquad (21)$$

One readily verifies that (19) satisfies (21) with the boundary condition

$$\begin{bmatrix} x(T) \\ p(T) \end{bmatrix} = \begin{bmatrix} X \\ P \end{bmatrix} . \qquad (22)$$

We remark that (19) represents the form of the solution usually obtained when one employs Pontryagin's Maximum Principle and (20). At that point, it is necessary to "turn (19) inside out" in order to obtain (12). Once (12) is found, then use of (4) and (18) readily yields (9). Although in the linear case it is always relatively straightforward to turn (19) or its equivalent "inside out," in the nonlinear case this may be an impossible task. That is why one finds so few actual nonlinear problems which have been completely solved using Pontryagin's principle. In contrast, <u>if</u> the H. J. Eq. (6)

or its equivalent can be solved explicitly, e.g., as (7), then (9) or its equivalent follows at once via (8). Therefore, what is needed is a slick way of solving the H. J. equation.

Unfortunately, the ideas to be presented below apply only to the linear case, so that we have a possible breakthrough only on the one front where we do not really need it. However, it contains philosophical implications which may be significant.

We start by returning to the H. J. Eq. (6). If we did not already know the solution is (7), the most effective way of finding it is to assume a quadratic form:

$$S(x, X, t) = \frac{1}{2} \begin{bmatrix} x & X \end{bmatrix} \begin{bmatrix} \pi_{11}(t) & \pi_{12}(t) \\ \pi_{12}(t) & \pi_{22}(t) \end{bmatrix} \begin{bmatrix} x \\ X \end{bmatrix} . \qquad (23)$$

Substitution of (23) into (6) yields a Ricatti equation for $\pi_{11}(t)$. When $\pi_{11}(t)$ is found, $\pi_{12}(t)$ is obtained as a solution of a linear differential equation, and $\pi_{22}(t)$ simply by a quadrature. The perplexing part of the process is putting the proper boundary conditions on π_{11}, π_{12}, and π_{22}. Using the benefits of hindsight, we find that for $t \simeq T$, the known correct solution (7) behaves like

$$S(x, X, t) \Big|_{t \approx T} \approx \frac{(x - X)^2}{2b^2 (t - T)} . \qquad (24)$$

It is hard to see how one should arrive at the conclusion a priori that (24) is the appropriate boundary condition for (6). There seems to be no suggestive interpretation for (24).

Let us now introduce the function

$$R(x, X, t) = \phi(t)\, e^{S(x, X, t)} . \qquad (25)$$

Here $\phi(t)$ is a function to be determined below. One finds that

$$\frac{\partial R}{\partial t} = \frac{\dot{\phi}}{\phi} R + \frac{\partial S}{\partial t} R \tag{26}$$

$$\frac{\partial R}{\partial x} = \frac{\partial S}{\partial x} R \tag{27}$$

$$\frac{\partial^2 R}{\partial x^2} = \frac{\partial^2 S}{\partial x^2} R + \left(\frac{\partial S}{\partial x}\right)^2 R \; . \tag{28}$$

Let us still assume that the form (23) is valid for $S(x, X, t)$. Then

$$\frac{\partial^2 S}{\partial x^2} = \pi_{11}(t) \; , \tag{29}$$

i.e., a function of t only.

Multiply through Eq. (6) by R:

$$R \frac{\partial S}{\partial t} + \frac{1}{2} b^2 \left(\frac{\partial S}{\partial t}\right)^2 R + ax \frac{\partial S}{\partial x} R = 0. \tag{30}$$

Suppose now that we require R to satisfy

$$\frac{\partial R}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 R}{\partial x^2} + ax \frac{\partial R}{\partial x} = 0. \tag{31}$$

Using (26) through (29), we have

$$\frac{\dot{\phi}}{\phi} R + R \frac{\partial S}{\partial t} + \frac{b^2}{2} \left(\frac{\partial S}{\partial x}\right)^2 R + \frac{b^2}{2} \pi_{11} R + ax \frac{\partial S}{\partial x} R = 0 \tag{32}$$

We now arbitrarily impose a condition in the form of a differential equation relating $\phi(t)$ to $\pi_{11}(t)$:

$$\phi(t) + \frac{b^2}{2} \pi_{11}(t) \phi(t) = 0 \; . \tag{33}$$

The solution to (33) is, of course,

$$\phi(t) = A \exp\left[-\int \frac{b^2}{2} \pi_{11}(t)\, dt \right] \tag{34}$$

where A is an arbitrary constant.

Thus, provided (29) and (34) hold, it appears that (30) and (31) are equivalent. In other words, the transformation (25) provides a way of transforming the nonlinear H. J. Eq. (6) into the linear diffusion-type Eq. (31).

Inserting the functions S(x, X, t) from (7) and $\phi(t)$ from (34) into (25) yields R(x, X, t) which is a solution to (31). In this connection (7) may be conveniently rewritten as

$$S(x, X, t) = -\frac{a}{2b^2 \sinh a(T-t)} \begin{bmatrix} x & X \end{bmatrix} \begin{bmatrix} e^{a(T-t)} & -1 \\ -1 & e^{a(t-T)} \end{bmatrix} \begin{bmatrix} x \\ X \end{bmatrix} . \tag{35}$$

The additive constant C in (7) is dropped since it can be absorbed by the multiplicative constant A in (34), as far as R is concerned. Comparing (23) and (35),

$$\pi_{11}(t) = \frac{-a e^{a(T-t)}}{b^2 \sinh a(T-t)} = -\frac{a}{b^2} \frac{2}{1 - e^{-2a(T-t)}} . \tag{36}$$

Then (34) gives

$$\phi(t) = A \exp\left[\int \frac{a\, dt}{1 - e^{2a(t-T)}} \right] = A \exp\left[\frac{1}{2} \int \frac{d\xi}{1 - e^{\xi}} \right] \tag{37}$$

where $\xi = 2a(t-T)$. Since A is arbitrary we take the indefinite integral.

Then

$$\phi = A \exp\left\{\frac{1}{2} \log\left(\frac{e^{\xi}}{1 - e^{\xi}}\right)\right\} = \frac{A\, e^{a(t-T)}}{\sqrt{1 - e^{2a(t-T)}}} \quad . \tag{38}$$

Thus we have

$$R(x, X, t) = \frac{A\, e^{a(t-T)}}{\sqrt{1 - e^{2a(t-T)}}} \exp - \left\{ \frac{a\left[x\, e^{\frac{a(T-t)}{2}} - X\, e^{\frac{a(t-T)}{2}}\right]^2}{2b^2 \sinh a(T - t)} \right\} . \tag{39}$$

Let us see whether R satisfies a more meaningful boundary condition than the somewhat perplexing condition (24) on S. For $t \approx T$ we have

$$R(x, X, t) \Bigg|_{t \approx T} \approx \frac{A}{\sqrt{2a(T - t)}} \exp\left\{ - \frac{(x - X)^2}{2b^2 (T - t)} \right\} . \tag{40}$$

Recall the known result[2] concerning a limiting approximation to the delta distribution:

$$\lim_{\alpha \longrightarrow \infty} \sqrt{\frac{\alpha}{\pi}} \; e^{-\alpha x^2} = \delta (x) . \tag{41}$$

Comparing with (40), we see that

$$\lim_{t \longrightarrow T} R(x, X, t) = A b \sqrt{\frac{\pi}{a}} \; \delta (x - X) . \tag{42}$$

We are thus motivated to set

$$A = \frac{1}{b} \sqrt{\frac{a}{\pi}} \quad . \tag{43}$$

Throughout this development we are assuming $a > 0$ for convenience. If $a < 0$, the procedure works out similarly, after making the obvious necessary modifications.

Inspecting (39), we see that for all $t < T$

$$\int_{-\infty}^{\infty} R\,dx < \infty; \quad \int_{-\infty}^{\infty} R\,dX < \infty . \qquad (44)$$

As a matter of fact, with A given by (43) it is not hard to show that

$$\int_{-\infty}^{\infty} R(x, X, t)\,dX = 1 \qquad (45)$$

for all $t \leq T$, where the integral is taken in the sense of the theory of distributions for $t = T$.

One of the features that we are claiming for the transformation from Eq. (6) to Eq. (31) is thus that the boundary condition

$$\lim_{t \longrightarrow T} R(x, X, t) = \delta(x - X) \qquad (46)$$

seems very natural for the optimal control problem of Eqs. (1) and (2) in which the desired final state is $x(T) = X$.

Digressing now momentarily, some confusion may be present concerning the negative sign in (7), in view of the well-known significance of the function $S(x, X, t)$ (see Ref. 1) and the fact that the cost J in Eq. (2) must evidently always be non-negative. There is no contradiction, however, as we would like to show. The correct relation between S and J is

$$S[x(t), X, t]\Big|_{t=T} - S[x(t), X, t]\Big|_{t=0} = J^{o} \qquad (47)$$

where J^{o} is the optimum cost. To evaluate this, we must insert the solution $x(t)$ given by (14) into $S(x, X, t)$ given by (7). The most straightforward way to proceed is to rewrite (7) as

$$S(x, X, t) = C - \frac{a e^{a(T-t)} \left[x - X e^{a(t-T)}\right]^2}{2b^2 \sinh a(T-t)} . \tag{48}$$

Now square both sides of (13) to obtain the bracketed quantity in (48). Making this substitution yields

$$S[x(t), X, t] = C - \frac{a e^{a(T-t)} \left(x_o - X e^{-aT}\right)^2 \sinh a(T-t)}{2b^2 \sinh^2 a(T-t)} \tag{49}$$

Then from (49)

$$S[x(t), X, t] \Big|_{t=T} = C \tag{50}$$

$$S[x(t), X, t] \Big|_{t=0} = C - \frac{a e^{aT} \left(x_o - X e^{-aT}\right)^2}{2b^2 \sinh aT} \tag{51}$$

so that (47) gives

$$J^o = \frac{a e^{aT} \left(x_o - X e^{-aT}\right)^2}{2b^2 \sinh aT} \geq 0 \tag{52}$$

as advertised. We assume throughout that $T > 0$, of course, as well as $0 \leq t \leq T$.

Let us now consider an entirely different problem, belonging to the realm of noise theory and stochastic processes rather than control theory. We will presently relate the following material to the preceding. Suppose white noise is passed through a first-order linear filter. The situation may be described by

$$\dot{x}(t) = a x(t) + b v(t) \tag{53}$$

where v(t) is a sample function from a gaussian white completely random process and x(t) is the output of the filter. It is well known that x(t) will be a sample function from a scalar Markov process.[3]

Let X be a random variable denoting the value of the filter output at some fixed time $T > 0$. We wish to determine the conditional or transition probability density $\overline{p}(X;T \mid x;t)$ of X, given the observed value of the filter output x at any previous time $t \leq T$. It is well known that $\overline{p}(X;T \mid x;t)$ satisfies the backward Fokker-Planck or Kolmogorov equation[3,4] which in this case is

$$-\frac{\partial \overline{p}(X;T \mid x;t)}{\partial t} = ax \frac{\partial \overline{p}(X;T \mid x;t)}{\partial x} + \frac{b^2}{2} \frac{\partial^2 \overline{p}(X;T \mid x;t)}{\partial x^2} \qquad (54)$$

where we have assumed that the covariance of the noise v(t) is

$$E[v(t)\, v(\tau)] = \delta(t - \tau). \qquad (55)$$

The appropriate boundary condition on (54) is

$$\lim_{t \longrightarrow T} \overline{p}(X;T \mid x;t) = \delta(X - x). \qquad (56)$$

Comparing (54) and (56) with (31) and (46) respectively, we see that

$$\overline{p}(X;T \mid x;t) = R(x, X, t) \qquad (57)$$

or explicitly we may write

$$\overline{p}(X;T \mid x;t) = \frac{1}{\sqrt{2\pi}\,\sigma(t, T)} \exp -\left\{ \frac{\left[X - x e^{a(t-T)}\right]^2}{2\sigma^2(t, T)} \right\} \qquad (58)$$

in which

$$\sigma^2(t, T) = \frac{b^2}{2a} \left[e^{2a(T-t)} - 1 \right]. \qquad (59)$$

Thus the solutions of both the open-loop stochastic filtering problem (53) and the deterministic optimal control problem (1), (2) have been shown to hinge on the partial differential equation (31). We suspect that this represents a more basic aspect of the duality between the two classes of problems noted by Kalman and Bucy.[5]

We propose that this duality can be exploited to solve problems in optimal control. Let the noise filtering problem (53) be simulated on a computer, either an analog computer using a gaussian noise generator or a digital computer using tables of random digits. Since we want all trajectories (sample functions) to pass through the point X at time T, probably the best way to do this is to put $\tau = T - t$ and rewrite (53) as

$$\frac{dx(\tau)}{d\tau} = -ax(\tau) - bv(\tau) . \tag{60}$$

Then, always start the system in the "initial" state

$$x(\tau)\Big|_{\tau = 0} = X \tag{61}$$

and solve for $\tau > 0$. By repeating the solution a myriad of times, a function equivalent for our purposes to the density function $\overline{p}\,(X;T \mid x;t)$ could be generated experimentally. We thus also have the function $R(x, X, t)$.

In order to calculate the optimal control function u(t) for the problem (1), (2), we wish to proceed as in (8) and (9). By (27), this is easy, and (9) can equally well be obtained from

$$u = \frac{b}{R(x, X, t)} \; \frac{\partial R(x, X, t)}{\partial x} . \tag{62}$$

All of the above generalizes readily to the vector-valued case. In order to solve the deterministic optimal control problem we simulate the dual stochastic filtering problem and generate the function $R(x, X, t)$ essentially by a Monte Carlo technique. From a knowledge of the value of this function and its gradient we at once obtain the control function u.

It unfortunately appears at present that this duality only exists between the class of control problems involving linear plants with quadratic cost functions and the class of stochastic filtering problems in which the transition probability density is gaussian. A plausibility argument as to the underlying reason for the duality follows.

Consider two random variables (not processes now) y, z. Let them have a joint probability density which is gaussian:

$$p_2(y, z) = A e^{-(\alpha y^2 + \beta yz + \gamma z^2)} . \qquad (63)$$

Then the marginal density function $p_1(y)$ can of course be found by marginal integration

$$p_1(y) = \int_{-\infty}^{\infty} p_2(y, z)\, dz. \qquad (64)$$

Evaluating the integral by completing the square yields explicitly

$$p_1(y) = B \exp\left[- (\alpha - \beta^2/4\gamma)\, y^2 \right] . \qquad (65)$$

We are not concerned here with the multiplicative constants A, B in (63) and (65), but only with the form of the exponential. What we want to point out is that $p_1(y)$ can be obtained from $p_2(y, z)$ not only by marginal integration but also by a <u>maximization procedure</u>. To carry it out, it is required that for any given value of y, determine the value of $z = z(y)$ such that the function $p_2[y, z(y)]$ is maximized. The function $p_1(y)$ is then proportional to this maximal value of p_2.

$$p_{vn} = C_1 \exp\left\{-\frac{1}{N} \sum_{k=1}^{n} v_{t_k}^2\right\}. \qquad (69)$$

Now let the number of observations become infinite ($n \longrightarrow \infty$). Then the discrete index k (or t_k) may be replaced by a continuous index t, and the sum in (69) becomes an integral:

$$p_{v\,\infty}[v(t)] = C_2 \exp\left\{-\frac{1}{N}\int_{t_1}^{t_\infty} v^2(t)\,dt\right\}. \qquad (70)$$

For convenience, assume $t_1 = 0$, $t_\infty = T$. Then

$$p_{v\,\infty}[v(t)] = C_2 \exp\left\{-\frac{1}{N}\int_0^T v^2(t)\,dt\right\}. \qquad (71)$$

The quantity $p_{v\,\infty}$ may be considered either as a <u>function</u> of an <u>infinite</u> number of <u>variables</u> $v_{t_1}, v_{t_2}, \ldots,$ or as a <u>functional</u> of the <u>single function</u> $v(t)$.

Suppose similarly that observations were made at times t_1, $t_2, \ldots, t_n$ on <u>both</u> the value of the output of the filter sample function $x(t)$ <u>and</u> on its derivative $\dot{x}(t)$ to give a set of observations

$$\{x_{t_1}, x_{t_2}, \ldots, x_{t_n}; \dot{x}_{t_1}, \dot{x}_{t_2}, \ldots, \dot{x}_{t_n}\}.$$

We can conceive of a joint density function $p_{x,\dot{x},n}(x_{t_1}, x_{t_2}, \ldots, x_{t_n};$ $\dot{x}_{t_1}, \dot{x}_{t_2}, \ldots, \dot{x}_{t_n})$ on all these variates. Again let $n \longrightarrow \infty$, and put $t_1 = 0$, $t_\infty = T$. Then by (53), since $v = (\dot{x} - ax)/b$ we conjecture that, using (70)

$$p_{x,\dot{x},\infty}[x(t); \dot{x}(t)] = C_3 \exp\left\{-\frac{1}{N}\int_0^T \left(\frac{\dot{x} - ax}{b}\right)^2 dt\right\}. \qquad (72)$$

Now what it is desired to do is to obtain the joint bivariate density function $p_2(X, x_o)$ on the initial observation x_o and the final observation $x_T = X$ only. This seems to require some sort of marginal integration in an infinite number of dimensions over all the $\dot{x}_{t_k}$ and all the x_{t_k} except x_o and X. If marginal integration and a maximum procedure are equivalent for gaussian distributions, however, we propose that

$$p_2(X, x_o) = K_1 \max_{x(t)} p_{x, \dot{x}\, \infty} [x(t); \dot{x}(t)] \tag{73}$$

where only those x(t) are admissible which satisfy $x(0) = x_o$, $x(T) = X$.

Now inspecting (71), (72), and (73), we assert that our desired joint density function $p_2(X, x_o)$ can be found from the minimization problem $\min \int_0^T v^2(t)\, dt$ subject to the constraints that

$\dot{x} = ax + bv$ and $x(0) = x_o$, $x(T) = X$.

Clearly since

$$p_2(X_1, x_o) = p_c(X|x_o)\ p_1(x_o) \tag{74}$$

our procedure is tantamount to finding the transition probability density $p_c(X|x_o)$, which in turn is really our function $\overline{p}\ (X;T|x;t)$ lurking in disguise.

Therefore, heuristically it appears that $\overline{p}(X;T|x;t)$ may be found either by solving the Fokker-Planck or Kolmogorov Eq. (54); by carrying out an infinite dimensional marginal integration on the infinite dimensional joint density (72); or by noting that the latter can be accomplished via an equivalent extremization procedure.

It is apparent, of course, that the minimization problem at which we finally arrived is exactly the same as the statement of the original optimal control problem in Eqs. (1) and (2).

We submit that the underlying reason for the apparent duality between stochastic filtering problems and deterministic control problems lies in a generalization of the relation (66) for the gaussian density functions.

Let us terminate by making two final bold proposals. The first proposal concerns the so-called stochastic optimal control problem in which rather wide interest has recently been manifested. A good summary of these developments is contained in the report by Wonham.[4] Consideration of this problem leads to the so-called stochastic Hamilton-Jacobi equation. Suppose that our plant is now

$$\dot{x}(t) = ax + b_1 u(t) + b_2 v(t) \tag{75}$$

where $u(t)$ is the control and $v(t)$ is white noise. Let it now be required to minimize the expected value of the cost of control $E[J]$, where

$$J = \frac{1}{2} \int_0^T u^2(t)\, dt . \tag{76}$$

Without going into the details regarding the observations on $x(t)$ or the specific form of the stochastic Hamilton-Jacobi equation for this problem, we say that by means of a series of manipulations similar to those embodied in Eqs. (25 - 34) above, the stochastic Hamilton-Jacobi equation can likewise be transformed into a linear diffusion equation of the form (31).

Consequently, we propose that rather than considering two distinct categories of problems, namely stochastic filtering problems on the one hand and deterministic optimal control problems on the other, at least in the very restrictive "linear" cases considered in this note the correct point of view is to consider a whole continuum of problems. This continuum consists of problems of the stochastic optimal control variety discussed by Wonham. The two categories above simply lie at opposite extremes of this continuum.

Our second, and final proposal concerns the fact that the transformation of (6) into (31) is applicable to any similar problem in the calculus of variations, whether or not it originates in the theory of optimal control.

Thus, in classical mechanics the Hamiltonian for a harmonic oscillator is[6]

$$H(x, p) = \frac{p^2}{2m} + \frac{k^2}{2} x^2 . \tag{77}$$

The Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial x} \right)^2 + \frac{k^2}{2} x^2 = 0 . \tag{78}$$

By introducing the function

$$R(x, X, t) = \phi(t) e^{(i/\hbar) S(x, X, t)} \tag{79}$$

where $S(x, X, t)$ is assumed to have the form (23), and by using a series of manipulations similar to those discussed previously, Eq. (79) may be transformed into

$$i\hbar \frac{\partial R}{\partial t} = - \frac{\hbar^2}{2m} \frac{\partial^2 R}{\partial x^2} + \frac{k^2}{2} x^2 R . \tag{80}$$

Now, Eq. (80) is <u>precisely</u> the Schroedinger equation for a <u>quantum</u>

mechanical harmonic oscillator.[7] Our final proposal, therefore, is that the philosophical implication of the above equivalence is that all of the information obtained from the solution to the quantum mechanical harmonic oscillator problem must be implicitly contained in the solution to the classical mechanical harmonic oscillator problem found via Hamilton-Jacobi theory.

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