WAVE-MECHANICAL APPROACH TO
THE NONLINEAR THEORY OF O-TYPE
TRAVELING-WAVE TUBES

by

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Wave-mechanical Approach to the Nonlinear Theory
of O-type Traveling-wave Tubes[*]

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ABSTRACT

The one-dimensional motion of electrons in O-type traveling-wave devices is described in terms of wave mechanics. For the case of large-signal interaction it is found that in the final stage of interaction the electrons travel along with the phase velocity of the electromagnetic slow wave at discrete energy levels involving Planck's constant. By making use of the Pauli exclusion principle the saturation power and electronic efficiency of a large-signal traveling-wave tube are obtained.

I. INTRODUCTION

The nonlinear behavior of O-type traveling-wave tubes has intrigued many workers in the field of microwave tubes. Of particular interest is the derivation of expressions for the efficiency of the tube in terms of the tube parameters; a number of theories pertaining to this subject has been published in the past decade.[1,2,3] However, the problem turns out to be so complicated that only by resorting to computer calculations can reasonably correct answers be obtained, so that the advantages of an analytic expression in terms of the tube parameters are lost.

In the present theory the problem has been approached from the viewpoint of wave mechanics. It is felt that the Schrödinger equation of motion yields a more suitable starting point because its solutions contain all the relevant information about the motion of the electrons, the charge density, and, in particular, the energy of the electrons. Work on the wave-mechanical description of O-type interactions has been done earlier by Brillouin.[4,5]

It is shown that the method is capable of yielding an analytic expression for the efficiency in terms of the tube parameters, without the need for large-scale computer calculations; moreover, it is easy to keep track of the assumptions that are made. A comparison with experiments is made. The results of the theory agree encouragingly with the measured results.

II. WAVE-MECHANICAL FORMULATION OF THE EQUATIONS

If we introduce the potential

$$V = - \int E_z \, dz, \tag{1}$$

then the equations for one-dimensional, nonrelativistic motion that must be satisfied are

$$\frac{\partial^2 V}{\partial z^2} - \epsilon \mu \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon}\left[\epsilon \mu \frac{\partial}{\partial t}\int J_z \, dz + \rho\right] \qquad (2)$$

$$\frac{\partial J_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0 \qquad (3)$$

$$\frac{dv_z}{dt} = \frac{e}{m}\frac{\partial V}{\partial z} \quad . \qquad (4)$$

The wave-mechanical analog of the above equations is obtained (see Appendix) by replacing the equation of motion by the time-dependent Schrödinger equation for the wave function ψ, letting

$$\rho = -e\,\psi^*\psi \quad , \qquad (5)$$

and

$$J = \frac{-i\hbar e}{2m}\left\{\psi\psi^{*'} - \psi^*\psi'\right\} ; \qquad (6)$$

where $\hbar$ is Planck's constant divided by 2π and the primes denote differentiation with respect to z. With this definition of the current density J the continuity equation is automatically satisfied. The two equations with which we are left are

the equation of motion,

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial z^2} + eV\psi = i\hbar\frac{\partial \psi}{\partial t} ; \qquad (7)$$

and the interaction equation,

$$\frac{\partial^2 V}{\partial z^2} - \epsilon\mu\frac{\partial^2 V}{\partial t^2} = \frac{e}{\epsilon}\left[\frac{\hbar^2\epsilon\mu}{4m^2}(\psi\psi^{*''} - 2\psi'\psi^{*'} + \psi^*\psi'') + \psi^*\psi\right] \quad . \qquad (8)$$

We shall look at the case of weak reaction of the electron wave on the voltage wave. Then, although the electron wave may contain all harmonics of the basic frequency ω, the dominant harmonic in the voltage wave will be the zero-th harmonic (that is, has the basic frequency ω). The phase velocity ω/k of the voltage wave may differ from $1/\sqrt{\epsilon\mu}$.
Thus we write

$$V = V_o + 2 V_1 \cos(\omega t - kz); \tag{9}$$

where

$$e V_o = \frac{1}{2} m v_o^2 , \tag{10}$$

and v_o is the dc velocity of the electrons. The equation of motion then becomes:

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{2me}{\hbar^2}\left[V_o + 2V_1 \cos(\omega t - kz)\right]\psi - \frac{2m}{i\hbar}\frac{\partial \psi}{\partial t} = 0 . \tag{11}$$

We make the substitution

$$\psi(z,t) = e^{i(\omega_o t - k_o z)}\phi(\xi) ; \tag{12}$$

where

$$\xi = \frac{1}{2}(\omega t - kz) , \tag{13}$$

and ω_o and k_o are as yet undetermined constants.
Then the differential equation becomes

$$\frac{k^2}{4}\frac{d^2\phi}{d\xi^2} + (ik k_o - \frac{m\omega}{i\hbar})\frac{d\phi}{d\xi} + \left[-k_o^2 - \frac{2m\omega_o}{\hbar} + \frac{2me}{\hbar^2} \cdot \right.$$
$$\left. \cdot (V_o + 2V_1 \cos 2\xi)\right]\phi = 0. \tag{14}$$

We choose

$$k_o = -\frac{m}{\hbar}\frac{\omega}{k} = -\frac{m}{\hbar}v_p, \qquad (15)$$

where v_p denotes the phase velocity of the voltage wave.

Then the coefficient of the $d\phi/d\xi$ term vanishes and Eq. (14) becomes

$$\frac{d^2\phi}{d\xi^2} + \frac{4}{k^2}\left[-k_o^2 - \frac{2m\omega_o}{\hbar} + \frac{2me}{\hbar^2}(V_o + 2V_1\cos 2\xi)\right]\phi = 0. \qquad (16)$$

We define

$$\eta = -\frac{4}{k^2}\left[k_o^2 - \frac{2me}{\hbar^2}V_o + \frac{2m\omega_o}{\hbar}\right], \qquad (17)$$

$$\gamma = \frac{4}{k^2}\frac{4meV_1}{\hbar^2}. \qquad (18)$$

Then the differential equation reduces to the standard form of the Mathieu equation:

$$\frac{d^2\phi}{d\xi^2} + (\eta + \gamma\cos 2\xi)\phi = 0. \qquad (19)$$

It is important to recall that

$$k_o^2 = \frac{m^2}{\hbar^2}v_p^2 \qquad (15a)$$

and

$$\frac{m}{\hbar^2}2eV_o = \frac{m^2}{\hbar^2}v_o^2, \qquad (16a)$$

so that

-4-

$$k_o^2 \cdot \frac{2me}{\hbar^2} V_o = \frac{m^2}{\hbar^2}(v_p^2 - v_o^2). \tag{20}$$

Thus we may write for η

$$\eta = -\frac{4m^2}{k^2\hbar^2}\left[v_p^2 - v_o^2 + 2\frac{\hbar\omega_o}{m}\right] . \tag{21}$$

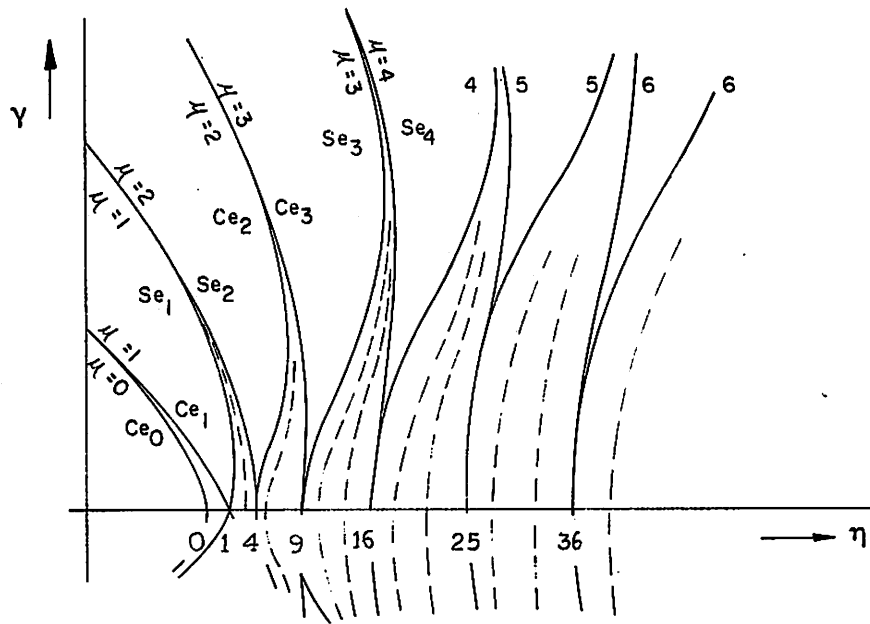## III. PROPERTIES OF THE SOLUTIONS OF THE MATHIEU EQUATION

Floquet's theorem for the Mathieu equation states that the general solution can be written as[5]

$$\phi = A\, u_1(\xi)\, e^{i\mu\xi} + B\, u_2(\xi)\, e^{-i\mu\xi} , \tag{22}$$

where $u_1(\xi)$ and $u_2(\xi)$ are periodic with period π. Along the η-axis in the η-γ plane the solutions are of the form $e^{\pm i\sqrt{\eta}\,\xi}$, so that $\mu = \sqrt{\eta}$ and the functions $u_1(\xi)$ and $u_2(\xi)$ are constants. Alternatively, we may write the solutions along the η-axis as $\cos\sqrt{\eta}\,\xi$ and $\sin\sqrt{\eta}\,\xi$ . .

For $\gamma \neq 0$ Mathieu functions $Ce_\mu(\gamma, \xi)$ and $Se_\mu(\gamma, \xi)$ have been defined. These functions are known as expansions in powers of γ, starting with either $\cos\mu\xi$ or $\sin\mu\xi$ as the independent term. Lines of constant μ run through the η-γ plane as shown in Fig. 1. On each line, two functions Ce_μ and Se_μ may be computed, except along the lines where μ is an integer, where only one periodic solution is obtained. These boundaries start from the points $\eta = m^2 = 1, 4, 9, 16, \ldots$ on the η axis. Two curves $\mu = m$ start from each of these points: one curve yields the function Ce_m, the other Se_m. On each curve there is an additional aperiodic solution. Outside the boundaries corresponding to μ = integer, as shown, the Mathieu equation has no periodic solutions. There $\mu = \alpha + i\beta$, $\beta \neq 0$; α may be zero.

There are no periodic solutions for any value of $\eta < -\gamma$. We can make the following statements about the character of the solutions:

(1) $\eta < -\gamma$ no periodic solutions, μ complex or imaginary;

(2) $-\gamma < \eta < +\gamma$ $\begin{cases} \text{narrow bands of periodic solutions,} \\ \text{broad bands without periodic solutions;} \end{cases}$

(3) $\eta > \gamma$ $\begin{cases} \text{broad bands of periodic solutions,} \\ \text{narrow bands without periodic solutions.} \end{cases}$

The bands with periodic solutions all become straight lines parallel to $\eta = -\gamma$ at infinity. These lines intersect a line parallel to the η-axis at the points

$$\eta = -\gamma + (2n + 1)\sqrt{2\gamma} \quad (n = 0, 1, 2, \ldots), \qquad (23)$$

if γ is greater than $2n^2$.

IV. ENERGY LEVELS

We are now in a position to discuss the behavior of the space charge $\rho = -e\psi^* \psi$. From Eq. (12) it is clear that

$$\rho = -e\,\psi^*\,\psi = -e\,\phi^*\,\phi \ . \qquad (24)$$

Thus, from the discussion on the behavior of ϕ in the previous section, we find:

(1) $\eta < -\gamma$, that is,

$$\frac{1}{2}\,m\,(v_p^2 - v_o^2) + \hbar\omega_o > 2\,e\,V_1 \ . \qquad (25)$$

In this case no periodic propagation is possible.

(2) $-\gamma < \eta < +\gamma$, that is,

$$-2 eV_1 < \frac{1}{2} m (v_p^2 - v_o^2) + \hbar\omega_o < 2 eV_1 .\qquad (26)$$

In this case there are narrow bands of periodic propagation and broad bands where no periodic propagation is possible.

(3) $\eta > +\gamma$, that is,

$$\frac{1}{2} m (v_p^2 - v_o^2) + \hbar\omega_o < -2 eV_1 .\qquad (27)$$

In this case there are broad bands of periodic propagation and narrow bands where no periodic propagation is possible.

We shall not take up the discussion of the reaction of the electron wave on the voltage wave. Instead, we shall draw a general conclusion about the periodic propagation in case (2) where

$$-2 eV_1 < \frac{1}{2} m (v_p^2 - v_o^2) + \hbar\omega_o < 2 e V_1;\qquad (28)$$

which will be the case for a strong interaction voltage $2V_1$ (see Eq. (37)).

We recall that the momentum operator is $-i\hbar\, \partial/\partial z$. If we differentiate Eq. (12) for $\psi(z, t)$ with respect to z and multiply by $-i\hbar$, we obtain

$$-i\hbar \frac{\partial \psi}{\partial z} = i\hbar\, e^{i(\omega_o t - k_o z)} \left[ik_o \phi(\xi) + \frac{1}{2} k \frac{d\phi}{d\xi} \right] .\qquad (29)$$

Substituting Eq. (15) for k_o and putting $k = \omega/v_p$, we find $|k_o/k| = m v_p^2/(\hbar\omega)$. For a practical case, v_p is of the order of one-tenth of the velocity of light, ω is of the order of 10^{10} sec^{-1}. Then, $|k_o/k|$ is of the order of 10^9. Considering that ϕ is a smooth function of ξ,[7] we find that the term $1/2\, kd\phi/d\xi$ in Eq. (29) is, to excellent approximation, negligible compared to the term $ik_o \phi(\xi)$. Eq. (29) then becomes

$$- i\hbar \frac{\partial \psi}{\partial z} = -\hbar k_o \psi = m v_p \psi . \tag{30}$$

Thus, the momentum of the electron is

$$p = m v_p ; \tag{31}$$

which means that the electron travels at the phase velocity of the voltage wave.

Similarly, the energy operator is $i\hbar \, \partial \psi / \partial t$. If we differentiate Eq. (12) for $\psi(z, t)$ with respect to t and multiply by $i\hbar$, we obtain

$$i\hbar \frac{\partial \psi}{\partial t} = i\hbar \, e^{i(\omega_o t - k_o z)} \left[i\omega_o \, \phi(\xi) + \frac{1}{2} \omega \frac{d\phi}{d\xi} \right] . \tag{32}$$

If we neglect the term $1/2 \, \omega \, d\phi/d\xi$, we find

$$i\hbar \frac{\partial \psi}{\partial t} = -\hbar \omega_o \psi ; \tag{33}$$

that is, the total energy of the electron is given by

$$W = -\hbar \, \omega_o . \tag{34}$$

We shall find that (see Eq. (37)), in the case of a strong interaction voltage, a measure for W is $2eV_1$. Taking ω of the order of 10^{10} sec^{-1}, we find that ω_o / ω is of the order of $10^5 \, V_1$. V_1 may be of the order of 10^3 Volt, so neglecting the term $1/2 \, \omega \, d\phi/d\xi$ in Eq. (32) compared to the term $i\omega_o \, \phi(\xi)$ is, again to excellent approximation, justified.

The bands of periodic propagation in the region $-\gamma < \eta < \gamma$ are given approximately by

$$\eta = -\gamma + (2n + 1)\sqrt{2\gamma} \quad (n = 0, 1, 2, \dots) ; \tag{35}$$

which approximation is the better the higher the value of γ. According to Eq. (18), $\gamma = 16\,eV_1\,mv_p^2/(\hbar\omega)^2$. For the practical values quoted, γ is of the order of 10^{18}. The approximation made in Eq. (35) is, therefore, utterly justified. Thus, the bands of periodic propagation are given by

$$\frac{8\,m}{k^2\hbar^2}\left[\frac{1}{2}m\,(v_o^2 - v_p^2) - \hbar\omega_o\right] =$$

$$= -\frac{16\,m\,eV_1}{k^2\hbar^2} + (2n+1)\sqrt{\frac{32\,m\,eV_1}{k^2\hbar^2}}\,, \qquad (36)$$

or,

$$\frac{1}{2}m(v_o^2 - v_p^2) - \hbar\omega_o = -2\,eV_1 + \frac{2n+1}{2}\,k\,\hbar\sqrt{\frac{2eV_1}{m}}\,. \qquad (37)$$

We may define a velocity v_1 in terms of the potential V_1 defined in Eq. (9):

$$\frac{1}{2}mv_1^2 = 2\,eV_1\,. \qquad (38)$$

Then, writing $k = \omega/v_p$, we obtain

$$\frac{1}{2}m(v_o^2 - v_p^2) - \hbar\omega_o = -\frac{1}{2}mv_1^2 + \frac{2n+1}{2\sqrt{2}}\,\hbar\,\omega\frac{v_1}{v_p}\,. \qquad (39)$$

The total energy of the electron, $W = -\hbar\omega_o$, then becomes

$$W = \frac{1}{2}m(v_p^2 - v_o^2 - v_1^2) + \frac{2n+1}{2\sqrt{2}}\,\frac{v_1}{v_p}\,\hbar\,\omega \quad (n = 0, 1, 2, \ldots). \qquad (40)$$

The term $1/2\,mv_p^2$ represents the kinetic energy of the electron, the term $-1/2\,mv_o^2$ is the dc potential energy. The term $-1/2mv_1^2 = -2\,eV_1$ indicates that, in the state of lowest energy ($n = 0$), the electron travels in a minimum of the interaction voltage $2V_1\cos(\omega t - kz)$. From Eqs. (31) and (40) we conclude that the electrons

travel along with the voltage wave <u>at selected levels of total energy</u>, [*]
W_n.

The sharpness of the energy levels gradually decreases as n increases, and finally disappears as $|\eta|$ becomes larger than γ; in which case we enter the region of broad bands of periodic propagation.

V. SATURATION POWER AND ELECTRONIC EFFICIENCY OF THE LARGE-SIGNAL O-TYPE TRAVELING-WAVE AMPLIFIER

We assume that the electrons leave the cathode with zero velocity, that is, we neglect thermal velocities. Then the total energy W in the final stage of interaction (Eq. (40)) is the energy gained from the field by an electron if W is positive, or the energy given up to the field if W is negative. In the case of the traveling-wave tube we are interested in the case $W < 0$. Let $I_o = + Ne$ be the dc cathode current, where N is the total number of electrons leaving the cathode per second. Then, because of the conservation of charge, N electrons pass the exit plane of the tube per second.

In the final stage of interaction the electrons travel in bunches centered at the potential minima in the voltage wave. The number of bunches passing the exit plane per second is $\omega/2\pi$, so that the number of electrons per wavelength is $2\pi N/\omega$. Because we have neglected

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[*]   This is not the same result as that obtained by Brillouin (Ref. 4), who found that the energy in the interaction fields is quantized. Not only does the present analysis show that one might more conveniently consider the energy of the <u>electrons</u> to be quantized, but in addition, the energy levels depend (<u>unlike in Brillouin's</u> analysis) on the phase velocity v_p and on the interaction potential V_1.

thermal velocities these $2\pi N/\omega$ electrons per wavelength occupy the energy levels given by Eq. (40) in such a way that each energy level is occupied by two electrons from the lowest possible energy up to some maximum energy, so as to accomodate all electrons. (The Pauli exclusion principle states that each possible state may be occupied by one electron; because the electron has spin 1/2 each energy level corresponds to two possible states, so that each energy level may be occupied by two electrons.)

Thus we find that the total energy given up to the field by the electrons passing the exit plane per second is given by

$$P = - 2 \frac{\omega}{2\pi} \left[\frac{2\pi}{\omega} \frac{N}{2} \frac{1}{2} m(v_p^2 - v_o^2 - v_1^2) + \frac{1}{2\sqrt{2}} \hbar \omega \frac{v_1}{v_p} \sum_j (2j + 1) \right] \quad , \quad (41)$$

where j extends from 0 to $\left[(2\pi/\omega) (N/2) - 1 \right]$.

The sum in this expression can be evaluated immediately

$$\sum_j (2j + 1) = \left(\frac{2\pi}{\omega} \frac{N}{2} \right)^2 . \quad (42)$$

Then we obtain

$$P = - N \frac{1}{2} m(v_p^2 - v_o^2 - v_1^2) - N^2 \frac{\pi \hbar}{2\sqrt{2}} \frac{v_1}{v_p} . \quad (43)$$

Now P is the power propagating in the tube in the final stage of interaction. The amplitude of the voltage wave is $2V_1$, so we may introduce a coupling impedance K by

$$K = 4 V_1^2 / (2P) . \quad (44)$$

Then we may write

$$2 V_1 = \sqrt{2 P K} \; . \tag{45}$$

From the definition of v_1 (Eq. (38)) we then have

$$\frac{1}{2} m v_1^2 = e \sqrt{2 P K} \; , \tag{46}$$

and

$$v_1 = \sqrt{\frac{2e}{m}} \; (2 P K)^{1/4} \; . \tag{47}$$

Substituting Eqs. (43) and (47), and putting $N = I_o/e$, we obtain

$$P = \frac{I_o}{e} \left[\frac{1}{2} m (v_o^2 - v_p^2) + e (2PK)^{1/2} - \frac{I_o}{e} \frac{\pi \hbar}{2 v_p \sqrt{2}} \sqrt{\frac{2e}{m}} (2PK)^{1/4} \right] \; . \tag{48}$$

From Eq. (10) we have $1/2 \, m v_o^2 = e V_o$, so that we may write

$$P = I_o V_o \left[\left(\frac{1 - v_p^2}{v_o^2} \right) + \frac{(2PK)^{1/2}}{V_o} - \frac{I_o}{V_o} \frac{h}{4 e^2 v_p} \sqrt{\frac{e}{m}} (2PK)^{1/4} \right] \; , \tag{49}$$

where we have substituted $2 \pi \hbar = h$.

We define the dc beam power P_o by

$$P_o \equiv I_o V_o \tag{50}$$

and the dc beam resistance R_o by

$$R_o \equiv V_o / I_o \; . \tag{51}$$

Then Eq. (49) becomes

$$\frac{P}{P_o} = \left(1 - \frac{v_p^2}{v_o^2}\right) + \left(\frac{2PK}{P_oR_o}\right)^{1/2} + \frac{h}{2mev_o v_p\sqrt{2}} I_o \left(\frac{2PK}{P_oR_o}\right)^{1/4} . \qquad (52)$$

We define the electronic efficiency η_e by

$$\eta_e \equiv \frac{P}{P_o} , \qquad (53)$$

the coupling parameter $\mathcal{K}$ by[*]

$$\mathcal{K}^4 = 2K/R_o, \qquad (54)$$

the synchronism parameter β

$$\beta = 1 - (v_p^2/v_o^2) , \qquad (55)$$

and the parameter q

$$q = \frac{h\, I_o}{2mev_o v_p\sqrt{2}} . \qquad (56)$$

Then we obtain the following equation

$$\eta_e - \mathcal{K}^2 \eta_e^{1/2} + q\,\mathcal{K}\eta_e^{1/4} - \beta = 0 , \qquad (57)$$

or, in terms of a new variable $\eta' \equiv \eta_e^{1/4}$,

$$\eta'^4 - \mathcal{K}^2 \eta'^2 + q\,\mathcal{K}\,\eta' - \beta = 0 \qquad (58)$$

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[*] We note that $\mathcal{K}$ is simply related to Pierce's gain parameter C, by $\mathcal{K}^4 = (2C)^3$.

This is an equation of the fourth degree in η'. For fixed tube parameters the electronic efficiency η_e may be computed immediately. For $I_o = 10$ ma and v_o and v_p of the order of one-tenth of the velocity of light, q is of the order of 0.02. We see, therefore, that under ordinary conditions, the quantum effects are negligible, so that we may write Eq. (58) as

$$\eta'^4 - \hbar^2 \eta'^2 - \beta = 0 . \qquad (59)$$

The coefficients in this equation are entirely determined by Pierce's parameters C and QC.

Table I. $(v_o - v_p)/(Cv_o)$ versus QC for maximum amplification of the linearly operated traveling-wave tube. [*]

QC	$(v_o - v_p)/(Cv_o)$
0.00	0.50
0.003	0.50
0.010	0.52
0.023	0.55
0.041	0.59
0.065	0.64
0.081	0.67
0.099	0.71
0.121	0.76
0.147	0.82
0.181	0.89
0.228	0.98
0.295	1.10
0.412	1.29
0.490	1.40
0.608	1.56
0.814	1.81
0.995	2.00
1.32	2.30

[*] K. Mouthaan, Master's Thesis, Technological University Delft, 1963; Appendix A.

Table I gives $(v_o - v_p / (Cv_o)$ as a function of QC for maximum amplification of the linearly operated traveling-wave tube. Table II shows a comparison between measured and theoretical results obtained by Cutler[8] and efficiencies calculated from Eq. (59). The results obtained by the present method are at least as close as those obtained by the more laborious methods utilized by Cutler. The present results have not been corrected for circuit attenuation, a correction that would tend to lower them. On the other hand, values of $v_o^2 - v_p^2$ used in the present computation were calculated from the given values of C and QC for maximum gain; if these values were obtained for maximum efficiency rather than maximum gain the results tabulated in the last column would need to be increased. Using the wrong value of v_p might also account for the numerical discrepancies shown in Table II.

TABLE II. Efficiency, %

Tube	C	QC	Cutler[8]		Measured	From Eq. (59)
			Intrinsic	Corrected for circuit attenuation		
McDowell	0.078	0.27	26.0	21.6	19.5	18.0
McDowell	0.058	0.29	16.2	12.5	13.2	13.7
Brangaccio and Cutler	0.041	0.61	6.0	6.0	11.0	13.5
Danielson and Watson	0.05	0.35	7.0	4.8	6.6	12.7
R. Warnecke	0.125	0.32	33.0	33.0	27.0	30..8
W. Kleen and W. Friz	0.05	0.5	11.5	5.7	7.8	15.6
W. Kleen	0.10	0.20	26.0	22.0	20.0	21.3
L. Bruck	0.065	0.19	23.0	18.5	15.0	13.1
Hughes Aircraft Co.	0.12	0.19	31.0	29.0	39.0	26.7
Hughes Aircraft Co.	0.11	0.15	15.5	12.7	25.0	22.3

VI. CONCLUSION

The major assumption in our calculation of the electronic efficiency of nonlinearly operated traveling-wave tubes is that saturation has been reached; that is to say, a stable periodic situation prevails in the final stage of interaction.

Our theory may be summarized as follows. The transformation described by Eqs. (12), (13) and (15) physically means the transformation to the case where the electron moves in the static potential well $eV = 1/2 \, m(v_o^2 - v_p^2) + 2eV_1 \cos 2\xi$. We have calculated the energy levels for the electron, assuming that V_1 is high enough to prevent the electron from escaping the well. Assuming a stable periodic situation in the final stage of interaction, there are $2\pi/\omega N$ electrons per potential well. We have calculated the total energy given up to the field by these $2\pi/\omega N$ electrons. Then we have taken the sum over the $\omega/2\pi$ potential wells, passing the exit plane of the tube per second, to obtain the saturated output power; which in turn gives the electronic efficiency.

The most important result of the present theory is that the electrons, in passing through the tube, not only loose the amount of kinetic energy $1/2 \, m(v_o^2 - v_p^2)$, but, in addition, the amount of potential energy $2eV_1$. The latter contribution is decisive in obtaining the correct result for the efficiency of the tube.

APPENDIX

Schrödinger Equation of Motion

Consider the one-dimensional motion of a particle of mass m in a conservative field of force. Let V be the potential energy of the particle in this field of force.

We have

$$\frac{p_z^2}{2m} + V = W ,$$

where p_z is the momentum and W the total energy of the particle.

The Schrödinger equation of motion is obtained by introducing operators

$$p_z \longrightarrow \frac{h}{2\pi i} \frac{\partial}{\partial z}$$

$$W \longrightarrow -\frac{h}{2\pi i} \frac{\partial}{\partial t}$$

and operating on the wave function ψ:

$$-\frac{h^2}{8\pi^2 m} \frac{\partial^2 \psi}{\partial z^2} + V\psi = -\frac{h}{2\pi i} \frac{\partial \psi}{\partial t} ;$$

Where h denotes Planck's constant.

In this formalism the quantity

$$\psi^* \psi \, dz$$

is interpreted as the probability that the particle has position between z and $z + dz$; whence the normalization condition

$$\int_{-\infty}^{+\infty} \psi^* \psi \, dz = 1 ,$$

that is, the probability that the particle will be found somewhere in the range $-\infty < z < +\infty$ is equal to unity.

For an electron moving in the field derived from an electric potential V we have

$$\frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial z^2} - e V \psi = i\hbar \frac{\partial \psi}{\partial t} \quad ,$$

where $-eV$ is the potential energy of the particle (negative) and where we have written $\hbar = h/2\pi$. In the case of an electron "beam" we normalize the wave function ψ such that

$$\int_{-\infty}^{+\infty} - e \psi^* \psi \, dz = Q \quad ,$$

where Q is the total charge in the beam, and interpret then the density $-e \psi^* \psi$ as the charge density

$$- e \psi^* \psi \longleftrightarrow \rho \quad .$$

Probability Current

Let $\psi^* \psi \, dz$ be the probability of finding the particle in the range z to $z + dz$ at the instant t and let ψ be normalized to unity

$$\int_{-\infty}^{+\infty} \psi^* \psi \, dz = 1 \quad .$$

The probability of finding the particle somewhere at the left of a point z at the instant t is

$$\int_{-\infty}^{z} \psi^* \psi \, dz \quad , \tag{1}$$

and the time rate of change of this probability is

$$\frac{\partial}{\partial t} \int_{-\infty}^{z} \psi^* \psi \, dz \; . \tag{2}$$

The function ψ is normalized at all instants t; therefore, a decrease in the quantity (1) is accompanied by an equal increase in the probability of finding the particle at the right of the point z, and vice versa.

So whenever the quantity (2) is negative the probability flows from left to right past the point z and vice versa.

We define the probability current at the point z at the instant t to mean the time rate at which the probability flows from left to right past the point z at the instant t. This rate, denoted by $J(z, t)$ is then given by

$$J(z, t) = - \frac{\partial}{\partial t} \int_{-\infty}^{z} \psi^* \psi \, dz \; . \tag{3}$$

In the case of an electron beam we define the current density $J = J(z, t)$ in a similar way as

$$J(z, t) = - \frac{\partial}{\partial t} \int_{-\infty}^{z} e \, \psi^* \psi \, dz.$$

Note that the current density J and the charge density ρ so defined satisfy the continuity equation

$$\frac{\partial J}{\partial z} + \frac{\partial \rho}{\partial t} = 0.$$

We can obtain an expression for the probability current (3) that is more suitable for use by differentiating the right-hand side of (3) under the integral sign and making use of the equation of motion

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial z^2} + V\psi = i\hbar\frac{\partial\psi}{\partial t} \, ,$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial z^2} + V\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t} \, .$$

Thus,

$$J(z,t) = -\frac{\partial}{\partial t}\int_{-\infty}^{z}\psi^*\psi\,dz$$

$$= -\int_{-\infty}^{z}\left[\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right]dz$$

$$= -\frac{i\hbar}{2m}\int_{\infty}^{z}\left[\psi^*\psi'' - \psi\psi^{*}{}''\right]dz$$

$$= \frac{i\hbar}{2m}\left\{\psi\psi^{*}{}'\Big|_{-\infty}^{z} - \int_{-\infty}^{z}\psi^{*}{}'\psi'\,dz + \right.$$

$$\left. - \psi^*\psi'\Big|_{-\infty}^{z} + \int_{-\infty}^{z}\psi'\psi^{*}{}'\,dz\right\}$$

$$= \frac{i\hbar}{2m}\left[\psi\psi^{*}{}' - \psi^*\psi'\right] \, ,$$

because the integrals cancel and the integrated parts are zero at the lower limit. So

$$J(z,t) = \frac{i\hbar}{2m}\left[\psi\psi^{*}{}' - \psi^*\psi'\right] \, .$$

Similarly for the case of an electron beam

$$J(z,t) = \frac{-i\hbar}{2m}\,e\left[\psi\psi^{*}{}' - \psi^*\psi'\right] \, .$$

Let us look at the quantity $\partial J / \partial t$.

$$\frac{\partial J}{\partial t} = \frac{-i\hbar e}{2m}\left[\psi\frac{\partial \psi^{*}{}'}{\partial t} + \psi^{*}{}'\frac{\partial \psi}{\partial t} - \psi^{*}\frac{\partial \psi'}{\partial t} - \psi'\frac{\partial \psi^{*}}{\partial t}\right]$$

$$= \frac{-i\hbar e}{2m}\left[\left(\psi^{*}{}' - \psi^{*}\frac{\partial}{\partial z}\right)\frac{\partial \psi}{\partial t} + \left(\psi\frac{\partial}{\partial z} - \psi'\right)\frac{\partial \psi^{*}}{\partial t}\right]$$

$$= \frac{-i\hbar e}{2m}\left[\left(\psi^{*}{}' - \psi^{*}\frac{\partial}{\partial z}\right)\left(\frac{i\hbar}{2m}\psi'' + \frac{eV\psi}{i\hbar}\right) + \right.$$

$$\left. + \left(\psi\frac{\partial}{\partial z} - \psi'\right)\left(-\frac{i\hbar}{2m}\psi^{*}{}'' + \frac{eV\psi^{*}}{i\hbar}\right)\right],$$

where we have made use of the equation of motion,

$$= \frac{-i\hbar e}{2m}\left[\frac{i\hbar}{2m}\psi^{*}{}'\psi'' - \frac{eV}{i\hbar}\psi^{*}{}'\psi - \frac{i\hbar}{2m}\psi^{*}\psi''' + \frac{eV}{i\hbar}\psi^{*}\psi' + \right.$$

$$\left. - \frac{i\hbar}{2m}\psi\psi^{*}{}''' + \frac{eV}{i\hbar}\psi\psi^{*}{}' + \frac{i\hbar}{2m}\psi'\psi^{*}{}'' - \frac{eV}{i\hbar}\psi'\psi^{*}\right]$$

$$= \frac{-i\hbar e}{2m}\cdot\frac{i\hbar}{2m}\left[\psi^{*}{}'\psi'' + \psi'\psi^{*}{}'' - \psi^{*}\psi''' - \psi\psi^{*}{}'''\right].$$

We write this in the form

$$\frac{\partial J}{\partial t} = \frac{\hbar^{2}e}{4m^{2}}\left[(\psi^{*}{}'\psi')' - (\psi^{*}\psi'' + \psi\psi^{*}{}'')' + (\psi^{*}{}'\psi'' + \psi'\psi^{*}{}'')\right]$$

Then we have

$$\frac{\partial}{\partial t} \int J(z, t)\, dz = \int \frac{\partial J}{\partial t}\, dz$$

$$= \frac{h^2 e}{4m^2} \left[\psi^{*\prime}\, \psi^\prime - (\psi^* \psi^{\prime\prime} + \psi\, \psi^{*\prime\prime}) + \psi^{*\prime}\, \psi^\prime + \right.$$

$$\left. - \int \psi^{*\prime\prime}\, \psi^\prime\, dz + \int \psi^\prime\, \psi^{*\prime\prime}\, dz \right],$$

where we have integrated the term $\psi^{*\prime}\, \psi^{\prime\prime}$ by parts,

$$= \frac{h^2 e}{4m^2} \left[2\, \psi^{*\prime}\, \psi^\prime - (\psi^* \psi^{\prime\prime} + \psi\, \psi^{*\prime\prime}) \right].$$

The final result is

$$\frac{\partial}{\partial t} \int J(z, t)\, dz = \frac{h^2 e}{4m^2} \left(\psi\, \psi^{*\prime\prime} - 2\, \psi^\prime\, \psi^{*\prime} + \psi^*\, \psi^{\prime\prime} \right).$$

REFERENCES

1. A. Nordsieck, Proc. IRE, 41, 630, 1953.

2. J. E. Rowe, IRE Trans., Electron Devices, January 1956, pp. 39-56.

3. L. A. Wainstein, Trav. Congr. Int., "Tubes hyperfrequences," Paris 1956; Tome 1, 4.1.09.

4. L. Brillouin, Proc. Nat. Acad. Sci., 41, 401, 1955.

5. L. Brillouin et M. Parodi, Propagation des ondes dans les milieux périodiques, (Masson et Cie, Paris, 1956).

6. L. Brillouin, Wave propagation in periodic structures (Dover Pubs., 1953).

7. E. Jahnke and F. Emde, Tables of functions with formulae and curves (Dover Publications, 1945).

8. C. C. Cutler, B.S.T.J., 35, 841, 1956.