Copyright © 1964, by the author(s).
All rights reserved.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission.

Electronics Research Laboratory
University of California
Berkeley, California
Internal Technical Memorandum M-69

SIGNAL DESIGN FOR SEQUENTIAL DETECTION SYSTEMS*

by

G. L. Turin

*The work reported herein was supported by the Department of the Air Force under grant AF-AFOSR-230-64.

May 6, 1964

SIGNAL DESIGN FOR SEQUENTIAL DETECTION SYSTEMS*

by

G. L. Turin**

ABSTRACT

We consider a coherent, white, Gaussian channel, through which one of two signals is sent to a receiver which operates as an optimum sequential detector. A noiseless feedback link is assumed, which continuously informs the transmitter of the state of the receiver's uncertainty concerning which signal was sent, and which also synchronizes the transmitter when the receiver has reached a decision. The transmitter, in turn, uses the output of the feedback link to modify its transmission so as to hasten the receiver's decision.

The following problem is posed: given average- and peak-power constraints on the transmitter and a prescribed probability of error for the receiver, what signal waveforms should the transmitter use in order to minimize the average transmission time, and how should it utilize the fed-back value of the receiver's uncertainty to modify these waveforms while transmission is in progress? We give partial solutions to these questions. In particular, we have shown that if the peak-to-average power ratio is sufficiently large, significant improvement of performance may be achieved through the use of uncertainty feedback.

---

*The work reported herein was supported by the Department of the Air Force under grant AF-AFOSR-230-64.

**Department of Electrical Engineering, University of California, Berkeley, California.

INTRODUCTION

Consider a channel disturbed only by additive, white, Gaussian noise n, defined on the time interval $(-\infty, \infty)$. Through this channel, one of two signal waveforms s_+ and s_-, defined on $[t_o, \infty)$, may be transmitted with a priori probabilities P_+ and P_-, respectively. The output, z, of the channel may therefore be either of the form $z = s_+ + n$ (hypothesis H_+) or of the form $z = s_- + n$ (hypothesis H_-). Suppose that z is observed over the time interval $[t_o, t]$, where t is a parameter taking values in $[t_o, \infty)$; denote this observation by z_t.

The optimal sequential test for deciding between hypotheses H_+ and H_-, given the sequence of observations $\{z_t\}$--- i.e., the procedure which minimizes the expected time to make a decision for given probabilities of erroneously deciding H_+ and H_- --- is known[1] to be the following: We first form

$$y(t) \triangleq \log \frac{\Pr[H_+/z_t]}{\Pr[H_-/z_t]} \quad (t \geq t_o) , \tag{1}$$

where $\Pr[H_{\pm}/z_t]$ are the a posteriori probabilities of $H_{\pm}$, given z_t, and

$$y(t_o) \triangleq \log \frac{P_+}{P_-} \triangleq y_o \quad . \tag{2}$$

No decision is made for $t_o \leq t < t_o + T$, where

$$t_o + T = \sup \{t : Y_- < y(t) < Y_+\} . \tag{3}$$

It is decided that H_+ is the true hypothesis if $y(t_o + T) = Y_+$; it is decided that H_- is true if $y(t_o + T) = Y_-$. Since $y(t)$ is clearly a random process, T is a random variable, and it is its average,

$E[T] = P_+ E[T/H_+] + P_- E[T/H_-]$, which is a minimum for the receiver utilizing this procedure.

The thresholds $Y_{\pm}$ may be related to the given error probabilities as follows. We note that whenever it is decided that H_+ is true, i.e., the test stops with $y = Y_+$, we have from (1):

$$\Pr[H_-/z_T] = e^{-Y_+} \Pr[H_+/z_T] . \tag{4}$$

Averaging (4) over all observations z_T which lead to decision H_+, we obtain

$$P_- P_+^e = e^{-Y_+} P_+(1 - P_+^e) , \tag{5}$$

where $P_+^e (P_-^e)$ is the probability that H_+ (H_-) is chosen incorrectly. A similar expression involving Y_- may be obtained. Solving these expressions for $Y_{\pm}$, and using (2), we obtain

$$Y_{\pm} = y_o \pm \log \frac{1 - P_{\pm}^e}{P_{\pm}^e} . \tag{6}$$

The average error probability is then $P^e = P_- P_+^e + P_+ P_-^e$.

For the assumed channel, one may easily derive an expression for $y(t)$:[2]

$$y(t) = y_o + \frac{2}{N_o} \int_{t_o}^{t} \{s_+[y(\tau), \tau] - s_-[y(\tau), \tau]\} z(\tau) d\tau$$

$$- \frac{1}{N_o} \int_{t_o}^{t} \{s_+^2 [y(\tau), \tau] - s_-^2 [y(\tau), \tau]\} d\tau , \tag{7}$$

where N_o is the (single-ended) noise power density (watts/cps). In (7), we have written the dependence of $s_{\pm}$ on t both directly and also through

$y(t)$, the latter in order to allow for what we shall call uncertainty feedback. That is, we allow the receiver (observer) to feed back to the transmitter the current value of y, thus indicating the state of the receiver's uncertainty concerning the transmitted signal. The transmitter may then make modifications of the two signals $s_{\pm}$ --- which modifications are known to the receiver --- the object being to improve the progress of the test. The feedback link also allows the transmitter to be informed when the test has terminated, so that transmission of a new binary digit may begin.[3]

We consider in this paper the problem of choosing the signals $s_{\pm}$, subject to certain constraints on power, so as to minimize $E[T]$ for given thresholds $Y_{\pm}$ (i.e., given $P^e_{\pm}$); it is assumed that the receiver, which has previously been optimized for a particular pair of signals, remains optimized as the signals are varied so as to find the best pair.

FORMULATION OF THE PROBLEM

If we differentiate (7) with respect to t and substitute $z = s_{\pm} + n$ in the result, we obtain

$$\frac{dy}{dt} = \pm \frac{1}{N_o} \Delta^2 + \frac{2}{N_o} \Delta n , \qquad (8)$$

where $\Delta = \Delta(y,t) = s_+(y,t) - s_-(y,t)$. This is a generalized Langevin equation[4] and, since n is a white, Gaussian process, it follows that y is a Markov process. If we let $p_{\pm}(y,t/y_o,t_o)$ be the transition probability densities of y under the hypotheses $H_{\pm}$, respectively (i.e., $p_{\pm}(y,t/y_o,t_o)$ is the probability density of $y(t)$, given that $y(t_o) = y_o$ and that $H_{\pm}$ is true), then $p_{\pm}$ satisfy, respectively, the Fokker-Planck equations[4]

$$\frac{\partial p_{\pm}}{\partial t} = \mp \frac{\partial}{\partial y}\left[\frac{\Delta^2}{N_o} p_{\pm}\right] + \frac{\partial^2}{\partial y^2}\left[\frac{\Delta^2}{N_o} p_{\pm}\right] , \qquad (9)$$

subject to the initial conditions $p_{\pm}(y, t_o/y_o, t_o) = \delta(y - y_o)$ and suitable boundary conditions on $p_{\pm}$ at $y = \pm\infty$.

We are not interested in the probability densities $p_{\pm}$ themselves, but rather in the probability densities $q_{\pm}(y, t/y_o, t_o)$ defined by the statements: $q_{\pm}(y_1, t_1/y_o, t_o)\, dy$ = probability that if $H_{\pm}$ is true and $y(t_o) = y_o$, then $y(t_1)$ lies in $(y_1, y_1 + dy)$ and $Y_- < y(t) < Y_+$ for all $t_o \leq t \leq t_1$. These are the densities which describe the uncertainty of the receiver at a time t_1 when the receiver is still testing. Notice, however, that they are not proper probability densities, since

$$\int_{Y_-}^{Y_+} q_{\pm}(y, t_1/y_o, t_o)\, dy$$

$$= \Pr\left[Y_- < y(t) < Y_+, \text{ all } t_o \leq t \leq t_1/y(t_o) = y_o; H_{\pm} \right]$$

$$\triangleq F_{\pm}(t_1/y_o, t_o) \quad , \tag{10}$$

which is not generally equal to unity. ($F_{\pm}$ are so-called "first-passage" probability distributions of $y(t)$; in the present context they are the probabilities that the receiver has not made a decision by time t_1, given that $H_{\pm}$ is true, respectively.)

It has been shown[5] that $q_{\pm}$ satisfy the same differential equations as $p_{\pm}$, i.e.,

$$\frac{\partial q_{\pm}}{\partial t} = \mp \frac{\partial}{\partial y}\left[\frac{\Delta^2}{N_o} q_{\pm}\right] + \frac{\partial^2}{\partial y^2}\left[\frac{\Delta^2}{N_o} q_{\pm}\right] \tag{11}$$

subject to the initial conditions $q_{\pm}(y, t_o/y_o, t_o) = \delta(y - y_o)$ and the boundary conditions $q_{\pm}(Y_{\pm}, t/y_o, t_o) = q_{-}(Y_{\pm}, t/y_o, t_o) = 0$. Thus, if we can solve the equations (11), we can use (10) to compute $F(t/y_o, t_o)$, the probability that the test has not stopped by time t:

$$F(t/y_o, t_o) \triangleq P_+ F_+(t/y_o, t_o) + P_- F_-(t/y_o, t_o) \quad . \tag{12}$$

Further, since $f(t/y_o, t_o) \triangleq -\partial F(t/y_o, t_o)/\partial t$ is the probability density distribution[6] of the stopping time, we have

$$t_o + \overline{T}(y_o) \triangleq t_o + E[T/y_o] = \int_{t_o}^{\infty} t\, f(t/y_o, t_o)\, dt \ . \tag{13}$$

Thus, we must try to choose $s_{\pm}$, subject to constraints, so as to minimize (13).

A SPECIAL CLASS OF SIGNALS

The equations (11) are difficult to solve in general because $\Delta(y, t)$ may depend in quite a complicated way upon its arguments. However, some progress may be made by considering only the class of signals of the form

$$s_{\pm}(y, t) = \pm\, U_{\pm}(y) \quad \sigma(t) \quad , \tag{14}$$

where we take $U_{\pm}$ to be positive functions of y. By limiting ourselves to this class of signals, we restrict the transmitter, on learning the state of the receiver's uncertainty, to an adjustment only of its instantaneous power output. If the transmitter sees that the receiver is heading toward a wrong decision, it may, for example, increase the power drastically, while leaving the power small if the receiver is doing well.

Substituting (14) into (11), we obtain the differential equations

$$\frac{N_o}{\sigma^2} \frac{\partial q_{\pm}}{\partial t} = \mp \frac{\partial}{\partial y} [U^2 q_{\pm}] + \frac{\partial^2}{\partial y^2} [U^2 q_{\pm}] \ , \tag{15}$$

where we have set $U = U_+ + U_-$. These equations may now be solved by the usual separation-of-variable technique, i.e., by assuming solutions of the form $q_{\pm}(y, t/y_o, t_o) = u(y)\, v(t)$. It is easily demonstrated[7] that the solutions obtained in this way depend on t and t_o only through the function

$$R(t, t_o) \triangleq \frac{1}{N_o} \int_{t_o}^{t} \sigma^2(\tau)\, d\tau \quad . \tag{16}$$

Thus, the transition densities $q_{\pm}$ are "nonstationary" --- i.e., do not depend on t and t_o only through their difference --- unless $\sigma(t)$ is a constant function of t.

If we now define two new functions $\hat{q}_{\pm}$ by

$$q_{\pm}(y, t/y_o, t_o) = \hat{q}_{\pm}\left[y, R(t, t_o)/y_o\right] \quad , \tag{17}$$

and make the change of variable

$$t' = R(t, t_o) \quad , \tag{18}$$

it follows from (15) that $\hat{q}_{\pm}$ satisfy

$$\frac{\partial \hat{q}_{\pm}}{\partial t'} = \mp \frac{\partial}{\partial y}\left[U^2 \hat{q}_{\pm}\right] + \frac{\partial^2}{\partial y^2}\left[U^2 \hat{q}_{\pm}\right] \; . \tag{19}$$

These are recognizable as Fokker-Planck equations for Markovian processes for which the transition densities are stationary.[4] Thus, the nonstationarity of the transition densities of $y(t)$ derives from local expansions or compressions of the time scale: stationarity can be obtained by expanding the time scale at time t by a factor $\partial R/\partial t$.

It is important to recognize that the solutions $\hat{q}_{\pm}$ of (19), and the functions derived from them below, are not dependent on $\sigma(t)$.

We define

$$\hat{F}_{\pm}(t'/y_o) = \int_{Y_-}^{Y_+} \hat{q}_{\pm}(y, t'/y_o)dy \quad , \tag{20a}$$

$$\hat{F}(t'/y_o) = P_+ \hat{F}_+(t'/y_o) + P_- \hat{F}_-(t'/y_o) \quad , \tag{20b}$$

$$\hat{f}_{\pm}(t'/y_o) = - \frac{\partial}{\partial t'} \hat{F}_{\pm}(t'/y_o) \quad , \tag{21a}$$

$$\hat{f}(t'/y_o) = - \frac{\partial}{\partial t'} \hat{F}(t'/y_o) \quad . \tag{21b}$$

Then, from (10), (12), (17) and (18) we have

$$F_{\pm}(t/y_o, t_o) = \hat{F}_{\pm}[R(t, t_o)/y_o] \quad , \tag{22a}$$

$$F(t/y_o, t_o) = \hat{F}[R(t, t_o)/y_o] \, , \tag{22b}$$

$$f_{\pm}(t/y_o, t_o) \triangleq - \frac{\partial}{\partial t} F_{\pm}(t/y_o, t_o) = \hat{f}_{\pm}[R(t, t_o)/y_o] \frac{\partial}{\partial t} R(t, t_o) \, , \tag{23a}$$

$$f(t/y_o, t_o) = \hat{f}[R(t, t_o)/y_o] \frac{\partial}{\partial t} R(t, t_o) \quad . \tag{23b}$$

Hence, we may rewrite (13) as

$$t_o + \overline{T}(y_o) = - \int_{t_o}^{\infty} t \, \hat{f}[R(t, t_o)/y_o] \frac{\partial}{\partial t} R(t, t_o) \, dt \quad . \tag{24}$$

If we assume $\sigma^2(t)$ to be almost nowhere zero, then R is monotone increasing in t, and we may write a unique inverse of the transformation (18):

$$t = t_o + r(t') . \tag{25}$$

Then, using (18) and (25), (24) becomes

$$\overline{T}(y_o) = \int_0^{\infty} r(t') \hat{f}(t'/y_o) \, dt' , \tag{26a}$$

or, integrating by parts and using (21),[6]

$$\overline{T}(y_o) = \int_0^{\infty} \frac{dr(t')}{dt'} \hat{F}(t'/y_o) \, dt' . \tag{26b}$$

We now derive an expression for the average transmitted power. First consider the class $\mathcal{Y}_t$ of all member functions of the process y which stop --- i.e., first reach one of the thresholds --- at time t. The instantaneous power transmitted at time τ $(t_o \leq \tau \leq t)$, given that $H_{\pm}$ is true and that $y(\tau) = \eta$, is $U_{\pm}^2(\eta)\, \sigma^2(\tau)$. If we average this over the conditional distribution of η, given $y \in \mathcal{Y}_t$, i.e., over

$$pr(\eta, \tau/y_o, t_o; H_{\pm}; y \in \mathcal{Y}_t) = \frac{f_{\pm}(t/\eta, \tau) \; q_{\pm}(\eta, \tau/y_o, t_o)}{f_{\pm}(t/y_o, t_o)} , \tag{27}$$

we obtain the average instantaneous power transmitted at τ, given that $H_{\pm}$ is true and that the test stops at t. If we now take the <u>time</u> average, over the interval $[t_o, t]$, of this (statistical) average instantaneous power, we obtain

$$S_{\pm}(t) \triangleq \frac{1}{t - t_o} \int_{t_o}^{t} d\tau\, \sigma^2(\tau) \int_{Y_-}^{Y_+} d\eta\, U_{\pm}^2(\eta) \frac{f_{\pm}(t/\eta, \tau)\, q_{\pm}(\eta, \tau/y_o, t_o)}{f_{\pm}(t/y_o, t_o)}, \tag{28}$$

which is the average power expended over the interval $[t_o, t]$ for all tests ending at t, given that $H_{\pm}$ is true. Now, further averaging $S_{\pm}$ over the distribution of the stopping time t, i.e., over $f_{\pm}(t/y_o, t_o)$, we have

$$\overline{S}_{\pm} = \int_{t_o}^{\infty} \frac{dt}{t - t_o} \int_{t_o}^{t} d\tau\, \sigma^2(\tau) \int_{Y_-}^{Y_+} d\eta\, U_{\pm}^2(\eta)\, f_{\pm}(t/\eta, \tau)\, q_{\pm}(\eta, \tau/y_o, t_o), \tag{29}$$

which is an expression for the average transmitted power, given that $H_{\pm}$ is true. Next, transforming the variable t in (29) according to (18) and (25), and also transforming the variable τ according to

$$\tau' = R(\tau, t_o), \qquad \tau = t_o + r(\tau'), \tag{30}$$

we obtain, using (17) and (23),

$$\overline{S}_{\pm} = N_o \int_0^{\infty} \frac{dt'}{r(t')} \int_0^{t'} d\tau' \int_{Y_-}^{Y_+} d\eta\, U_{\pm}^2(\eta)\, \hat{f}_{\pm}(t' - \tau/\eta)\, \hat{q}_{\pm}(\eta, \tau'/y_o). \tag{31}$$

Finally, averaging (31) over the a priori probabilities of $H_{\pm}$, we have an expression for the average transmitted power,

$$\overline{S} = N_o \int_0^{\infty} \frac{\hat{g}(t'/y_o)}{r(t')}\, dt', \tag{32}$$

where we have set

$$\hat{g}(t'/y_o) = P_+\, \hat{g}_+(t'/y_o) + P_-\, \hat{g}_-(t'/y_o) \tag{33a}$$

and

$$\hat{g}_{\pm}(t'/y_o) \triangleq \int_0^{t'} d\tau' \int_{Y_-}^{Y_+} d\eta \; U_{\pm}^2(\eta)\, \hat{f}_{\pm}(t' - \tau' /\eta)\, \hat{q}_{\pm}(\eta, \tau'/y_o) \; . \tag{33b}$$

Suppose now that the transmitter is peak-power limited, so that

$$s_{\pm}^2 (y, t) = \sigma^2(t)\, U_{\pm}^2 (y) \leq P_{peak} \; . \tag{34}$$

We denote by A^2 the maximum of the values attained by U_+^2 and U_-^2. Then we must have

$$\sigma^2(t) \leq \frac{P_{peak}}{A^2} \tag{35a}$$

or

$$\frac{\partial}{\partial t} R(t, t_o) \leq \frac{P_{peak}}{A^2 N_o} \triangleq a \tag{35b}$$

or

$$\frac{d}{dt'} r(t') \geq \frac{1}{a} \; . \tag{35c}$$

Notice that the average and peak powers cannot be chosen completely independently, for, noting that $r(0) = 0$, we have from (35c)

$$r(t') \geq \frac{t'}{a} \; , \tag{36}$$

and hence, from (32),

$$\overline{S} \leq \frac{P_{peak}}{A^2} \int_0^{\infty} \frac{\hat{g}(t'/y_o)}{t'} dt' \; . \tag{37}$$

We may now reformulate the problem stated after (13) for the class of signals defined by (14): Given an average-power constraint $\overline{S} = P_{av}$, with $\overline{S}$ given by (32), and given the peak-power constraint of (34) or

(35), how should $\sigma^2(t)$ and $U_{\pm}(y)$ be chosen so as to minimize the average test length, (26)?

PARTIAL SOLUTION OF THE PROBLEM

Let us first consider the case in which there is only an average-power constraint, but no peak-power constraint, so that the quantity a in (35) is infinite. Then, applying the Schwarz inequality to the product of (26a) and (32), we obtain

$$\frac{\overline{S}\,\overline{T}}{N_o} \geq \left[\int_0^{\infty} \sqrt{\hat{f}(t'/y_o)\,\hat{g}(t'/y_o)}\; dt'\right]^2 , \qquad (38)$$

the equality holding if and only if

$$r(t') = c \sqrt{\frac{\hat{g}\,(t'/y_o)}{\hat{f}\,(t'/y_o)}} , \qquad (39)$$

where c is a constant. Notice from (19) et seq. that the lower bound of (38) depends on the signal waveforms only through $U(y)$, but not through $\sigma^2(t)$. Thus, if $U_{\pm}(y)$ are given, and $\overline{S}$ is constrained to equal P_{av}, $\overline{T}$ achieves its minimum value only if r is of the form (39), where c is adjusted, using (32), so as to satisfy the average-power constraint. The optimal $\sigma^2(t)$ corresponding to (39) may then be found through use of (16), (18) and (25).

In the other extreme, when there is a peak-power constraint but no average-power constraint, it is clear from (26b) that $\overline{T}$ is minimized when dr/dt' is taken at its minimum value for all t', i.e.,

$$\frac{d\,r(t')}{dt'} = \frac{1}{a} , \quad \text{all } t' \geq 0 , \qquad (40a)$$

or

$$\sigma^2(t) = \frac{P_{peak}}{A^2} \quad , \quad \text{all } t \geq t_o \ . \tag{40b}$$

When the peak- and average-power constraints are both operative, the problem becomes considerably more difficult to solve. It is shown in Appendix III, however, that if we assume $U_{\pm}$ (and hence $\hat{f}$ and $\hat{g}$) are given, then the following assertion may be made: If r yields a minimum of $\overline{T}$, it must be possible to write it in the form (39) in some intervals of t', and its derivative in the form (40) in the remaining intervals, the entire solution being subject to the two power constraints and also to the conditions that r must be continuous in t' and $r(0) = 0$. Such an assertion narrows down the class of functions r that must be considered. We defer further discussion of the doubly constrained problem until the examples of the next two sections, remarking here only that the introduction of a peak-power constraint, by preventing the satisfaction of (38) with the equality sign, results in a larger value of $\overline{T}$ (for a given $\overline{S}$) than could be achieved without such a constraint.

Note that we have been assuming at critical points that the functions $U_{\pm}(y)$ are given. Under this assumption, we may be able to find the optimal $\sigma^2(t)$ associated with the given $U_{\pm}(y)$. The question remains, for what pair of functions $U_{\pm}(y)$ will the $\overline{T}$ associated with the optimal $\sigma^2(t)$ thus found be absolutely the smallest? The explicit answer to this question appears unobtainable, for it would seem to involve knowledge of a general solution of (19), explicit in terms of $U(y)$, together with an ability to evaluate (20), (21) and (33) in terms of this solution.

We therefore restrict ourselves to consideration and comparison of two specific pairs of feedback functions. One of these pairs corresponds to the case in which the receiver's uncertainty is not fed back to the transmitter. The other pair corresponds to linear feedback of the uncertainty variable y.

SYNCHRONIZATION FEEDBACK

We first investigate the case $U_+(y) = U_-(y) = 1/2$, $U(y) = U_+(y) + U_-(y) = 1$. In this case, the state of the receiver's uncertainty does not affect the transmission; the feedback channel is used only to notify the transmitter that the receiver has made a decision, so that the transmitter and receiver may be synchronized for transmission of the next bit of information.

The functions $\hat{q}_{\pm}$, $\hat{F}_{\pm}$ and $\hat{f}_{\pm}$ for the case $U = 1$ have been derived in Appendix II.

We note first that, since y is a Markovian process, the inner integral of (33b), with $U_{\pm}^2 = 1/4$, is just equal to $\hat{f}(t'/y_o)/4$. Hence,

$$\hat{g}_{\pm}(t'/y_o) = \frac{1}{4} t' \hat{f}_{\pm}(t'/y_o) \tag{41}$$

and

$$\hat{g}(t'/y_o) = \frac{1}{4} t' \hat{f}(t'/y_o) \ . \tag{42}$$

From the assertion given subesequent to (40), it follows that, if a solution exists, it must be of the form

$$r(t') = \begin{cases} \dfrac{2C\sqrt{t'_p}}{a}\sqrt{t'}, & 0 \le t' \le t'_p \\[2ex] \dfrac{1}{a}\left[t' + (2C-1)\,t'_p\right], & t' \ge t'_p \ , \end{cases} \tag{43}$$

where, from (35b),

$$a = \frac{4\,P_{peak}}{N_o} \ , \tag{44}$$

and the constant C must not be less than unity, so that (35c) will not be violated.

The parameters C and t'_p may be related by substituting (43) into (32) and invoking the average-power constraint:

$$\frac{1}{2C\sqrt{t'_p}} \int_0^{t'_p} t'\,\hat{f}(t'/y_o)dt' + \int_{t'_p}^{\infty} \frac{t'}{t'+(2C-1)t'_p}\,\hat{f}(t'/y_o)dt' = \frac{P_{av}}{P_{peak}} \,. \tag{45a}$$

We may also substitute (43) into (26a) to obtain:

$$\overline{T}(y_o) = \frac{2C\sqrt{t'_p}}{a} \int_0^{t'_p} \sqrt{t'}\,\hat{f}(t'/y_o)dt' + \frac{1}{a} \int_{t'_p}^{\infty} \left[t'+(2C-1)t'_p\right]\hat{f}(t'/y_o)dt' \,. \tag{45b}$$

We must now solve (45a) for C as a function of t'_p (remembering that $C \geq 1$), substitute the result into (45b), and then find the value or values of t'_p which minimize $\overline{T}$; the value(s) of t'_p thus found may then be used to evaluate C. For each pair of parameters t'_p and C so derived, we have a possible solution of the form (43). If we assume -- as is physically reasonable -- that a solution does indeed exist, any one of these solutions will serve as an optimum.

Through use of (16), (18), (25) and (44), we may convert (43) into an expression for $\sigma^2(t)$:[8]

$$\sigma^2(t) = \begin{cases} \dfrac{2a\,P_{peak}}{C^2\,t'_p}(t-t_o)\,, & 0 \leq t-t_o \leq \dfrac{2C\,t'_p}{a} \\[2ex] 4\,P_{peak} & ,\ t-t_o > \dfrac{2C\,t'_p}{a} \,. \end{cases} \tag{46}$$

Thus, the optimum policy for the transmitter is to increase its instantaneous power linearly from zero until $t-t_o = 2C\,t'_p/a$, and to transmit at peak power thereafter.[9]

The two extreme cases we have previously discussed, viz., those in which only one power constraint is operative, are easily reduced to special cases of the general case above. When there is no peak-power limitation, i.e., $P_{peak} \longrightarrow \infty$, then from (45a) we have $t'_p \longrightarrow \infty$ (in fact, $t'_p/a \longrightarrow \infty$). (Then, from (45b) we see that $\overline{T}$ is minimized by taking $C = 1$.) The transmitter's policy is thus to increase power linearly as long as necessary to terminate the test.

The other extreme, in which there is no average-power constraint other than that implied by the peak-power constraint, corresponds to the condition $P_{av}/P_{peak} = 1$ -- see (37) and (42). In this situation, (45a) has the solution $t'_p = 0$, and (46) reduces to (40b). The transmitter's policy is thus to transmit at the allowable peak power at all times.

LINEAR FEEDBACK

We henceforth restrict ourselves for simplicity to the symmetrical case of equal a priori probabilities, $P_+ = P_- = 1/2$, and equal error probabilities, $P^e_+ = P^e_-$. Then, from (2) and (6), $y_o = 0$ and $Y_+ = -Y_- \triangleq Y$.

In this section we consider the class of feedback functions described by

$$U_{\pm}(y) = \frac{1}{2}\left[1 \pm k \frac{y}{Y}\right], \tag{47}$$

where $|k| \leq 1$. The case $k = 0$ is the case of synchronization feedback, previously discussed. When $k < 0$, the transmitter increases its power quadratically as it sees the receiver progressing toward a wrong decision; when $k > 0$, the power is increased quadratically as the receiver progresses toward the correct decision.

Note that $U_{\pm}$ have been chosen so that again $U_+ + U_- = 1$; thus the expressions for $\hat{q}_{\pm}$ and $\hat{f}_{\pm}$ derived in Appendix II again hold. Substitution of (A2-5) and (A2-9) into (33), followed by considerable manipulation, leads to the following asymptotic formula for $\hat{g}(t'/0)$:

$$\hat{g}(t'/0) \xrightarrow[Y \to \infty]{} \frac{1}{4}(1 + k)^2\, t'\, \hat{f}(t'/0) . \tag{48}$$

We have seen in (42) that this limiting form is indeed exact for all Y (and for the nonsymmetric case also) when $k = 0$. When $k \neq 0$, the result holds as $Y \longrightarrow \infty$, which, from (6), corresponds to small error probability.

It is clear that (43) and (46) still hold asymptotically ($Y \longrightarrow \infty$) in the present case, where (44) is now replaced by

$$a = \frac{P_{peak}}{A^2 N_o} = \frac{4 P_{peak}}{N_o(1 + |k|)^2}, \tag{49}$$

and (45a, b) become

$$\frac{1}{2C\sqrt{t'_p}} \int_0^{t'_p} \sqrt{t'}\, \hat{f}(t'/0)\, dt' + \int_{t'_p}^{\infty} \frac{t'}{t' + (2C - 1)t'_p}\, \hat{f}(t'/0)\, dt'$$

$$= \left(\frac{1 + |k|}{1 + k}\right)^2 \frac{P_{av}}{P_{peak}} \tag{50a}$$

$$\overline{T}(0) = \frac{2C\sqrt{t'_p}}{a} \int_0^{t'_p} \sqrt{t'}\hat{f}(t'/0)dt' + \frac{1}{a}\int_{t'_p}^{\infty} \left[t' + (2C - 1)t'_p\right] \hat{f}(t'/0)dt' . \tag{50b}$$

From (37) we see that, as in the case of (45a), the right-hand side of (50a) cannot exceed unity. The discussion following (45) holds equally here; notice in particular that the optimum values of C and t'_p obtained from (50) do not depend on k when $k \geq 0$.

As before, when there is no peak-power limitation, i.e., $P_{peak} \longrightarrow \infty$, then $t'_p \longrightarrow \infty$ and $C = 1$, and the optimum $\sigma^2(t)$ increases linearly and indefinitely. In the other extreme, when the peak-power constraint is the controlling factor (i.e., $P_{av} = \left(\frac{1 + k}{1 + |k|}\right)^2 P_{peak}$), then $t'_p = 0$, and

and $\sigma^2(t)$ becomes a constant function of t. Note, however, that for the present case of linear feedback, the instantaneous power actually transmitted is no longer of the form $\sigma^2(t)$, as in the case of synchronization feedback; now the transmitted power has the form $\sigma^2(t)\, U_{\pm}^2\, [y(t)]$, which, except when $k = 0$, is a random process.

It would be of interest to find, for various peak-to-average power ratios, the value of k which minimizes $\overline{T}$. We shall consider here only the two extreme cases in which only one or the other power constraint is operative.

In the case in which the transmitter is strongly peak-power limited, (50b) becomes

$$\overline{T}(0) = \frac{N_o(1+|k|)^2}{4\, P_{peak}} \int_0^{\infty} t'\, \hat{f}(t'/0)dt', \tag{51}$$

whence it is clear that we should optimally take $k = 0$. This, of course, makes good sense, since any other choice would result in the transmitter's not always utilizing the full peak power available to it. No advantage is gained in the strictly peak-power-limited case by using the feedback link for anything but synchronization feedback.

On the other hand, when there is essentially no peak-power limitation, so that $t'_p \longrightarrow \infty$ and $C = 1$, we have, from (49) and (50a),

$$\frac{a}{2\sqrt{t'_p}} \int_0^{t'_p} \sqrt{t'}\ \hat{f}(t'/0)dt' \longrightarrow \frac{4\, P_{av}}{N_o(1+k)^2}\ ; \tag{52}$$

whence (50b) becomes

$$\overline{T}(0) = \frac{N_o(1+k)^2}{4\, P_{av}} \left[\int_0^{\infty} \sqrt{t'}\ \hat{f}(t'/0)dt'\right]^2 . \tag{53}$$

In this case, taking $k = -1$ minimizes $\overline{T}$, and in fact reduces it to zero. This is a misleading result, however; for, from (49) and (52) one may easily show that (46) now becomes

$$\sigma^2(t) = \frac{8\,P_{av}}{N_o}\left[\int_0^{\infty}\sqrt{t'}\,\hat{f}(t'/0)dt'\right]^{-2}\frac{(1+|k|)^2}{(1+k)^4}(t-t_o)\,, \qquad (t \geq t_o) \quad (54)$$

so that taking $k = -1$ would require the transmitter immediately to transmit infinite power (although for an infinitesimally short time).[10]

In practical situations, of course, the peak power is constrained. None the less, if the peak-to-average power ratio is sufficiently large, we may take (53) and (54) to be approximately correct, so long as k is sufficiently removed from -1; (54) will then violate the peak-power constraint only during very improbable tests of very great length. Under these conditions, we see from (53) that use of uncertainty feedback can result in an appreciable power advantage over the synchronization-feedback case, $k = 0$. For example, if the peak-to-average power ratio is sufficiently large, and we take $k = -1/2$, linear uncertainty feedback can result in the same average transmission time and the same probability of error as synchronization feedback, but with 6 db less average power;[11] equivalently, with a given average power and error probability, one could transmit four times faster by using this type of uncertainty feedback than by not using it. Note, however, that a choice of $k > 0$ would result in a power disadvantage. Thus, the transmitter should use its knowledge of the receiver's uncertainty in such a way as to increase power as the receiver tends toward a wrong decision.

THE RADAR CASE

We close by commenting on the modifications that must be made in the foregoing analysis to accomodate it to the radar case. In these comments, we identify H_+ with the hypothesis "target present" and H_- with the hypothesis "target absent."

There are four major differences which distinguish the radar from the communication case:

D_1 - One cannot usually specify a priori probabilities in the radar case.

D_2 - Unlike a communication transmitter, a radar transmitter cannot know which is the correct hypothesis. In fact, a radar transmitter always transmits the same signal, which we here assume is of the form $\sigma(t)\ U\left[y(t)\right]$. The function U, rather than being the sum of two feedback functions, as previously, is now the feedback function which is always used by the transmitter.

D_3 - From the point of view of the radar receiver, there are none the less two distinct signals possible: $s_+(y,t) = \sigma(t)\ U(y)$ and $s_-(y,t) = 0$.

D_4 - The postulation of a noiseless feedback link[3] is always valid in the radar case, since the transmitter and receiver are at the same location.

Difference D_1 leads to a redefinition of $y(t)$ of (1) as a likelihood ratio rather than a ratio of a posteriori probabilities. The test defined in (3) et seq. is then Wald's sequential probability ratio test, which simultaneously minimizes $\overline{T}_+ \triangleq E\left[T/H_+\right]$ and $\overline{T}_- \triangleq E\left[T/H_-\right]$ for given false-alarm and detection probabilities, P^e_+ and $1 - P^e_-$, respectively.[1] These probabilities are related to the thresholds $Y_\pm$ by (6), with $y_o = 0$.

From the new definition of $y(t)$ and from D_3 it follows that we must take $y_o = 0$ and $s_- = 0$ in (7). Then the analysis leading to (19) carries through, where U is now the function defined in D_2, and y_o is always replaced by 0. As before, differential equations (19) must be solved for $\hat{q}_\pm$, and $\hat{f}_\pm$ of (21a) must be evaluated in terms of these solutions. Now, however, separate expressions analogous to (26a) must be written for $\overline{T}_+$ and $\overline{T}_-$ in terms of $\hat{f}_+$ and $\hat{f}_-$, respectively; (26a) itself cannot be written because of D_1.

Similarly, we cannot write the expression for $\overline{S}$ in (32), but must be content with the values of $\overline{S}_+$ and $\overline{S}_-$ given by (31), into which we may substitute (33b) for brevity. In these latter equations, we must replace $U_\pm$ with the function U defined in D_2, for the calculation of average power is based on the transmitted signal of D_2, rather than the received signals given by D_3.

The problem now becomes the following: For a given feedback function U, and subject to specified values of $\overline{S}_\pm$ and to inequality (35a) (where A^2 is now the maximum value of U^2), find the $\sigma(t)$ which simultaneously minimizes $\overline{T}_+$ and $\overline{T}_-$. Unfortunately, it may no longer be true that there is an answer to this problem, i.e., a single $\sigma(t)$ which results in a simultaneous minimization. For example, if we consider the case in which there is no peak-power limitation, we may carry through much as before, arriving at two inequalities of the form (38), one for $\overline{S}_+\overline{T}_+$ and one for $\overline{S}_-\overline{T}_-$. We would thus find that in order to perform the simultaneous minimization, it would be necessary to make $r(t')$ proportional to both $\sqrt{\hat{g}_+/\hat{f}_+}$ and $\sqrt{\hat{g}_-/\hat{f}_-}$. In the case of synchronization feedback -- i.e., $U(y)$ = constant -- this is possible, since, from (41), both $\sqrt{\hat{g}_+/\hat{f}_+}$ and $\sqrt{\hat{g}_-/\hat{f}_-}$ are equal to $\sqrt{t'/2}$. In other situations, however, such an ideal solution may not be available, and a compromise solution must be sought.

ACKNOWLEDGMENT

The author is grateful to Drs. E. Eisenberg and M. Zakai for tolerating his many attempted monologues on this work, and for converting them into fruitful dialogues.

APPENDIX I: SOLUTIONS OF (15)

We assume solutions to (15) of the form $q_{\pm}(y, t/y_o, t_o) = u_{\pm}(y)v_{\pm}(t)$, and substitute this form into (15), obtaining

$$\frac{N_o}{\sigma^2 v_{\pm}} \frac{d v_{\pm}}{dt} = \frac{1}{u_{\pm}} \left\{ \mp \frac{d}{dy} \left[U^2 u_{\pm}\right] + \frac{d^2}{dy^2} \left[U^2 u_{\pm}\right] \right\} . \tag{A1-1}$$

Since the left-hand side of (A1-1) is a function only of t, and the right-hand side only of y, both sides must equal a constant, say $-\lambda$. Then $u_{\pm}$ and $v_{\pm}$ satisfy the differential equations

$$L_{\pm} u_{\pm} = -\lambda u_{\pm} , \tag{A2-2a}$$

$$\frac{d v_{\pm}}{dt} = - \frac{\lambda}{N_o} \sigma^2 v_{\pm} , \tag{A1-2b}$$

where the differential operators $L_{\pm}$ are defined by

$$L_{\pm} u(y) \triangleq \mp \frac{d}{dy} \left[U^2(y)\, u(y)\right] + \frac{d^2}{dy^2} \left[U^2(y)\, u(y)\right] . \tag{A1-3}$$

Equations (A1-2a) must be solved using boundary conditions $u_{\pm}(Y_+) = u_{\pm}(Y_-) = 0$. Discrete sets of solutions exist, which we label $\{u_+^{(n)}\}$ and $\{u_-^{(n)}\}$, respectively; we denote the corresponding sets of eigenvalues by $\{\lambda_+^{(n)}\}$ and $\{\lambda_-^{(n)}\}$.

Consider now the linear space $\mathcal{S}$ of twice-differentiable functions defined on $[Y_-, Y_+]$ and vanishing at the endpoints of this interval. Define an inner product between two elements $u, w \in \mathcal{S}$ as

$$\langle u, w \rangle_U \triangleq \int_{Y_-}^{Y_+} U(y)\, u(y)\, w(y)\, dy \quad , \tag{A1-4}$$

where we take U to be a positive and twice-differentiable function defined on $[Y_-, Y_+]$. Now, L_+ and L_- are linear operators on $\mathcal{S}$, and in fact it is easily shown, using integration by parts, that L_- is the adjoint of L_+, i.e., for $u, w \in \mathcal{S}$,

$$\langle u, L_+ w \rangle_U = \langle L_- u, w \rangle_U \quad . \tag{A1-5}$$

It follows[13] that the eigenvalue sets $\{\lambda_+^{(n)}\}$ and $\{\lambda_-^{(n)}\}$ of L_+ and L_- are identical; we denote this common set by $\{\lambda^{(n)}\}$. Further, integration by parts shows that for $u \in \mathcal{S}$, $\langle L_+ u, u \rangle_U \leq 0$, with equality if and only if u is the zero function. Thus, there are no zero-valued eigenvalues, and, in fact, $\lambda^{(n)} > 0$ for all n. Therefore, $\{u_+^{(n)}\}$ and $\{u_-^{(n)}\}$ both span $\mathcal{S}$ and form reciprocal bases, i.e.,

$$\langle u_+^{(m)} , u_-^{(n)} \rangle_U = \delta_{mn} \quad , \tag{A1-6}$$

where we have normalized the eigenfunctions by assuming that $\langle u_+^{(n)}, u_-^{(n)} \rangle = 1$, and we have assumed that if, corresponding to an eigenvalue λ, there exist more than one pair of eigenfunctions u_+ and u_-, these have been orthogonalized so as to satisfy (A1-6).

For any eigenvalue $\lambda^{(n)}$, (A1-2b) may be solved, yielding

$$v_{\pm}^{(n)}(t) = k_{\pm}^{(n)} \quad e^{-\lambda^{(n)} R(t, t_o)} \quad , \tag{A1-7}$$

where R is defined as in (16), and $k_{\pm}^{(n)}$ are constants.

The general solutions of (A1-1) may be written as

$$q_{\pm}(y, t/y_o, t_o) = \sum_{n=1}^{\infty} c_{\pm}^{(n)} \; u_{\pm}^{(n)}(y) \;\; e^{-\lambda^{(n)} R(t, t_o)} \quad , \tag{A1-8}$$

where $c_{\pm}^{(n)}$ are constants. We now impose the initial conditions $q_{\pm}(y, t_o / y_o, t_o) = \delta(y - y_o)$, i.e.,

$$\delta(y - y_o) = \sum_{n=1}^{\infty} c_{\pm}^{(n)} u_{\pm}^{(n)} (y) . \tag{A1-9}$$

Then, using (A1-6), we have

$$U(y_o) u_{\mp}^{(m)} (y_o) = \int_{Y_-}^{Y_+} U(y) u_{\mp}^{(m)} (y) \delta(y - y_o) dy$$

$$= \sum_{n=1}^{\infty} c_{\pm}^{(n)} \langle u_{\mp}^{(m)} , u_{\pm}^{(n)} \rangle_U = c_{\pm}^{(m)} , \tag{A1-10}$$

so that

$$q_{\pm} (y, t / y_o, t_o) = U(y_o) \sum_{n=1}^{\infty} u_{\mp}^{(n)} (y_o) u_{\pm}^{(n)} (y) e^{-\lambda^{(n)} R(t, t_o)} . \tag{A1-11}$$

Correspondingly, using (17), we have

$$\hat{q}_{\pm}(y, t' / y_o) = U(y_o) \sum_{n=1}^{\infty} u_{\mp}^{(n)} (y_o) u_{\pm}^{(n)} (y) e^{-\lambda^{(n)} t'} . \tag{A1-12}$$

APPENDIX II: THE CASE U = 1

The functions $\hat{q}_\pm$ may be evaluated through the Laplace-transform method of Darling and Siegert.[14] We continue, however, in the framework established in Appendix I, which has certain advantages here.

In particular, we consider the case in which $U = U_+ + U_- = 1$. Then the solutions of (A1-2a) are easily shown to have the form

$$u_\pm^{(n)}(y) = a_\pm^{(n)} e^{\pm \frac{y}{2}} \begin{cases} \cos n\omega_o (y - y_1), & n = 1, 3, 5, \ldots \\ \sin n\omega_o (y - y_1), & n = 2, 4, 6, \ldots \end{cases} , \quad \text{(A2-1)}$$

where

$$\omega_o = \frac{\pi}{Y_+ - Y_-} , \quad \text{(A2-2a)}$$

$$y_1 = \frac{Y_+ + Y_-}{2} . \quad \text{(A2-2b)}$$

The corresponding eigenvalues are

$$\lambda^{(n)} = n^2 \omega_o^2 + \frac{1}{4} \quad (n = 1, 2, \ldots) . \quad \text{(A2-3)}$$

In order to satisfy the normalization assumed in (A1-6), we must have

$$a_+^{(n)} a_-^{(n)} = \frac{2\omega_o}{\pi} . \quad \text{(A2-4)}$$

We therefore have from (A1-12), after some manipulation,

$$\hat{q}_\pm(y, t'/y_o) = \frac{\omega_o}{\pi} e^{\pm (y - y_o)/2} \sum_{n=1}^{\infty} \left[\cos n\omega_o (y - y_o) - \cos n\omega_o(y + y_o - 2 Y_+)\right] e^{-\lambda^{(n)} t'} . \quad \text{(A2-5)}$$

Equation (A2-5) may be rewritten in terms of the Jacobi Theta function[15]

$$\theta_3(\upsilon,\ i\pi\tau) \triangleq 1 + 2 \sum_{n=1}^{\infty} e^{-n^2\pi^2\tau} \cos 2n\pi\upsilon$$

$$= \frac{1}{\sqrt{\pi\tau}} \sum_{n=-\infty}^{\infty} e^{-\frac{(\upsilon+n)^2}{\tau}}, \qquad \text{(A2-6)}$$

which is a tabulated function.[16] Then, using (A2-3), (A2-5) becomes

$$\hat{q}_{\pm}(y, t'/y_o) = \frac{\omega_o}{2\pi} e^{\pm\frac{y-y_o}{2}} e^{-t'/4}$$

$$= \left\{\theta_3\left[\frac{\omega_o}{2\pi}(y-y_o),\ \frac{i\omega_o^2 t'}{\pi}\right] - \theta_3\left[\frac{\omega_o}{2\pi}(y+y_o-2Y_+),\ \frac{i\omega_o^2 t'}{\pi}\right]\right\}. \qquad \text{(A2-7)}$$

Note from the second equality of (A2-6) that, for very small t', the functions $\hat{q}_{\pm}$ in (A2-7) are approximately Gaussian with mean y_o and variance $(\omega_o^2 t'/2\pi^2)$. As $t' \longrightarrow \infty$, $\hat{q}_{\pm}$ spread out over the interval $[Y_+, Y_-]$, with areas decreasing exponentially with t'.

We now evaluate $\hat{F}_{\pm}(t'/y_o)$ by integrating (A2-5):

$$\hat{F}_{\pm}(t'/y_o) = \int_{Y_-}^{Y_+} \hat{q}_{\pm}(y, t'/y_o)\, dy = \frac{2\omega_o^2}{\pi} \sum_{n=1}^{\infty} \frac{n}{\lambda^{(n)}} e^{-\lambda^{(n)} t'}$$

$$\left[e^{\pm\frac{Y_+ - y_o}{2}} \sin n\omega_o(Y_+ - y_o) + e^{\mp\frac{y_o - Y_-}{2}} \sin n\omega_o(y_o - Y_-)\right]. \qquad \text{(A2-8)}$$

Differentiating (A2-8) with respect to t' and using (21a), we have

$$\hat{f}_{\pm}(t'/y_o) = \frac{2\omega_o^2}{\pi} \sum_{n=1}^{\infty} n\, e^{-\lambda^{(n)}t'}$$

$$\left[e^{\pm \frac{Y_+ - y_o}{2}} \sin n\omega_o(Y_+ - y_o) + e^{\mp \frac{y_o - Y_-}{2}} \sin n\omega_o(y_o - Y_-) \right] . \quad \text{(A2-9)}$$

Notice from (A2-6) that

$$\theta_3'(\upsilon, i\pi\tau) = -4\pi \sum_{n=1}^{\infty} n\, e^{-n^2\pi^2\tau} \sin 2n\pi\upsilon \ , \qquad \text{(A2-10)}$$

where the prime denotes differentiation with respect to υ. Thus,

$$\hat{f}_{\pm}(t'/y_o) = -\frac{\omega_o^2}{2\pi^2} e^{-\frac{t'}{4}} \left\{ e^{\pm \frac{Y_+ - y_o}{2}} \theta_3'\left[\frac{\omega_o}{2\pi}(Y_+ - y_o), \frac{i\omega_o^2 t'}{\pi}\right] \right.$$

$$\left. + e^{\mp \frac{y_o - Y_-}{2}} \theta_3'\left[\frac{\omega_o}{2\pi}(y_o - Y_-), \frac{i\omega_o^2 t'}{\pi}\right] \right\} . \qquad \text{(A2-11)}$$

We remark that in the symmetric case in which $Y_+ - y_o = y_o - Y_- = Y$, the bracketed factor in the summands of (A2-8) and (A2-9) becomes simply $2 \cosh (Y/2) \sin (n\,\pi/2)$, and the terms corresponding to even-valued n vanish.

APPENDIX III: MINIMIZATION OF $\overline{T}$

We seek an $r(t')$ such that

$$\overline{T} = \int_0^\infty r(t')\,\hat{f}(t'/y_o)\,dt' \tag{A3-1}$$

is minimized, subject to

$$\int_0^\infty \frac{\hat{g}(t'/y_o)}{r(t')}\,dt' = P_{av} \tag{A3-2}$$

and

$$\frac{d}{dt'}\,r(t') = r'(t') \geq \frac{1}{a}\ . \tag{A3-3a}$$

We first rewrite (A3-3a) as

$$r'(t') = \frac{1}{a} + x^2(t')\ . \tag{A3-3b}$$

We than have a standard variational problem in three functions -- $r(t')$, $r'(t')$ and $x(t')$ -- and two constraints. Any solution must therefore[17] be a minimum of the functional

$$I = \int_0^\infty H\left[r(t'),\ r'(t'),\ x(t')\right]\,dt'\ , \tag{A3-4}$$

where

$$H \triangleq r(t')\,\hat{f}(t'/y_o) + \lambda_1 \frac{\hat{g}(t'/y_o)}{r(t')} + \lambda_2(t') \left[r'(t') - x^2(t') - \frac{1}{a}\right], \tag{A3-5}$$

with λ_1 a constant and λ_2 a function, both to be determined.

A minimum of the functional must satisfy the two Euler equations:[17]

$$\frac{\partial H}{\partial r} - \frac{d}{dt'} \frac{\partial H}{\partial r'} = 0 \qquad \text{(A3-6a)}$$

$$\frac{\partial H}{\partial x} = 0 \quad , \qquad \text{(A3-6b)}$$

or

$$\hat{f} - \lambda_1 \frac{\hat{g}}{r^2} - \frac{d\lambda_2}{dt'} = 0 \qquad \text{(A3-7a)}$$

$$2 \lambda_2 x = 0 \quad . \qquad \text{(A3-7b)}$$

If we assume that λ_2 and x are reasonably well behaved, then, except for isolated points, (A3-7b) can only be satisfied if either $x \equiv 0$ over an interval or $\lambda_2 \equiv 0$ over an interval.[18] In the former case, we have from (A3-3b)

$$r' = \frac{1}{a} \quad . \qquad \text{(A3-8)}$$

In the latter case, we must have $d\lambda_2/dt' \equiv 0$ over the interval, whence, from (A3-7a)

$$r = \sqrt{\lambda_1 \hat{g}/\hat{f}} \quad . \qquad \text{(A3-9)}$$

Thus, any r which qualifies as a solution of the problem must satisfy (A3-8) over some intervals of t' and (A3-9) over the remaining intervals, with λ_1 of (A3-9) being adjusted so that (A3-2) holds. Of course, (A3-9) cannot apply in an interval in which it would violate (A3-3a). In addition, it is clear from (16), (25) and (35a) that only those solutions which are continuous and for which $r(0) = 0$ are acceptable.

FOOTNOTES

1. D. Blackwell and M. A. Girshick, "Theory of Games and Statistical Decisions," John Wiley and Sons, Inc., New York, N. Y., Chap. 9; 1954.

2. P. M. Woodward, "Probability and Information Theory, with Applications to Radar," McGraw-Hill Book Co., Inc., New York, N. Y., Chap. 4; 1953.

3. We assume that the feedback channel is noiseless. This may be a reasonable assumption in such cases as that of telemetry from a satellite, in which the transmitter, on the satellite, is of low power, but the ground-based receiver may have associated with it an extremely high-powered feedback transmitter. See also the discussion of the radar case at the end of the paper.

4. J. L. Doob, "Stochastic Processes," John Wiley and Sons, Inc., New York, N. Y., Chap. VI, Sec. 3; 1953.

5. R. Fortet, "Les fonctions aléatoires du type de Markoff associées à certaines équations lineáres aux dérivées partielles du type parabolique," J. Math. Pures Appl., vol. 22, pp. 177-243; 1943.

6. We assume that $F(\infty) = 0$, so that f is a proper density. This assumption is equivalent to assuming that $R(t, t_o)$ of (16), below, satisfies $R(\infty, t_o) = \infty$; see Appendix I.

7. See Appendix I.

8. Note that (46) specifies low-pass waveforms as the optimum choices. The usual narrow-band approximations show, however, that little is lost by using instead an equivalent band-pass waveform, of which (46) is the squared envelope.

9. It is tempting to conjecture that C should always be set equal to unity, so that the optimum transmitter strategy is to increase power linearly until peak power is attained, and then to remain at peak power. However, it has not proved possible to prove this conjecture, and, in fact, similar problems in optimal control theory lead to discontinuities such as that shown by (46) when $C \neq 1$.

10. Notice that, since the receiver would immediately ($\overline{T} = 0$) and with high probability ($P^e \longrightarrow 0$) approach the correct threshold, the effect of the feedback control of the transmitter power is immediately to cut the power down to zero in a quadratic manner (see (47)).

    The result given in (54) is of considerable theoretical interest, for it implies that with proper feedback it is possible, with finite average power, to obtain arbitrarily small error probability in an arbitrarily short average time, so long as the allowable peak power is unlimited. Such a result does not obtain in the absence in uncertainty feedback.

11. Further, Viterbi has shown that sequential detection with synchronization feedback is itself 6 db superior to nonsequential detection in the symmetric, asymptotic situation being considered, when constant signals are used. This superiority will be even greater when the optimal signals derived here are used. See A. J. Viterbi, "Improvement of coherent communication over the Gaussian channel by error-free decision feedback," Jet Propulsion Lab., Cal. Inst. of Tech., Pasadena, Calif., Space Programs Summary No. 37-23, vol. IV, pp. 179-180; October 31, 1963.

12. We have assumed throughout that all expressions for the signals $s_{\pm}$ are scaled so as to take into account the channel attenuation.

13. B. Friedman, "Principles and Techniques of Applied Mathematics," John Wiley and Sons, Inc., p. 131; 1956.

14. D. A. Darling and A. J. F. Siegert, "The first passage problem for a continuous Markov process," Ann. Math. Stat., vol. 24, pp. 624-639; December, 1953.

15. W. Magnus and F. Oberhettinger, "Formulas and Theorems for the Functions of Mathematical Physics," Chelsea Publishing Co., New York, N. Y., pp. 98-99; 1954.

16. F. Jahnke and F. Emde, "Tables of Functions with Formulae and Curves," Dover Publications, New York, N. Y., pp. 41-45; 1945.

17. L. E. Elsgolc, "Calculus of Variations," Addison-Wesley Publishing Co., Reading, Mass., Chap. IV; 1962.

18. The case in which both x and λ_2 both are zero in an interval can be subsumed in the first case.