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AN ALGEBRAIC CHARACTERIZATION OF CONTROLLABILITY*

by

A. Chang

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The purpose of this note is to establish an algebraic characterization of complete controllability for linear differential systems. We shall consider systems whose state at time t is described by an n-dimensional vector $\underline{x}(t)$ that satisfies the differential equation

$$\dot{\underline{x}}(t) = \underline{A}(t)\,\underline{x}(t) + \underline{B}(t)\,\underline{u}(t)\,. \tag{1}$$

The function $\underline{u}(t)$, called the control, is assumed to be an r-dimensional vector, and $\underline{A}(t)$ and $\underline{B}(t)$ are nxn and nxr matrices respectively. A comprehensive discussion of the controllability of the system (1), hereafter called Σ, may be found in Ref. 1. The criteria for controllability of Σ presented there involve the solution of the matrix differential equation

$$\dot{\underline{\Phi}}(t, t_0) = \underline{A}(t)\,\underline{\Phi}(t, t_0)$$

with initial condition $\underline{\Phi}(t_0, t_0) = \underline{I}$, the identity matrix. However, it is seldom possible to obtain an analytic expression for $\underline{\Phi}$, unless $\underline{A}(t)$ is a constant or periodic in t. The content of this note, stated in Theorem 1, is an algebraic criterion for controllability involving only the matrices $\underline{A}(t)$ and $\underline{B}(t)$.

Before proceeding to Theorem 1, let us recall the definition of complete controllability. The general solution of (1) with initial condition $\underline{x}(t_0) = \underline{x}_0$ is given by

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** Electronics Research Laboratory, University of California, Berkeley, California.

$$\underline{x}(t, \underline{u}) = \underline{\Phi}(t, t_o)\,\underline{x}_o + \int_{t_o}^{t} \underline{\Phi}(t, s)\,\underline{B}(s)\,\underline{u}(s)\,ds \; .$$

If $u(t)$ is a control defined over the interval $t_o \leq t \leq t_1$ and $\underline{x}(t_1, \underline{u}) = 0$, $\underline{u}(t)$ is said to transfer $\underline{x}_o$ to the origin. Following Kalman,[1] we shall say Σ is completely controllable at time t_o if there exists a $t_1 > t_o$ such that for every initial state $\underline{x}_o$ there is some control $\underline{u}(t)$, $t_o \leq t \leq t_1$, which transfers $\underline{x}_o$ to the origin. In the sequel, t_o will be considered a constant, and for brevity the phrase "at time t_o" will often be omitted.

Theorem 1: Suppose $\underline{A}(t)$ and $\underline{B}(t)$ are (n-2) and (n-1) times continuously differentiable, respectively. Let

$$\underline{B}_1(t) = \underline{B}(t)$$

$$\underline{B}_i(t) = -\underline{A}(t)\,\underline{B}_{i-1}(t) + \frac{d\,\underline{B}_{i-1}}{dt}, \quad i = 2, 3, \ldots$$

Let

$$\underline{Q}(t) = [\,\underline{B}_1(t), \underline{B}_2(t), \ldots, \underline{B}_n(t)\,] \; .$$

Then

(i) A sufficient condition for Σ to be completely controllable at time t_o is for rank $\underline{Q}(t) = n$ for some $t > t_o$.

(ii) If the elements of $\underline{A}(t)$ and $\underline{B}(t)$ are analytic functions, then the latter condition is also necessary.

The proof of theorem 1 is based on the following result due to LaSalle.[2]

Lemma 1: A necessary and sufficient condition for Σ to be completely controllable at time t_o is that for every $\underline{y} \in R^n$, $\underline{y} \neq 0$, the r-vector $\underline{y} \cdot \underline{\Phi}(t_o, t)\,\underline{B}(t) \neq 0$ for some $t > t_o$.

We note that the row vector $\underline{y} \cdot \underline{\Phi}(t_o, t)$ is the solution to the differential equation

$$\dot{\underline{z}}(t) = -\underline{z}(t)\,\underline{A}(t) \tag{2}$$

which satisfies the initial condition $\underline{z}(t_o) = \underline{y}$.

<u>Proof of Theorem 1:</u>

(i) Suppose rank $\underline{Q}(t) = n$ for some $t > t_o$ but Σ is not completely contollable. Then, by Lemma 1, there exists $\underline{y}_o \neq 0$ such that

$$\underline{y}_o \cdot \underline{\Phi}(t_o, t) \cdot \underline{B}(t) = 0 \qquad t > t_o$$

or putting $\underline{z}(t, \underline{y}_o) = \underline{y}_o \cdot \underline{\Phi}(t_o, t)$,

$$\underline{z}(t, \underline{y}_o) \cdot \underline{B}(t) = 0 \qquad t > t_o . \tag{3}$$

Differentiating (3) and using (2) and the definition of the B_i,

$$0 = \dot{\underline{z}}\,\underline{B} + \underline{z}\,\dot{\underline{B}} = -\underline{z}\,\underline{A}\,\underline{B} + \underline{z}\,\dot{\underline{B}} = \underline{z}\,\underline{B}_2 . \tag{4}$$

Repeated differentiation of (4) yields,

$$\underline{z}(t, \underline{y}_o) \cdot \underline{B}_i(t) = 0 \qquad t > t_o, \quad i = 1, 2, \ldots, n. \tag{5}$$

But for all $t > t_o$, $\underline{z}(t, \underline{y}_o) \neq 0$; so (5) contradicts the assumption rank $\underline{Q}(t) = n$ for some $t > t_o$. This proves (i).

(ii) In order to prove (ii), we shall need

<u>Lemma 2:</u> Suppose $\underline{A}(t)$ and $\underline{B}(t)$ are analytic, let

$$\underline{Q}_j(t) = \left[\underline{B}_1(t), \underline{B}_2(t), \ldots \underline{B}_j(t)\right] \quad j = 1, 2, \ldots$$

Then there exists $k \leq n$, and nonempty open set $O \subset (t_o, \infty)$ such that for each $t \in O$,

$$\text{rank } \underline{Q}_k(t) = \text{rank } \underline{Q}_{k+j}(t), \qquad j = 1, 2, \ldots$$

Let us observe that $\underline{Q}_n(t) = \underline{Q}(t)$. Lemma 2 implies that the columns of $\underline{B}_j(t)$ for all $j > n$ are, for every $t \in O$, expressible as linear combinations of the columns of $\underline{Q}(t)$.

Assuming Lemma 2 for the moment, let us prove (ii). Suppose rank $\underline{Q}(t) < n$ for all $t > t_o$; it then suffices to show Σ is not completely controllable. Let O be the set in Lemma 2. Choose $t_1 \in O$ and let $\underline{b}_1, \underline{b}_2, \ldots \underline{b}_k$ be a maximal set of linearly independent column vectors of $\underline{Q}(t_1)$. Then, choose $\underline{y}_o \neq 0$ such that $\underline{y}_o \cdot \underline{\Phi}(t_o, t_1) = \underline{z}(t_1, y_o)$ is orthogonal to $\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_k$ (this is possible since $\underline{\Phi}(t_o, t_1)$ has an inverse). Then at t_1, in view of Lemma 2,

$$0 = \frac{d^j}{d_t{}^j}(\underline{z} \cdot \underline{B}) \Bigg|_{t = t_1} = \underline{z}(t_1, \underline{y}_o) \, \underline{B}_{j+1}(t_1), \quad j = 0, 1, \ldots \tag{6}$$

Since $\underline{A}(t)$ is analytic, $\underline{z}(t, \underline{y}_o)$ and therefore $\underline{z}(t, \underline{y}_o)$. $\underline{B}(t)$, are also analytic. From (6), it then follows that $\underline{z}(t, \underline{y}_o)$. $\underline{B}(t) = 0$ for all $t > t_o$, and therefore, in view of Lemma 1, Σ is not completely controllable at time t_o.

It remains to prove Lemma 2. For simplicity, let $\underline{B}(t) = \underline{b}(t)$, a column vector. The proof in the general case is analogous. For notational convenience we define the operator

$$\underline{L} = -\underline{A}(t) + \frac{d}{dt} \quad .$$

Then

$$\underline{Q}_j(t) = \left[\underline{b}(t), \; \underline{L}\,\underline{b}(t), \; \ldots, \; \underline{L}^{j-1}\,\underline{b}(t) \right]$$

Let $O_1 \subset (t_o \quad \infty)$ be the set of points for which rank $\underline{Q}_1(t) = 1$. O_1 is evidently open.* Consider $\underline{Q}_2(t)$. There are two possibilities: either rank $\underline{Q}_2(t) \leq 1$ for all t or rank $\underline{Q}_2(t) = 2$ for some t. If the first alternative holds, $\underline{L}\,\underline{b}(t) \in M_1(t)$ for every $t \in O_1$, where $M_1(t)$ is the linear manifold (in R^n) spanned by $\underline{b}(t)$. In this case, using the linearity of $\underline{L}$, it is easy to show $\underline{L}^j\,\underline{b}(t) \in \underline{M}_1(t)$ for $j = 1, 2, \ldots$ and all $t \in O_1$, and there is nothing more to prove. If the second alternative arises, let $O_2 \subset (t_o, \infty)$ be the set of points for which rank $\underline{Q}_2(t) = 2$. O_2 is open. It follows that we find ourselves in the same situation but with O_2 instead of O_1. Repeating this argument, let k be the smallest integer for which

$$\begin{aligned} &\text{rank } \underline{Q}_k(t) = k && t \in 0 \\ &\text{rank } \underline{Q}_{k+1}(t) \leq k && t \in (t_o, \infty) \end{aligned} \tag{7}$$

Evidently $k \leq n$. Denoting by $M_k(t)$ the linear manifold spanned by $\underline{b}(t), \underline{L}\,\underline{b}(t), \ldots, \underline{L}^{k-1}\,\underline{b}(t)$, it follows from (7) that $\underline{L}^k\,\underline{b}(t) \in M_k(t)$ for every $t \in O$, a nonempty open set using the linearity of L, we then can show, by induction, $\underline{L}^j\,\underline{b}(t) \in \underline{M}_k(t)$, $j = k, k+1, \ldots$ for all $t \in O$. This proves Lemma 2, and completes the proof of the theorem.

Comments: The sequence $\underline{B}_i(t)$ was used by Gamkrelidze in discussing the time optimal control of Σ (Ref. 3). When $\underline{A}(t)$ and $\underline{B}(t)$ are constant, $\underline{B}_i(t) = (-1)^i\,\underline{A}^{i-1}\,\underline{B}$, and Theorem 1 reduces to the well known criterion for controllability of constant coefficient systems. If $\underline{A}(t)$ and $\underline{B}(t)$ are analytic, then rank $\underline{Q}(t)$ achieves its maximum value except possibly on an isolated set of points. In this case, practically speaking, it suffices to check rank $\underline{Q}(t)$ for a single value of t, and not all $t > t_o$.

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* Since b(t) is analytic, the values of t for which b(t) = 0 are isolated points. We assume b(t) is not identically zero.

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