Electronics Research Laboratory
University of California
Berkeley, California
Internal Technical Memorandum M-68

EQUIVALENCE OF TIME-VARYING SYSTEMS

by

D. A. Calahan

* This work was supported in part by AFOSR Grant AF 177-63, University of Illinois.

May 5, 1964

EQUIVALENCE OF TIME-VARYING SYSTEMS*

Let a linear time-varying system be described by the state equations

$$\frac{d}{dt}(MX) = AX + Y \tag{1}$$

or

$$M\dot{X} = (A - \dot{M})X + Y \tag{2}$$

where M and A are n x n matrices, and

$$X = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad . \tag{3}$$

An "equivalence" variable z is now defined such that

$$M = M(z) \qquad A = A(z) \qquad X = X(z;t) \qquad z = z(t). \tag{4}$$

Hence, all time variations in M and A will be regarded as explicit functions of z rather than t. It is now proposed to find M(z) and A(z) such that one of the x_i's (x_p) be independent of z, <u>for any excitation Y</u>.

To begin, it is clear [letting dz/dt = k(t)]

$$\frac{dX}{dt} = k(t)\frac{\partial X}{\partial z} + \frac{\partial X}{\partial t} \tag{5}$$

*This work was supported in part by AFOSR Grant AF 177-63, University of Illinois.

and

$$\frac{dM}{dt} = k(t) \frac{\partial M}{\partial z} \quad . \tag{6}$$

Taking the partial derivative of Eq. (2) with respect to z yields

$$\frac{\partial M}{\partial z}\left(\frac{k\partial X}{\partial z} + \frac{\partial X}{\partial t} \right) + M\left(\frac{k\partial^2 X}{\partial z^2} + \frac{\partial^2 X}{\partial z \partial t} \right)$$

$$= \frac{\partial}{\partial t}\left[A - \frac{k\partial M}{\partial z} \right] X + \left[A - \frac{k\partial M}{\partial z} \right] \frac{\partial X}{\partial t} + \frac{\partial Y}{\partial x} \quad . \tag{7}$$

Sufficient conditions that x_p and Y not be functions of z are

$$\frac{\partial X}{\partial z} = aX \qquad\qquad \frac{\partial Y}{\partial z} = bY \tag{8}$$

where a, b are (1) $n \times n$ matrices, (2) functions of z only, and (3)

$$a_{pj} = 0 \qquad\qquad b_{jl} = 0 \qquad\qquad j = 1, 2, \ldots n. \tag{9}$$

Then, it is clear

$$\frac{\partial^2 X}{\partial z^2} = \frac{\partial a}{\partial z} X + a^2 X, \quad \frac{\partial X}{\partial z \partial t} = \frac{a\partial X}{\partial t} \quad . \tag{10}$$

Substituting Eqs. (8) and (10) in Eq. (7), and separating terms involving $\partial X/\partial t$ yields

$$\left\{ \frac{k\partial M}{\partial z} a + k M \frac{\partial a}{\partial z} + k M a^2 - \frac{\partial}{\partial z} \left[A - k \frac{\partial M}{\partial z} \right] \right.$$

$$\left. - \left[A - k \frac{\partial M}{\partial z} \right] a + b \left[A - k \frac{\partial M}{\partial z} \right] - k\, b\, M\, a \right\} X$$

$$+ \left\{ \frac{\partial M}{\partial z} + Ma - bM \right\} \frac{\partial X}{\partial t} = 0. \tag{11}$$

Eq. (11) can be satisfied independently of X if and only if both bracketed terms are identically zero. Thus

$$\frac{\partial M}{\partial z} = b\,M - Ma \;. \tag{12}$$

Before setting the other term to zero, it is helpful to note

$$\frac{\partial^2 M}{\partial z^2} = \frac{\partial b}{\partial z}\,M + b^2\,M - 2\,b\,Ma + Ma^2 - M\,\frac{\partial a}{\partial z} \;. \tag{13}$$

Now setting the other term to zero, one has

$$\begin{aligned} & k\,b\,Ma - k\,Ma^2 + k\,M\,\frac{\partial a}{\partial z} + k\,Ma^2 - \frac{\partial A}{\partial z} + k\,\frac{\partial b}{\partial z}\,M \\ & + k\,b^2\,M - 2\,k\,b\,Ma + k\,Ma^2 - k\,M\,\frac{\partial a}{\partial z} - Aa + k\,b\,Ma \\ & - b\,Ma^2 + b\,A - k\,b^2\,M + k\,b\,Ma - k\ \ b\,Ma = 0 \end{aligned} \tag{14}$$

or, simply

$$\frac{\partial A}{\partial z} = b\,A - Aa + k\,\frac{\partial b}{\partial z}\,M. \tag{15}$$

The solutions of Eqs. (12) and (15) yield $M(z(t))$ and $A(z(t))$ such that $x_p(t)$ is independent of z.

Several special cases are of interest:

1. $k(t) = 0$ so that M and A are varied only when the system is in a zero state. Eq. (15) then becomes

$$\frac{\partial A}{\partial z} = b\,A - Aa \tag{16}$$

an equation previously derived by somewhat similar means.[1, 2] Note that in this case, the matrix equations are similar in form to those which would be obtained by projection techniques, i.e., by evaluating the transfer function from y_1 to x_p in terms of s^* with coefficients $c_k(m_{ij}, a_{ij})$, and setting $dc_k = 0$. However, not only are Eqs. (12) and (15) more succinct since they are in terms of the parameter matrices themselves, but they show the differential equations in uncoupled form. Coupling through a and b will occur only when certain of the elements of M and A are required to remain invariant $(\partial m_{ij}/\partial z = 0)$.

2. $x = t$ so $k(t) = 1$. Equations then become

$$\frac{dM}{dt} = b\,M - Ma \tag{17}$$

$$\frac{dA}{dt} = b\,A - Aa + \frac{db}{dt}\,M\,. \tag{18}$$

If b is a discontinuous at t_0 [as would occur if an element in M had a discontinuous derivative at $t = 0$, and so requiring a discontinuity in b to satisfy Eq. (17)], then integrating Eq. (18) yields

$$A_0^+ = A_0^- + (b_0^+ - b_0^-)\,M_0\,. \tag{19}$$

Here, $b_0^+ - b_0^-$ is the discontinuity in b. Thus, a discontinuity in the derivative of M must be accompanied by a discontinuity in A as given by Eq. (19). For example, if

1. a is the zero matrix for all t

2. b is the zero matrix for $t \leq 0^-$ and is a matrix of constants b_0^+ for $t \geq 0^+$,

then

$$M(t) = M_0\,e^{-b_0^+ t} \tag{20}$$

---

*since the system is now time invariant when $k = 0$.

and

$$\frac{dA}{dt} = b_0^+ A \tag{21}$$

so that

$$A = (A_0^- + b_0^+ M_0) e^{-b_0^+ t} \quad . \tag{22}$$

Several rather obvious applications are the following:

1. Given a single time-varying element, find the time variations of other elements required to make the system behave as the time-invariant system described by M_0, A_0^- .

2. Given a time-varying system with undetermined stability, find an equivalent system (possibly time-invariant) for which stability is more readily determined.

D. A. Calahan
Electronic Research Laboratory
University of California
Berkeley, California
and Department of Electrical Engineering
University of Illinois
Urbana, Illinois

BIBLIOGRAPHY

1. J. D. Schoeffler, "Continuously equivalent networks and their application," Tech. Rpt. No. 5, Case Institute of Technology; December 2, 1962.

2. D. A. Calahan, "Computer generation of equivalent networks," IEEE Convention Record, 1964