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ANALYSIS OF A NEW CLASS OF PULSE-FREQUENCY MODULATED FEEDBACK SYSTEMS*

by

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SUMMARY: A new class of Pulse Frequency Modulated Systems is presented in this paper. These systems referred to here as ΣPFM have many advantages over previously used schemes such as Integral PFM. Most significant advantages are improved stability and simpler physical implementation of the modulator.

The major part of this paper is concerned with the study of sustained oscillations using a specially developed quasi-describing function. One important feature of this kind of PFM systems is that they often present a limit annulus and not a limit cycle, a feature which is common in most nonlinear discrete feedback systems. Few examples with experimental verification are presented and the limitations of the method are discussed.

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I. INTRODUCTION

The application of pulse frequency modulation to communication systems and control problems is becoming an important factor in present day technology.[1-3] In this work, the analysis and stability study of a certain class of these systems will be attempted. For the most part the analysis is based on the quasi-describing function developed for such systems. To acquaint the reader with the basic operation of such systems the basic definitions and concepts used in this connection will be first introduced.

In pulse frequency modulation, the information carrier is the time between the emission of two pulses with identical shape and amplitude. In most practical cases the shape of the pulses is of minor significance and thus for their mathematical study, they are usually approximated by impulses. A common feature of all pulse frequency modulated schemes (or coding procedures) is that their input has to be observed for a finite time before the emission of a pulse is decided. The simplest way to implement this is to integrate the input and decide on the emission of a pulse by observing the value of the integral at certain times [δ-modulation] or when it reaches a certain level [Integral Pulse Frequency Modulation, IPFM] These two schemes have been studied and certain methods for their analysis have been developed.[4-5]

A more general scheme, and to the authors' knowledge reported for the first time in this paper, is to feed the signal to a higher-order low-pass filter (possibly nonlinear or time varying) and decide on the emission of a pulse when its output reaches a certain level.

This class of systems is referred to in this paper as Sigma (Σ) pulse frequency modulation (ΣPFM) in order to indicate the summing up properties of the encoder used.

The ΣPFM systems present many advantages over the IPFM systems. Most important among them is improved stability and ease of physical implementation (see Appendix I). Because PFM presents a high degree of noise immunity it can be used in control systems where certain parts are contaminated by strong noise. Further applications of PFM lie in the uses of adaptive control[3] and of attitude control of space crafts[2] because of convenience of implementation.

Finally, another motivation for the study of such systems lies in their potential promise in providing a model for a number of functions of the nervous systems of animals.[6] This is because neurons also use pulse frequency modulation for the transmission of information.

Although extensive work has been done on PAM (or Sampled-data Systems), comparatively little work was done on IPFM and practically none on ΣPFM, discussed in this paper. The mathematical difficulty lies in the fact that in PFM systems the pulses are emitted at intervals of time computed by the system (a nonlinear operation) and not fixed a priori. Hence, the application of the difference equation approach does not offer much insight except in a few limited cases.

The emphasis of this paper lies in presenting some general mathematical tools for the study of such systems. Though the emphasis is on ΣPFM systems, however the method developed is also applicable to the previously studied IPFM system. This will be shown in the few examples discussed in this paper.

II. DEFINITION AND DESCRIPTION OF A ΣPFM MODULATOR

An integral Pulse Frequency Modulator (IPFM) has been previously defined as a device which emits an impulse whenever the absolute value of the integral of its input reaches a level r and then it resets the integral to zero.[4] If x denotes the input, y the output and p the value of the integral, the following two equations describe its behavior:

$$\frac{dp}{dt} = x - r\,\mathrm{sgn}(p)\ \delta(|p| - r) \qquad (1)$$

$$y = \delta(|p| - r) \qquad (2)$$

where $sgn(p) = \pm 1$ depending on the sign of p and δ is a unit impulse. (There is no loss of generality in assuming unit area of the emitted impulses).

The case where the modulator emits pulses of only one sign can be studied by direct extension of the methods developed here, therefore it will not be considered in this paper.

The second term of the RHS of Eq. (1) represents the resetting of p to the zero value. Indeed if t_o is the time of emission of an impulse (i.e., $|p(t_o)| = r$) and if both sides of Eq. (1) are integrated from t_o^- to t_o^+ we obtain

$$p(t_o^+) - p(t_o^-) = - r sgn p(t_o). \qquad (3)$$

By definition $p(t_o^-) = r sgn p(t_o)$, therefore

$$p(t_o^+) = 0. \qquad (4)$$

A generalization of the above scheme of PFM is to add an extra term in the left hand side of Eq. (1). Then a modulator described by the following equations is obtained:

$$\frac{dp}{dt} + g(p) = x - r\, sgn(p)\, \delta(|p| - r) \qquad (5)$$

$$y = \delta(|p| - r). \qquad (6)$$

The present discussion will be limited to the case where $g(p)$ is a nondecreasing, continuous, odd function of p (note that these constraints imply $p\, g(p) \geq 0$). This kind of modulation will be called Sigma Pulse Frequency Modulation (ΣPFM). The IPFM is obviously a special case of this for $g(p) = 0$.

If $x = x_o$ = const., it is easy to compute the time t_o of the emission of an impulse which will be generally referred to as firing time. Indeed,

$$\int_0^r \frac{dp}{|x_o| - g(p)} = t_o . \qquad (7)$$

Eq. (7) will have no solution, i.e., the modulator will never fire if

$$|x_o| < g(r) . \qquad (8)$$

For this reason $g(r)$ will be called the input threshold of the system and will be denoted by R.

When $g(p) = c\,p$ (linear) we have

$$t_o = \frac{1}{c} \ln \frac{|x_o|}{|x_o| - cr} \qquad (9)$$

and the input threshold is $R = cr$.

Systems with linear $g(p)$ have been previously defined as Neural NPFM because they were used in models of neural nets[6, 7]. Although the choice of this name is not a very successful one it is kept for the time being.

III. FREQUENCY RESPONSE OF A ΣPF MODULATOR

If the input to the modulator is a pulse of duration τ and amplitude x_o then obviously no firing will occur if

$$\tau < t_o . \qquad (10)$$

The equation

$$\tau = \int_0^r \frac{dp}{|x_o| - g(p)} \tag{11}$$

represents a curve in the (x_o, τ) plane which separates the points which cause a response from those which do not. If the input is a square wave with amplitude x_o and frequency $\omega = 2\pi/\tau$ (where $\tau/2$ is the duration of a half wave), Eq. (11) can be rewritten as

$$\omega = \frac{2\pi}{\tau} = \frac{\pi}{\int_0^r \frac{dp}{|x_o| - g(p)}} \tag{12}$$

The curve represented by Eq. (12) will be called the <u>(square wave) frequency cut off curve</u> of the modulator. One can easily see from Eq. (12) that the larger is "x_o" the larger is "ω". Hence the band-width of the modulator depends on the amplitude of the input. If an input of different wave shape is used then the computation becomes extremely involved but the results will be essentially the same.

For $g(p) = cp$ one can substitute from Eq. (9) and then Eq. (12) yields

$$\omega = \frac{\pi c}{\ln \frac{|x_o|}{|x_o| - c\,r}} \tag{13}$$

By defining normalized variables

$$\theta = \frac{\omega}{\pi c} \tag{14}$$

and

$$x = \frac{|x_o|}{c\,r} \; . \tag{15}$$

Eq. (13) is written

$$\theta = \frac{1}{\ln \frac{x}{x-1}} \qquad \text{for } x > 1 \; . \tag{16}$$

Obviously $\theta = 0$ for $x \leq 1$ (subthreshold input). A plot of Eq. (16) is shown in Fig. 1. It is easy to verify that the equation of the asymptote of the curve represented by Eq. (16) is

$$\theta = x - \frac{1}{2} \; . \tag{17}$$

IV. FEEDBACK SYSTEMS USING ΣPFM

Consider a feedback system with input u and output z. Then a ΣPF Modulator can be used as controller if its input x is the error, i.e. $(u-z)$. Such a case is shown in Fig. 2. Due to the constant amplitude of the pulses a memoryless nonlinearity following the modulator will not add any complexity to the system.

One can write in general the state equations of this system as

$$\Sigma \left\{ \begin{array}{ll} \dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{a}\,\delta(|p| - r) & (18) \\ \dot{p} = -g(p) + u - z - r \operatorname{sgn} \delta(|p| - r) & (19) \\ z = \underline{e}'\,\underline{x} & (20) \end{array} \right.$$

where the first equation represents the controlled plant, the second one the modulator and the last one the output. The quantities, $\underline{x}$, $\underline{a}$ and $\underline{e}$ are n-dim. column vectors and $\underline{A}$ a n x n matrix.

By using Laplace transform and noticing that the output of the modulator will have a transform

$$Y(s) = \mathcal{L}[Y(t)] = \mathcal{L}[\delta(|p(t_o)| - r)]$$

one can combine the first and third equation into

$$Z(s) = KF(s) \cdot Y(s). \tag{21}$$

It is very simple to check that the poles of F(s) are the same as the eigenvalues of $\underline{A}$ if no cancellations had occurred. This will be always true for simple plants.

Because of the peculiar behavior of ΣPFM systems we have first to define the concept of equilibrium. In the case of IPFM systems the position of equilibrium has been previously defined as the one where pulses are emitted in fixed pattern. However, it may be entirely undesirable from a technological point of view to have the modulator firing all the time. Hence we define as equilibrium the condition where no firing occurs and moreover $\dot{\underline{x}} = 0$, $\dot{p} = 0$.

In terms of Eq. (21) the following reasoning holds. If we have a constant input u_o we want z to have also a constant value after the emission of a finite number of pulses. In other words we want the limit

$$z_o = \lim_{s \to 0} s\, Z(s) = \lim_{s \to 0} Ks\, F(s)\, Y(s). \tag{22}$$

to exist and be a constant. This will obviously be satisfied if F(s) has one pole at the origin and all its other poles have negative real parts.

We therefore have to check also that

$$|u_o - z_o| < g(r) \tag{23}$$

otherwise the emission of pulses will not stop as one can see from Eq. (19). Let f(t) be the impulse response of the linear plant (i.e.,

$f(t) = \mathcal{L}^{-1}[F(s)s]$). and let $f_o = \lim_{t\to\infty} f(t)$ (it will always exist under the above mentioned conditions). Then if the number of positive pulses emitted is n_p and the number of negative pulses is n_n we define the <u>net number</u> of emitted pulses as

$$n = n_p - n_n \ . \tag{24}$$

Then we will have

$$z_o = nKf_o \ . \tag{25}$$

In general we will have

$$u_o = (n+v)\,Kf_o \tag{26}$$

where $-1/2 \leqslant v \leqslant 1/2$. Then inequality (23) can be written as

$$|vKf_o| < g(r) \tag{27}$$

or

$$2g(r) \geqslant Kf_o \tag{28}$$

In this way the following theorem has been proved.*

<u>Theorem</u>: A ΣPFM unity feedback system with a linear plant having transfer function F(s) and a gain element K (as in Fig. 2) has an equilibrium position defined by no pulse firing, constant output and constant value of p(t) if and only if the limit for $s \to 0$ of $s\,F(s)$ exists, it is different than zero and satisfies the inequality

$$2g(r) \geqslant K \lim_{s\to 0} s\ F(s) \ . \tag{28a}$$

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* An alternative expression and derivation of the theorem in terms of the state equations only can be found elsewhere.[7]

Note that in an IPFM system condition (28) can never be satisfied, hence, firing will never stop. This indicates one of the main advantages of ΣPFM over IPFM. Figs. 3 and 4 give the comparison of the step responses of two such systems.

V. SUSTAINED OSCILLATIONS IN ΣPFM FEEDBACK SYSTEMS

In reference to Fig. 2, one would expect steady-state oscillations for this nonlinear system. It is observed that such systems exhibit what is called a "limit annulus". A typical form of these oscillations for a second-order system is shown in Fig. 5. It is of interest to note that such kind of oscillations are also common in IPFM and in certain types of nonlinear discrete systems, in particular in relay sampled-data systems[8, 9]. This peculiar behavior of oscillations is due to the quantized nature of PFM systems. When the number of pulses per half-period required for sustained oscillations (periodic) is not an integer then the system moves in a range of various values of pulses per half period. It may, however, happen (although rarely) that solutions with integer number of pulses exist and in the case that the system exhibits a true limit cycle. For practical applications the determination of the outer and inner boundaries of the "limit annulus" shown in Fig. 5, is essential. The quasi-describing function method to be presented in the next section will determine these boundaries.

We may note in passing that this kind of oscillation indicates "pseudo-random" behavior because no regular oscillatory pattern can be found by inspection of the system's output. Thus a certain relationship could be possibly found between a random process and the output of a ΣPFM System, as it has been already for nonlinear sampled data systems.[8] This equivalence would be an interesting topic for a future research problem.

VI. DEFINITION AND DERIVATION OF THE QUASI-DESCRIBING FUNCTION FOR ΣPFM SYSTEMS

The usual definition of the describing function of a nonlinearity is the ratio of the complex amplitude of the fundamental Fourier harmonic of the output to the amplitude of a sinusoidal input. The Quasi-describing function will be defined similarly by taking the ratio to the fundamental component of any periodic input.* In the derivation of the quasi-describing function of ΣPFM systems, it will be assumed that the input to the modulator is a square wave. The reason for the latter choice is the fact that only for such a wave form one can determine the output of the modulator in closed form.

However, because the modulator acts as a low pass filter (Fig. 1) the error introduced by neglecting all the higher harmonics of the input will not be too large. In a later section, we will discuss the limitations of the obtained results because of such a choice of the quasi-describing function and its effect in determining the system stability and response behaviors.

To derive the quasi-describing function of a ΣPFM Modulator we will apply to the input modulator a square wave of amplitude S_o and period T. The output of the modulator will exhibit the same number of n-pulses for both half periods, as one can verify by an elementary computation.[7]

Let t_k be the firing times in one period $(k = 1, 2, \ldots, 2n)$. Then the output of the modulator $y(t)$ will be given by

$$y(t) = \sum_{k=1}^{n} \delta(t - t_k) - \sum_{k=n+1}^{2n} \delta(t - t_k), \quad t_{2n} < T. \tag{29}$$

*This will be a complex number if there is a phase difference between the two.

The Fourier series expansion of $y(t)$ can be found in the distribution sense as

$$y(t) = \sum_{m=-\infty}^{+\infty} B_m e^{jm\omega t}. \tag{30}$$

The computation of the coefficients B_m is given in Appendix II. if we define

$$\tau = t_{n+1} - t_o - \frac{T}{2} \tag{31}$$

where t_o is the firing time of the modulator for S_o computed from Eq. (7) and

$$\tau' = \frac{T}{2} - t_{n+1}$$

then the following result is obtained for $n = \pm 1$.

$$B_1 = \frac{1}{T}(1 + e^{-j\omega\tau})(1 + e^{j\omega\tau'}) \frac{1}{e^{j\omega t_o} - 1}. \tag{32}$$

$$B_{-1} = \frac{1}{T}(1 + e^{j\omega\tau})(1 + e^{-j\omega\tau'}) \frac{1}{e^{j\omega t_o} - 1} \tag{33}$$

The significance of τ and τ' is also shown in Fig. 6. In an IPFM system it is always $\tau = \tau'$ and for any ΣPFM system $\tau = 0$ if and only if $\tau' = 0$ and moreover one is a monotonic function of the other. Therefore one can make the simplifying approximation, $\tau = \tau'$. Then the fundamental component $(B_1 e^{j\omega\tau} + B_{-1}e^{-j\omega\tau})$ of the output will be

$$\frac{2}{T} \frac{1 + \cos \omega\tau}{\sin\left(\frac{\omega t_o}{2}\right)} \sin(\omega t - \frac{\omega_o}{2}). \tag{34}$$

The fundamental component of the input can easily be found as:

$$\frac{4 S_o}{\pi} \sin \omega t. \tag{35}$$

Then the quasi-describing function $Q(S_o, \omega, \tau)$ of a ΣPF Modulator with firing time t_o will be given by:

$$Q(S_o, \omega, \tau) = \frac{\omega}{4S_o} \frac{1+\cos\omega\tau}{\sin(\omega t_o/2)} e^{-j(\omega t_o/2)} . \quad (36)$$

The above relation is valid only for

$$\omega t_o < \pi \quad (37)$$

(see Eq. (12) in Sec. III). Otherwise $Q(S_o, \omega, \tau) = 0$. For an IPFM system Eq. (7) gives $t_o = r/S_o$. Substituting into (32) and defining an auxiliary variable ξ one obtains:

$$\text{IPFM;} \quad Q(\xi, \tau) = \frac{1}{r} \frac{1+\cos\omega t}{2} \frac{\xi}{\sin\xi} e^{-j\xi} \quad \text{if} \quad \xi < \frac{\pi}{2}$$

$$Q(\xi, \tau) = 0 \qquad \text{otherwise} \qquad . \quad (38)$$

$$\xi = \frac{\omega r}{2S_0}$$

This polar plot of $-1/Q(\xi, \tau)$ is shown in Fig. 7*.

For a NPFM system (i.e., $g(p) = cp$) substitute t_o from Eq. (9) and use the same normalized variables as in Sec. III (Eqs. (14) and (15)) to obtain.

$$\text{NPFM:} \quad Q(x, \theta, \tau) = \frac{1}{r} \frac{1+\cos\omega\tau}{2} \frac{\pi\theta}{2x} \cdot \frac{1}{\sin\left(\frac{\pi}{2}\theta \ln\frac{x}{x-1}\right)} \exp\left[-j\left(\frac{\pi}{2}\right)\theta \ln\frac{x}{x-1}\right]$$

$$\text{if} \quad \theta \ln\frac{x}{x-1} < 1 \qquad . \quad (39)$$

$Q(x, \theta, \tau) = 0$ otherwise.

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* In both cases the factor $1/r\,(1+\cos\omega\tau/2)$ has been left out to be included with the Nyquist locus of the linear plant.

The polar plot of $-1/Q(x, \theta, \tau)$ is shown in Fig. 8.*

By a simple computation and under the assumption of square-wave oscillations one can prove that the angle at which the above plots are intersected by the Nyquist plot of the linear plant can give the number of pulses per half period.[7] This is indicated with the sectors of Figs. 7 and 8.

VII. LIMITATIONS ON THE APPLICATION OF THE QUASI-DESCRIBING FUNCTION

In view of the assumptions introduced in the derivation of the quasi-describing function, one would expect that in certain cases the method might yield false conclusions. These limitations appear under two conditions.

1. One can notice from Fig. 8 that wherever the linear plant contains an integrator, the describing function method yields that sustained oscillations exists. However, we know that when the gain is very small, such oscillations cannot exist. Hence, the conclusion from the describing function is false. A simple analysis shows that the following test can be used for remedy.[7]

"If the amplitude S and frequency ω of the oscillations indicated by the Q. D. function are "small", we compute the quantity $1/2\ g(\pi/\omega)$. If S is less than this quantity no oscillations can exist."

The failure of the Q. D. function in this case is due to the quantized nature of the system.

2. When the wave shape of the oscillations differs much from a square wave, the Q. D. function may yield erroneous results. A correction may be made by estimating the waveform from the number of pulses per half period. A check should be made to verify that the amplitude indicated by the Q. D. function is close to the mean value of the waveforms obtained by the impulse response. This check is necessary wherever the

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* See note page 13.

frequency of oscillations is high in comparison to the characteristic frequencies of the system.

VIII. EXAMPLES OF APPLICATION OF THE Q. D. FUNCTION

According to the well-known technic the points of oscillations will be determined from the intersection of the negative inverse of the Q. D. Function and the Nyquist plot of the linear plant. In this work the factor $(1+\cos\omega\tau/2)$ will be included also with the Nyquist plot as it was indicated in the footnote of Sec. VI. Also because of the use of normalized variables for the Q. D. in NPFM similar variables should be also used for the Nyquist plot.

In the application of this method it is not necessary to check for all values of τ. Usually a check for $\tau=0$ and $\omega r=\pi/2$ will be enough because it gives the boundaries shown in Fig. 5.

The radius of the Nyquist plot for gain K and $\tau = 0$ will be denoted by S_1^K, and by S_0^K the same quantity for $\omega\tau = \pi/2$. Note also the identity

$$S_0^K = S_1^{K/2} . \tag{40}$$

1st Example: IPFM with plant $20/s(s+1)$: In Fig. 7 the points of intersection are shown. From them the values of ω and ξ are determined and therefore the amplitude S_0. The results of the analog computer simulation are shown in Fig. 9, and a comparison of the predicted and observed values at the Table I.

2nd Example: NPFM with plant $K/s(s+1)$: For $K = 20$ both boundaries exist as it is shown in Fig. 8 (S_1^{20} and S_0^{20}). For $K = 10$ only one point is found because the intersection of S_0^{10} fails to satisfy the criterion proposed in Sec. VII. Indeed the result is $\theta = 0.2$ and x less than 1.03, i.e., $T = 20$ and $S_0 < 2.6$. But $1/2g(T/2) = 1/2 \; 10(1 - e^{-10})$ $= 5 > S_0$. In this case one expects that the system may present a true limit cycle. This is verified by the analog computer simulation (Figs. 10 and 11) is shown.

TABLE I (1st Example)

		Q. D. Function	Analog Computer
outer boundary	V_1	10	10
	T_1	3.3	3.6
	n_1	2-3	2
inner boundary	V_2	5.1	6
	T_2	4.8	4.6
	n_2	1-2	1

TABLE II (2nd Example)

	Q-D Function			Analog Computer		
	θ/T	x/V	Number of pulses	T	V	Number of pulses
S_1^{20}	1.15 / 3.5	3.5 / 8.8	2	2.0	13	2
S_0^{20}	0.74 / 5.4	2.2 / 5.5	1	3.6	4	1
S_1^{10}	0.74 / 5.4	2.2 / 5.5	1	3.5	5	1
S_0^{10}	-	-	0	-	-	0

CONCLUSION

In this paper, the basic operation and use of ΣPFM feedback systems has been introduced and studied. The main emphasis of this work is based on stability and sustained oscillations of such systems. The concept of limit annulus is discussed in detail and examined. This phenomenon is not only characteristic of ΣPFM system but also appears in most nonlinear discrete systems.[8, 9] Hence, this study will shed a light on the general periodic and aperiodic behavior of nonlinear discrete systems.

Few examples with their experimental verification have been introduced which clarify justification of the use and importance of the quasi-describing function. The limitation of this method has been critically examined and some of the modifications required for obtaining the correct results have been indicated. While the stability problem has been examined using the quasi-describing function, the application of Lyapunov stability method is also possible. The detailed study of such a method which represents an alternate approach has been achieved and will be presented in a forthcoming work. Other problems connected with statistical and optimal control of such ΣPFM systems are of significance and definitely warrant future investigation.

APPENDIX I

Hardware Implementation of a ΣPF Modulator

There are several ways of implementing a ΣPF Modulator, one of them is shown in Fig. 12. The break voltage of the gas tube V_g plays the role of r. The relays A and B are used to provide pulses of the regular rectangular wave shape. More examples can be found elsewhere.[7]

APPENDIX II

Computation of Fourier Series Coefficients of the Output of a ΣPF Modulator

Starting from Eqs. (29) and (30) we have:

$$B_m = \frac{1}{T}\int_0^T y(t)\, e^{-jm\omega t}\, dt = \frac{1}{T}\left[\sum_{k=1}^{n} e^{-jm\omega t_k} - \sum_{k=n+1}^{2n} e^{-jm\omega t_k}\right] \quad (41)$$

where $\omega = 2\pi/T$. Furthermore we note from Fig. 6 that

$$t_k = kt_0 \quad \text{for } k = 1, 2, \ldots n$$

and

$$t_k = \frac{T}{2} + \tau + (k - n)\, t_0 \quad \text{for } k = n + 1, \ldots 2n$$

where τ has been defined in Section VI. Then Eq. (41) yields

$$B_m = \frac{1}{T}\left[1 - (-1)^m e^{-jm\omega\tau}\right] \sum_{k=1}^{n} e^{-jm\omega k t_0} . \quad (42)$$

The finite sum in the above equation is a geometric progression, hence we have

$$B_m = \frac{1}{T} \left[1 - (-1)^m \; e^{-jm\omega\tau}\right] \; e^{-jm\omega t_0} \; \frac{e^{-jm\omega n t_0} - 1}{e^{-jm\omega t_0}} . \qquad (43)$$

By including τ' as defined in Section VI we obtain:

$$B_m = \frac{1}{T} \left[1 - (-1)^m \; e^{-jm\omega\tau}\right] \left[1 - (-1)^m \; e^{jm\omega\tau'}\right] \frac{1}{e^{jm\omega t_0} - 1} . \qquad (44)$$

Substituting $m = \pm 1$ in the above equation we obtain eht relations (32) and (33). Using the simplifying assumption $\tau = \tau'$ then Eqs. (32) and (33) become:

$$B_1 = \frac{2}{T} \; (1 + \cos \omega\tau) \; \frac{1}{e^{j\omega t_0} - 1} \qquad (45)$$

$$B_{-1} = \frac{2}{T} \; (1 + \cos \omega \; \tau) \frac{1}{e^{-j\omega t_0} - 1} \quad . \qquad (46)$$

Hence the fundamental component $B_1 \; e^{j\omega t} + B_{-1} e^{-j\omega t}$ will be given by:

$$\frac{2}{T} \; (1 + \cos \omega\tau) \left[\frac{e^{j\omega t}}{e^{j\omega t_0} - 1} + \frac{e^{-j\omega t}}{e^{-j\omega t_0} - 1} \right]$$

$$= \frac{2}{T} (1 + \cos \omega\tau) \frac{\cos \omega(t - t_0) - \cos \omega t}{1 - \cos \omega t_0}$$

$$= \frac{2}{T} (1 + \cos \omega\tau) \frac{\sin \left(\omega t - \frac{\omega t0}{2}\right)}{\sin \frac{\omega t0}{2}} \qquad (47)$$

This result has been used in Eq. (34).

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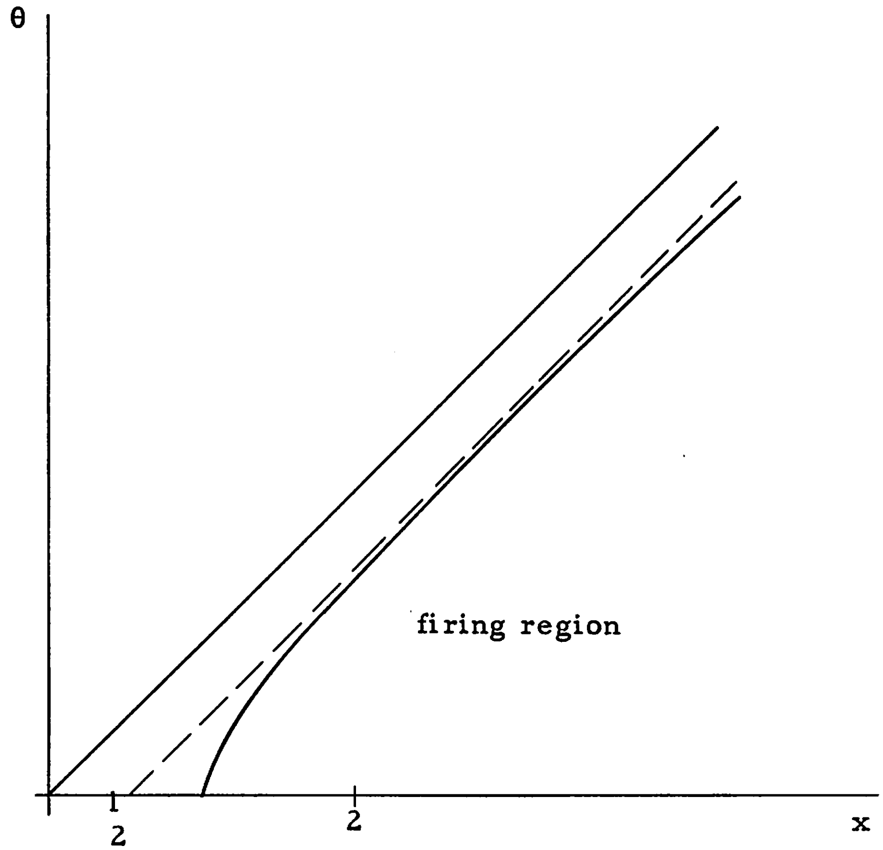

Fig. 1. Frequency cut-off curve.

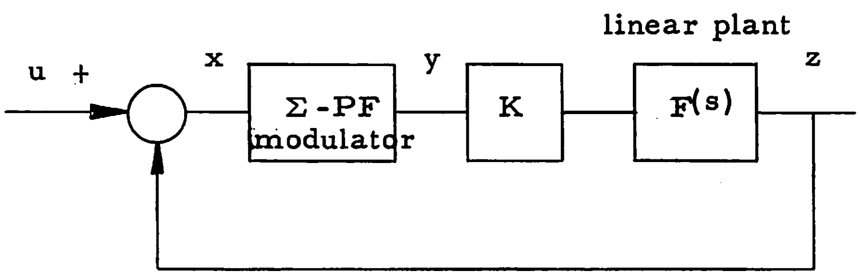

Fig. 2. A ΣPFM feedback system.

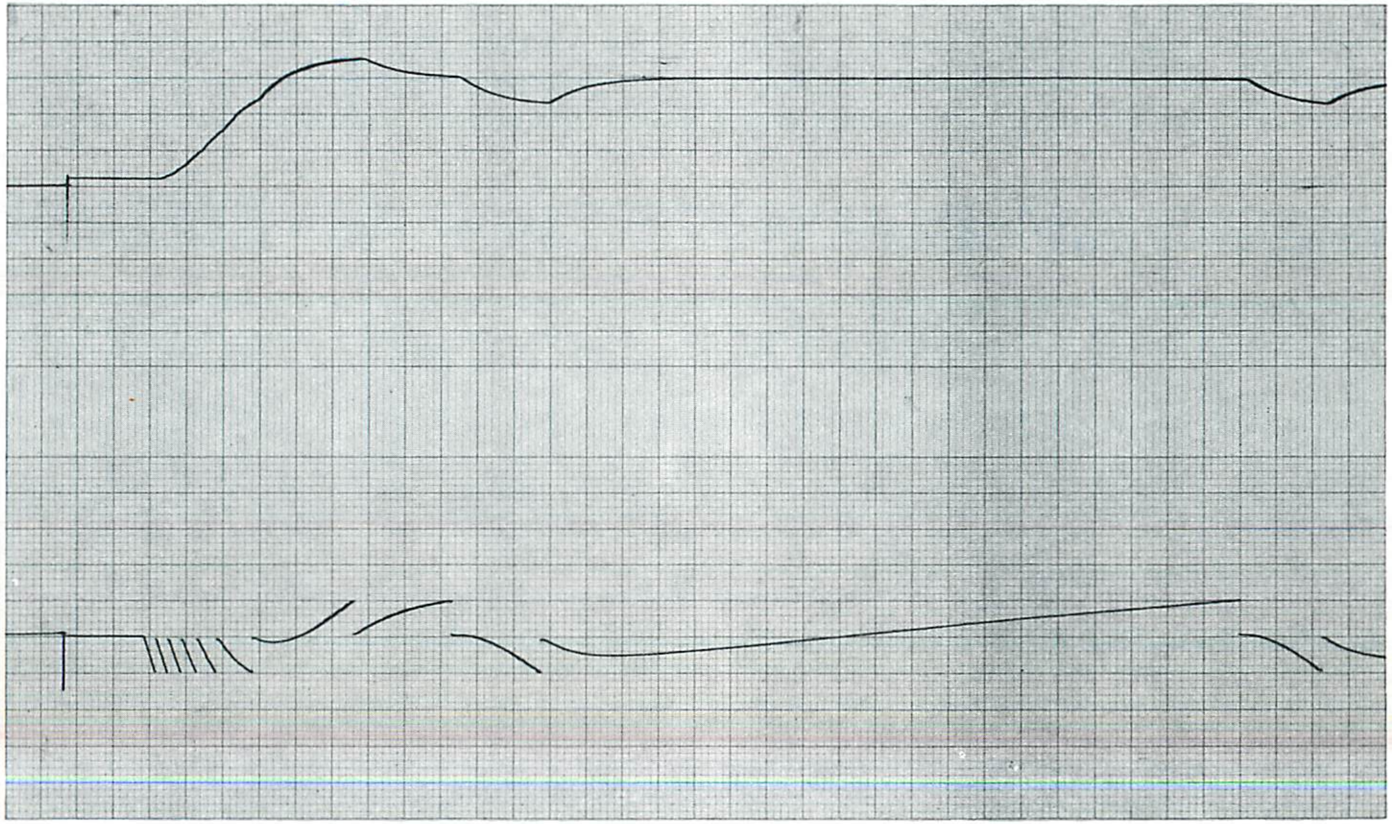

Fig. 3 Step response of IPFM System (upper trace output, lower trace p(t)).

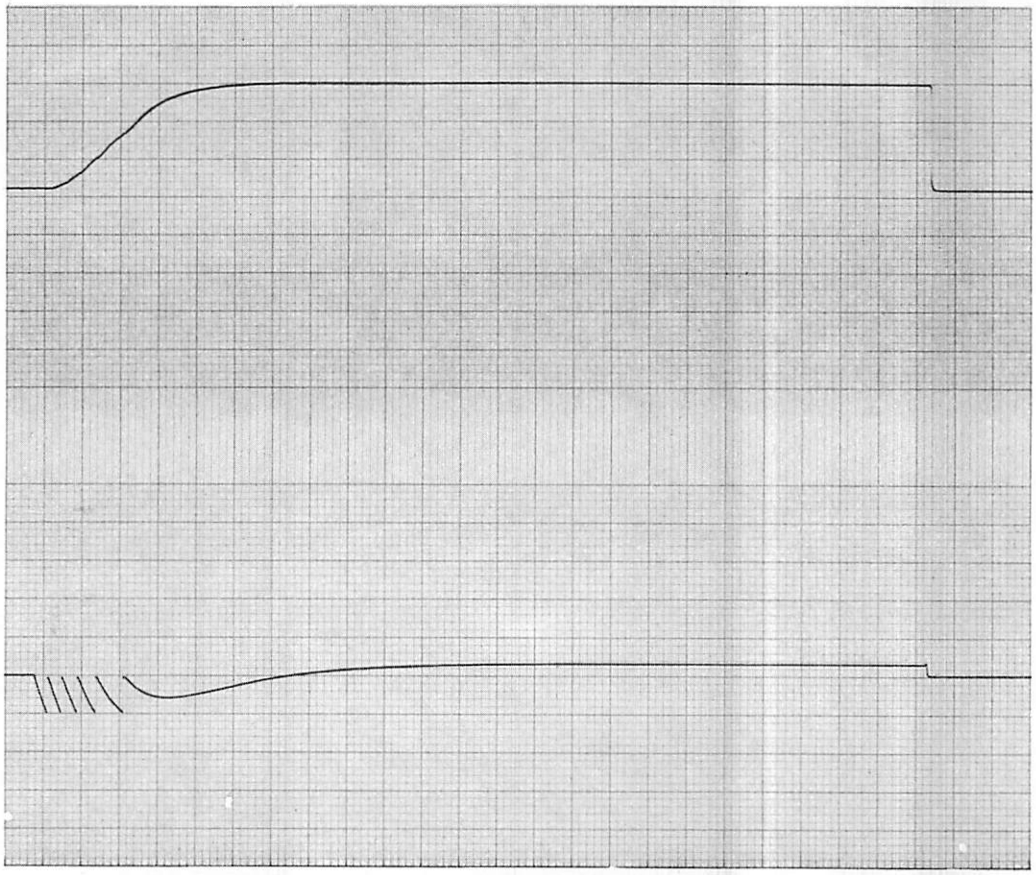

Fig. 4 Step response of NPFM System (traces as in Fig. 3).

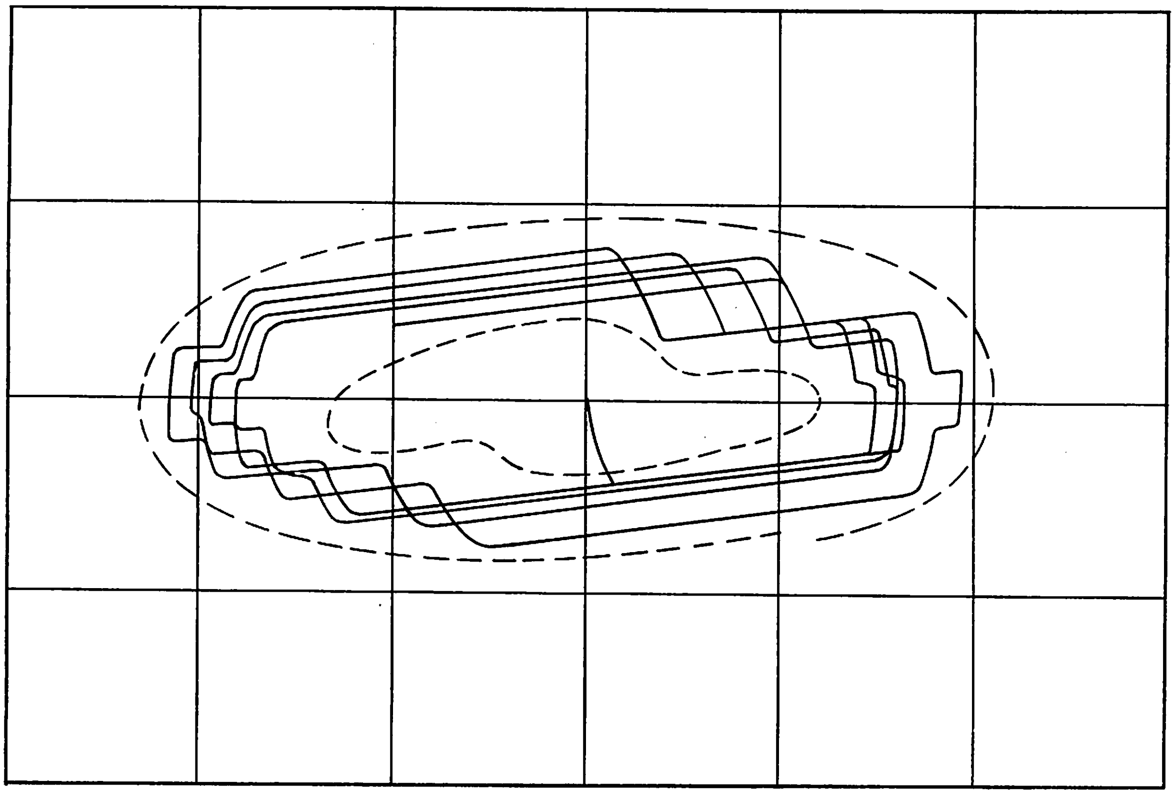

Fig. 5. Analog computer recordings of trajectories of a ΣPFM system under conditions of sustained oscillations.

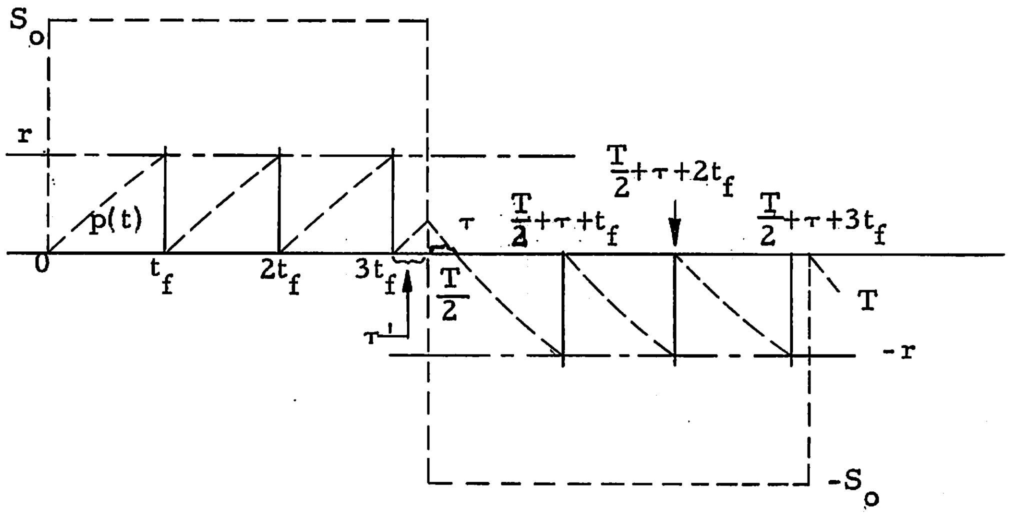

Fig. 6. Σ-PF Modulator response to a square wave input.

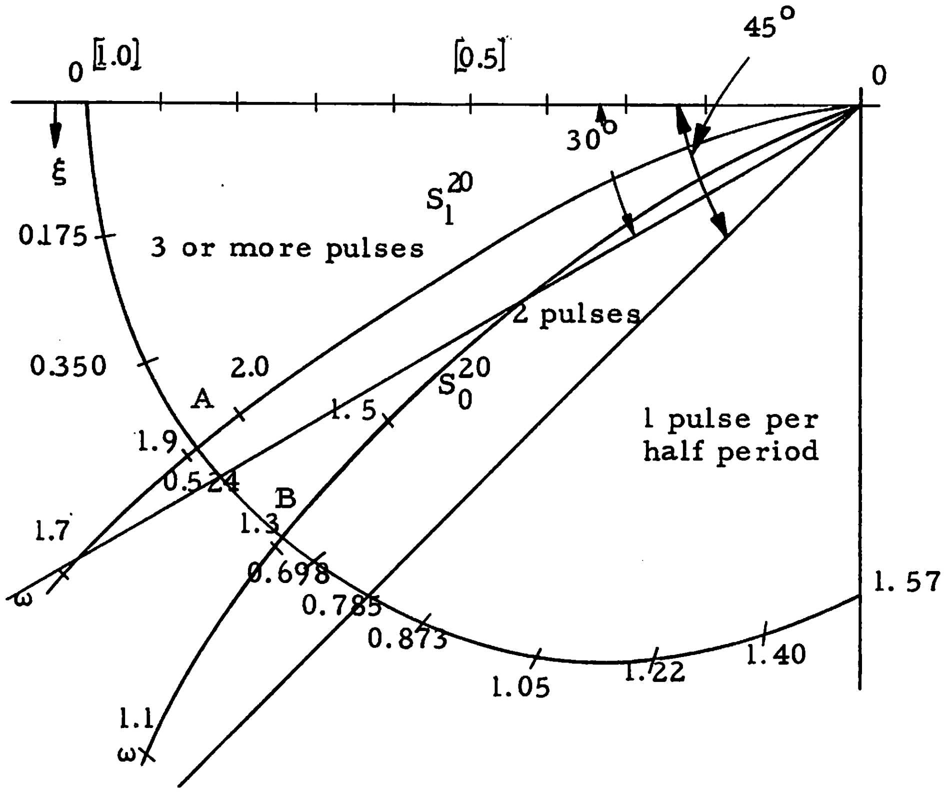

Fig. 7 Determining the characteristics of steady-state oscillations in an IPFM feedback system with linear plant transfer function 20/s(s+1). Points of oscillation A and B.

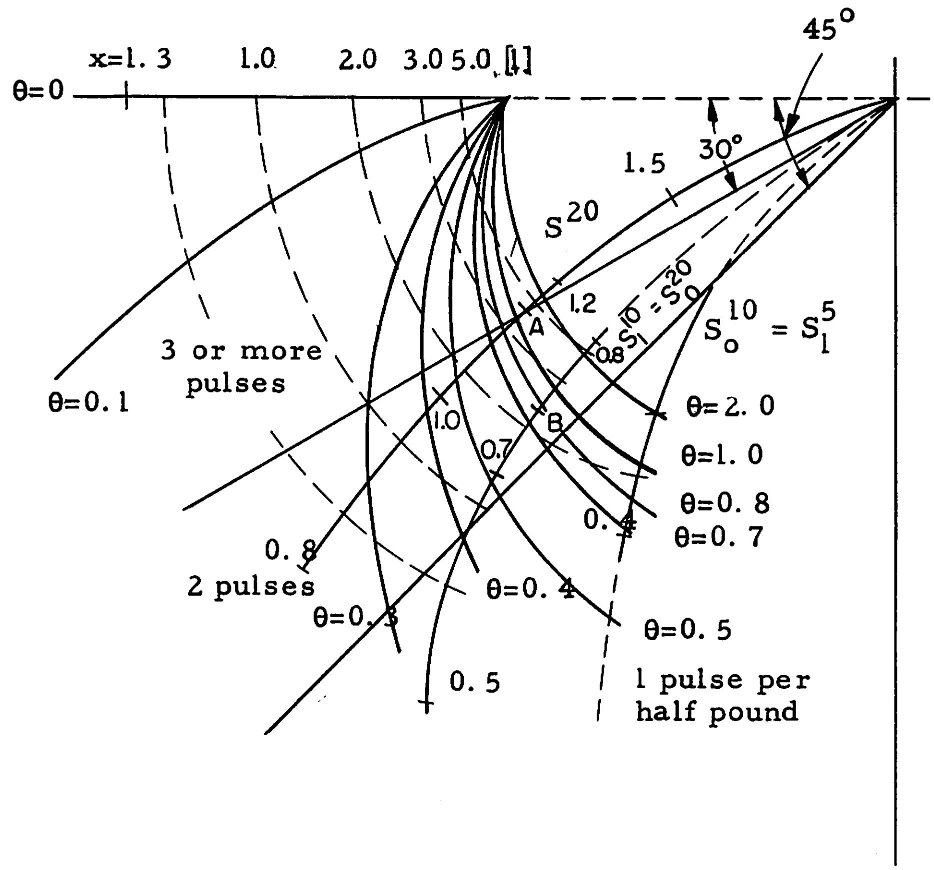

Fig. 8. Determining the characteristics of steady state oscillations in a NPFM feedback system with linear plant transfer function K/s(s+1). Points of oscillation A and B.

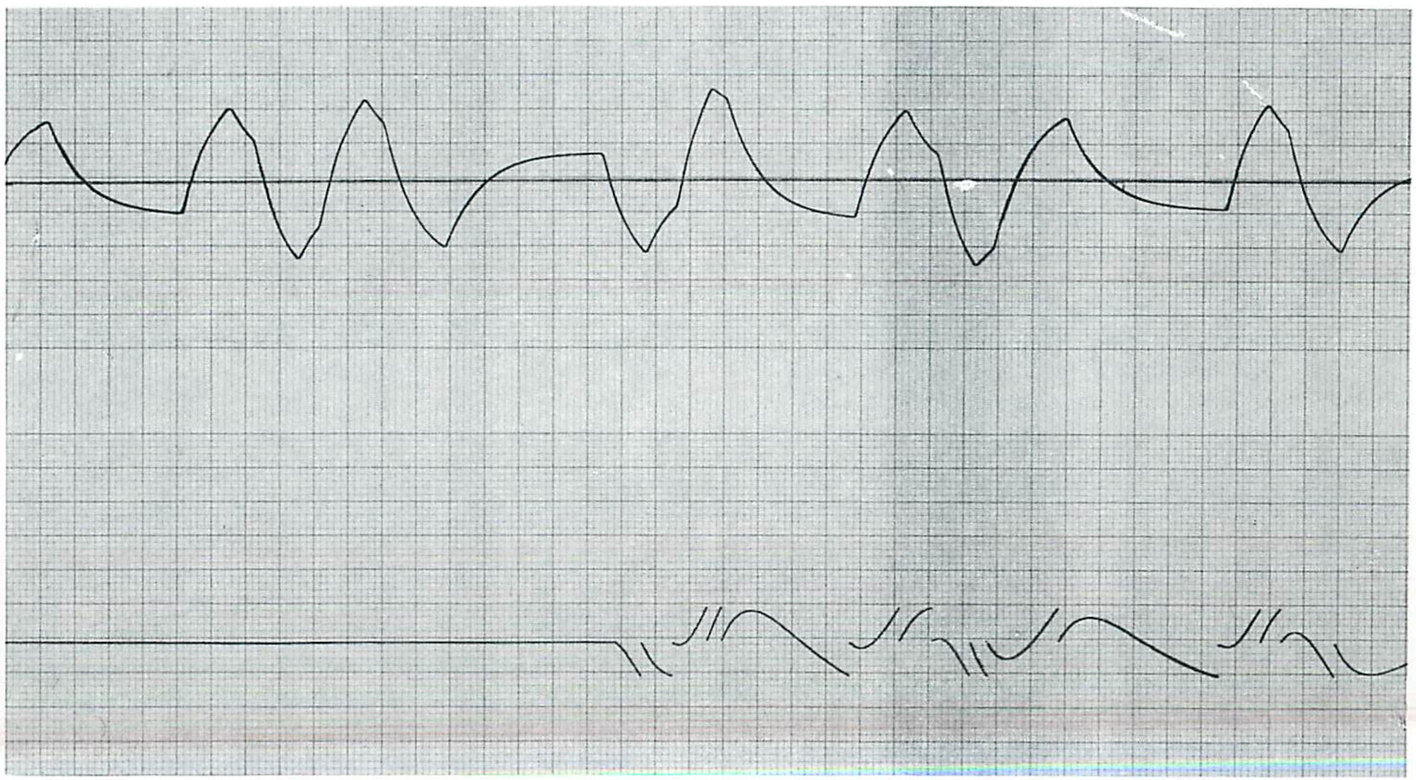

Fig. 9 Sustained oscillations in NPFM System with linear plant $20/s(s+1)$ (upper trace output, lower trace $p(t)$).

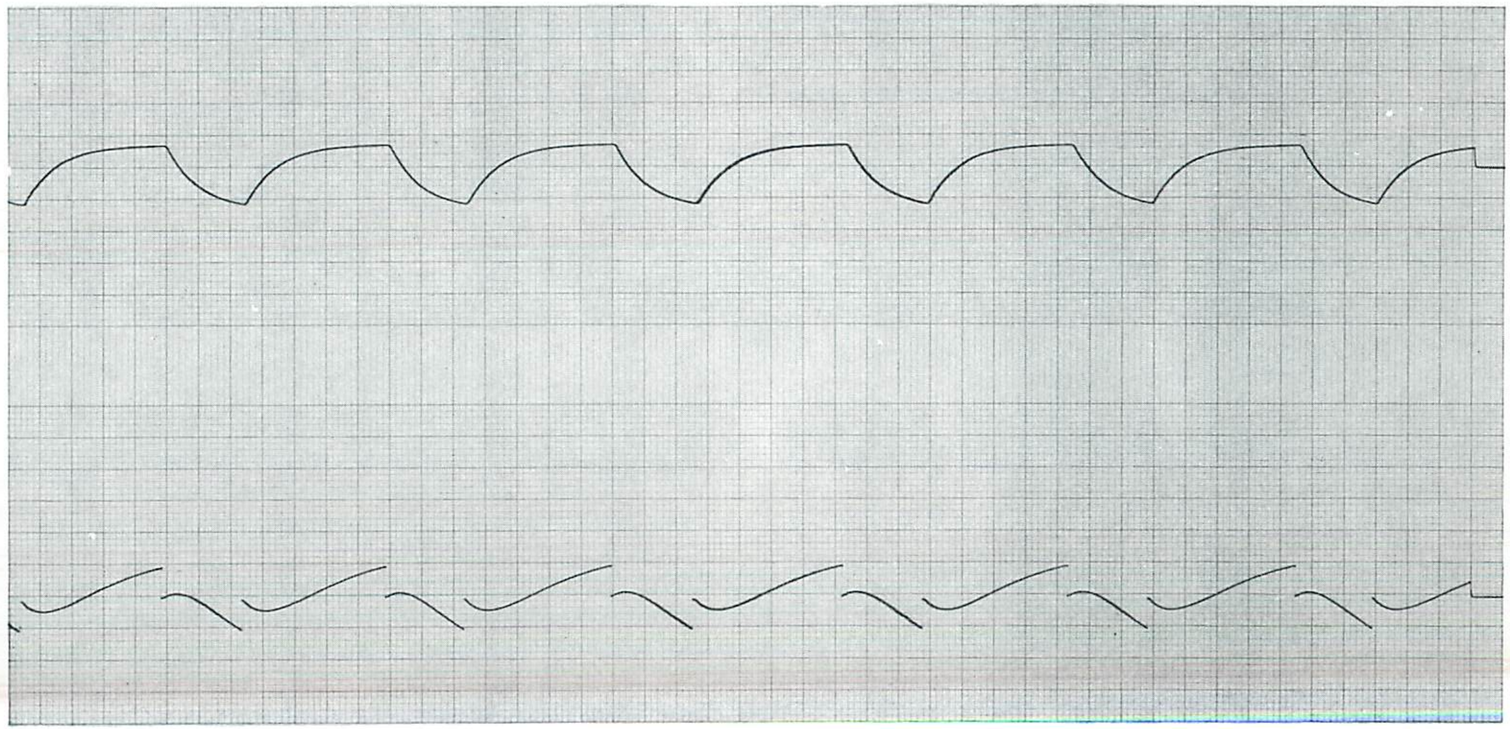

Fig. 10 As Fig. 9 with linear plant $10/s(s+1)$.

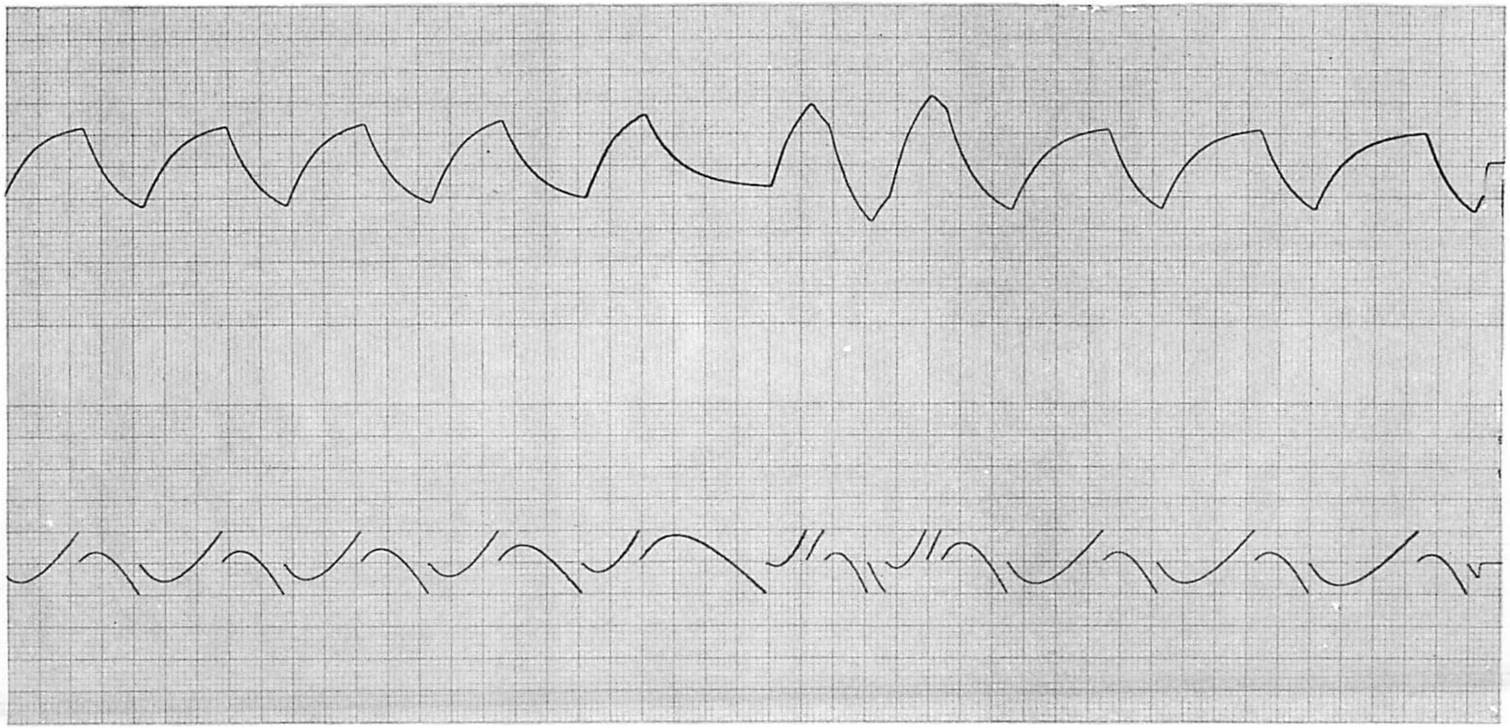

Fig. 11 Sustained oscillations in IPFM System with linear plant $20/s(s+1)$ (upper trace output, lower trace $p(t)$).

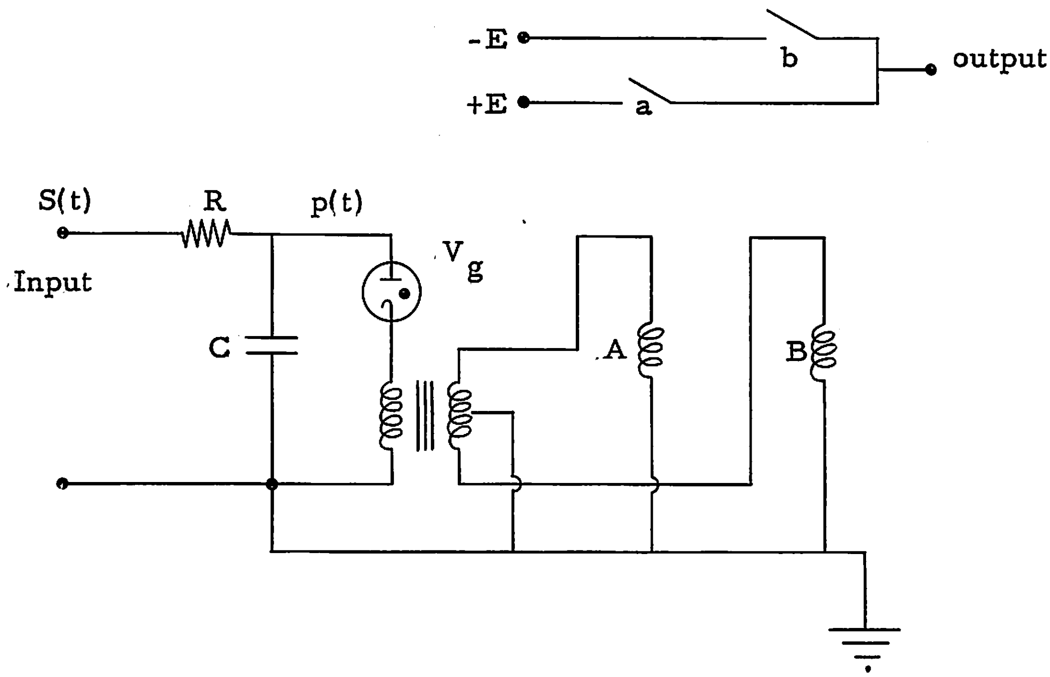

Fig. 12. Implementation of a Σ-pulse frequency modulator.