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PROPAGATION AND INSTABILITIES IN BOUNDED FINITE TEMPERATURE PLASMAS

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The quasi-static analysis is used to examine propagation and instabilities in a plasma in which both the transverse bounds and the temperature are important. A dispersion equation is derived by use of a dielectric tensor, which is correct within the quasi-static assumption, and the assumption of longitudinal velocity, only. It is found that the dispersion curves have a smooth transition from the zero temperature case at long wavelengths to the unbounded finite temperature case at short wavelengths. The stop-bands which appear in the zero temperature analysis become propagating regions in the more general case. Landau-type damping is calculated and it is found that the cyclotron wave is strongly damped. The effect of the boundaries and the magnetic field on a double humped velocity distribution is examined. The transverse boundaries are stabilizing with respect to space-charge waves, but the cyclotron interactions are strengthened. The temperature always exerts a stabilizing influence which is particularly marked for the interaction between the cyclotron waves. Curves of the limiting regions of stability, with respect to various parameters, are given, illustrating these effects. By the use of an example it is demonstrated that a moderate temperature can suppress the cyclotron instability. This result is in agreement with the experimental observation that a plasma, predicted to be unstable with respect to the cyclotron-wave interaction, was actually stable.

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I. INTRODUCTION

The problem of the propagation of waves on a cold plasma column within a waveguide has been analyzed utilizing both the complete set of Maxwell's equations[1, 2] and the quasi-static approximation.[3, 4] In the former case a coupled set of equations is obtained leading to hybrid modes consisting of mixed transverse magnetic (TM) and transverse electric (TE) type solutions. In the limits of the dc magnetic field, B_o, going either to zero or to infinity, the equations decouple and the solutions are pure TE or TM. In the general case of finite B_o, there are four types of modes present, the perturbed TE and TM waveguide modes, modes associated primarily with space-charge forces, and modes arising primarily from magnetic forces associated with the cyclotron motion of the particles. We will be concerned with the "space-charge" and "cyclotron" modes and with propagation at frequencies below the cutoff of the waveguide modes. In this regime one mode always has a velocity less than the speed of light, while the other has both slow and fast wave regions. In the region of slow-wave propagation, both modes are predominately TM.[2] For slow, TM modes the quasi-static approximation, $\nabla \times E = 0$, may be used. With the introduction of this approximation the problem is considerably simplified, the solution for the space-charge and cyclotron modes being found in terms of a scalar potential. Using this approximation several workers have studied the interactions in beam-plasma systems with finite dimensions.[5-8] Harrison has solved the warm plasma problem assuming an infinite magnetic field. Briggs and Bers assume a finite magnetic field and have included thermal effects in their analysis using a resonance velocity distribution;

$f_o(u) = v_T/\pi(u^2 + v_T^2)$ where u is the velocity directed along the dc magnetic field. The present work will introduce temperature into the problem assuming a Maxwellian distribution.

Much work has been devoted to the study of longitudinal plasma oscillations[9] and cyclotron waves[10] in a warm plasma in an infinite media. An equivalent dielectric tensor has been derived for warm plasmas in a magnetic field,[11, 12] and utilizing this tensor the dispersion relation for plasma waves can be found. For the special case of propagation along the magnetic field the dispersion relations for the space-charge and cyclotron modes decouple. For the space-charge mode in one dimension limiting values for streaming instabilities have been found[13-15] as functions of the product $k_z D$ and the ratio $\bar{u}/v_T$ where k_z is the propagation constant, D is the Debye length, $\bar{u}$ the relative drift velocity between two streams (assumed along B_o) and $v_T = (kT/m)^{1/2}$ the thermal velocity. A similar treatment will be used in this work to find instability limits in a finite system. In addition to the interactions between the space-charge waves of the two streams the cyclotron cyclotron and space-charge cyclotron interactions are also considered. The latter interaction occurs in a plasma with transverse bounds because the cyclotron mode has a longitudinal component and the space-charge mode transverse components; the two modes therefore couple.

In Section II a dispersion relation will be derived, within the quasi-static approximation, which includes longitudinal temperature and transverse boundaries. In Section III propagation in a stationary plasma will be discussed and in Section IV the limits of instability for a two-stream system are determined.

II. DERIVATION OF THE DISPERSION RELATION

The configuration we will consider here is that of a circular waveguide with a finite magnetic field along the axis of the guide, (the z direction). The guide is either completely or partially filled with one or more uniform plasma columns positioned concentric to the guide axis. Collisions will be neglected. An $\exp\left[j(k_z z - \omega t)\right]$ variation will be assumed with k_z taken as real; ω, as determined from the dispersion relation will in general be complex and an $\exp(jn\theta)$ variation will be taken for the azimuthal dependence.

The basic equations are Maxwell's

$$\nabla \times \underline{E} = j\,\omega\,\underline{B} \tag{1}$$

$$\nabla \times \underline{H} = -\,j\,\omega\epsilon_o\,\underline{E} + \underline{J} \tag{2}$$

and the equation of motion

$$\frac{\partial \underline{v}_i}{\partial t} + \underline{v}_i \cdot \nabla \underline{v}_i = \frac{e_i}{m_i}\,\underline{E} + (\underline{v}_i \times \underline{B})\,\frac{e_i}{m_i} \tag{3}$$

where the subscript refers to species i. Assuming no drift motion and linearizing the equation of motion we may use the resulting expression to express $\underline{J}$ in terms of $\underline{E}$. The right-hand side (RHS) of (2) can thus be written in terms of a dielectric tensor,

$$\nabla \times \underline{H} = -j\omega\underset{\approx}{\epsilon}\underline{E} \tag{4}$$

taking the divergence of (4) we obtain,

$$\nabla \cdot \underset{\approx}{\epsilon}\,\underline{E} = 0\ . \tag{5}$$

Assuming the constitutive relation

$$\underline{D} = \underset{\approx}{\epsilon}\,\underline{E}\ , \tag{6}$$

$\underline{D}$ and $\underline{E}$ can be transformed to a moving coordinate system and provided $(\underline{u}_i \times \underline{B})_\perp << E_\perp$ ($\underline{u}_i$ being the drift velocity), an $\underset{\approx}{\epsilon}$ is obtained in the new coordinate system in which Eqs. (5) and (6) are satisfied simultaneously.[16] For TM type waves it can be shown that $(\underline{u}_i \times \underline{B})_\perp$ is of the order of $(v_{ph}/c)^2 E_\perp$ and hence for slow-wave propagation the inequality holds. The requirements of TM-type slow waves are identical to those necessary for the quasi-static approximation. The dielectric tensor for the drifting plasma in cylindrical coordinates is,

$$\underset{\approx}{\epsilon} = \epsilon_o \begin{bmatrix} \epsilon_{rr} & j\epsilon_{r\theta} & 0 \\ -j\epsilon_{r\theta} & \epsilon_{\theta\theta} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} \tag{7}$$

where,

$$\epsilon_{rr} = \epsilon_{\theta\theta} = 1 + \sum_i \frac{\omega_{pi}^2}{\omega_{ci}^2 - (\omega - k_z\overline{u}_i)^2}$$

$$\epsilon_{zz} = 1 - \sum_i \frac{\omega_{pi}^2}{(\omega - k_z\overline{u}_i)^2} \tag{8}$$

$$\epsilon_{r\theta} = \sum_i \frac{\omega_{ci}\,\omega_{pi}^2}{(\omega - k_z\overline{u}_i)(\omega_{ci}^2 - (\omega - k_z\overline{u}_i)^2)}$$

This tensor differs somewhat from that derived from the equation of motion including drift velocity but both lead to the same result when substituted in Eq. (5).[16]

Consider each species to have an unperturbed velocity distribution in the longitudinal direction. (In this analysis transverse temperature is not included. The transverse temperature would not be expected to have a large effect except in circumstances where $T_\perp \gg T_\parallel$ in which case the system is unstable.)[17] Each species is divided into an infinity of streams each with an infinitesimal density,

$$d\rho \propto \omega_{pi}^{2} f_i(u)\, du \tag{9}$$

such that the sum, obtained by integrating over the infinitesimal streams, gives the total density for each species. Taking such a summation within the dielectric tensor and integrating by parts leads to,[16]

$$\epsilon_{rr} = 1 + \sum_i \omega_{pi}^{2} \int_{-\infty}^{\infty} \left[\frac{1}{2\omega_{ci}(\omega + \omega_{ci} - k_z u)} - \frac{1}{2\omega_{ci}(\omega - \omega_{ci} - k_z u)} \right] f_i(u)\, du$$

$$\epsilon_{r\theta} = \sum_i \omega_{pi}^{2} \int_{-\infty}^{\infty} \left[\frac{1}{2\omega_{ci}(\omega + \omega_{ci} - k_z u)} + \frac{1}{2\omega_{ci}(\omega - \omega_{ci} - k_z u)} - \frac{1}{\omega_{ci}(\omega - k_z u)} \right] f_i(u)\, du$$

$$\epsilon_{zz} = 1 - \sum_i \frac{\omega_{pi}^{2}}{k_z} \int_{-\infty}^{\infty} \left[\frac{1}{\omega - k_z u} \right] \frac{\partial f_i(u)}{\partial u}\, du \tag{10}$$

As a result of having chosen an $\exp[j(k_z z - \omega t)]$ variation rather than using an initial value approach, the above integrals are singular for real ω. However, this difficulty is easily overcome by following Landau's prescription for an infinite medium.[18] For a transformation of the time ω is located in the upper half of the ω-plane. The resulting integrals are then analytically continued so as to be defined on the real ω-axis and in the lower half of the ω-plane. Introducing a Maxwellian distribution,

$$f(u) = \frac{1}{\sqrt{2\pi}\, v_T} \exp\left[- \frac{(u - \bar{u})^2}{2 v_T^2} \right] \tag{11}$$

the dielectric tensor may be expressed in terms of the plasma dispersion function tabulated by Fried and Conte,[19]

$$\epsilon_{rr} = 1 + \sum_i \frac{\omega_{pi}^2}{2\omega_{ci} v_{Ti}\sqrt{2}\, k_z} \left[Z(\zeta_{2i}) - Z(\zeta_{1i}) \right]$$

$$\epsilon_{r\theta} = \sum_i \frac{\omega_{pi}^2}{2\omega_{ci} v_{Ti}\sqrt{2}\, k_z} \left[2\, Z(\zeta_{3j}) - Z(\zeta_{2i}) - Z(\zeta_{1i}) \right]$$

$$\epsilon_{zz} = 1 + \sum_i \frac{\omega_{pi}^2}{v_{Ti}^2 k_z^2} \left[1 + \zeta_{3j}\, Z(\zeta_{3i}) \right] \tag{12}$$

where,

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(-u^2)}{u - \zeta}\, du \tag{13}$$

in the upper half of the ζ plane. We can also express $Z(\zeta)$ and its analytic continuation, valid over the entire ζ plane, as[19]

$$Z(\zeta) = 2j\, e^{-\zeta^2} \int_{-\infty}^{j\zeta} \exp(-t^2)\, dt \tag{14}$$

with

$$\zeta_{1i} = \frac{\omega - \bar{u}_i k_z + \omega_{ci}}{k_z v_{Ti} \sqrt{2}}$$

$$\zeta_{2i} = \frac{\omega - \bar{u}_i k_z - \omega_{ci}}{k_z v_{Ti}\sqrt{2}}$$

$$\zeta_{3i} = \frac{\omega - \bar{u}_i k_z}{k_z v_{Ti}\sqrt{2}} \tag{15}$$

In the quasi-static approximation $\underline{E}$ is derivable from a scalar potential ($\underline{E} = -\nabla\phi$) which upon substituting in Eq. (5) gives,

$$\nabla \cdot (\underset{\approx}{\epsilon}\nabla\phi) = 0 \tag{16}$$

Using the dielectric tensor for the warm plasma Eq. (16) is solved subject to the boundary conditions that tangential $\underline{E}$ and normal $\underline{D}$ are continuous. For the configuration with guide radius b and beam radius a the result is,[3]

$$\epsilon_{rr_c} T_c a \frac{J'_n(T_c a)}{J_n(T_c a)} + n\epsilon_{r\theta_c}$$

$$= \epsilon_{rr_d} T_d b \left[\frac{J'_n(T_d a)N_n(T_d b) - J_n(T_d b)N'_n(T_d b)}{J_n(T_d a)N_n(T_d b) - J_n(T_d b)N_n(T_d a)} \right]$$

$$+ n\epsilon_{r\theta_d} \tag{17}$$

where,

$$(Ta)^2 = -a^2 k_z^2 \frac{\epsilon_{zz}}{\epsilon_{rr}} \tag{18}$$

J_n and N_n are the Bessel functions of the first and second kind n-th order. The subscript c denotes summation over all beams within radius a and subscript d denotes summation over all beams outside of radius a. We first consider the simpler dispersion relation of a completely filled waveguide for which the dispersion relation reduces to,

$$J_n(T_a) = 0 \tag{19}$$

hence,

$$Ta = p_{n\nu} \tag{20}$$

and from Eq. (18),

$$p_{n\nu}^2 = -a^2 k_z^2 \frac{\epsilon_{zz}}{\epsilon_{rr}} \tag{21}$$

Substituting from Eq. (12) the dispersion relation is,

$$\frac{D_1^2 p_{n\nu}^2}{a^2} + k_z D_1^2 = -\sum_i \left[\frac{D_1^2}{D_i^2} (1 + \zeta_i Z(\zeta_i)) \right.$$

$$\left. + \frac{D_1^2 p_{n\nu}^2 \, 1}{a^2 \, 2\sqrt{2} k_z D_i \frac{\omega_{ci}}{\omega_{pi}}} (Z(\zeta_{2i}) - Z(\zeta_{1i})) \right] \tag{22}$$

where we have normalized to the Debye length of stream one

$$D_1 = \frac{v_{T1}}{\omega_{p1}} \tag{23}$$

If the radius, a, is increased indefinitely (22) reduces to,

$$k_z^2 D_1^2 = -\sum_i \frac{D_1^2}{D_i^2} (1 + \zeta_{3i} Z(\zeta_{3i})) \tag{24}$$

the dispersion relation for longitudinal oscillations in a warm, unbounded plasma. We also recover another mode which for a cold plasma would be, $\omega = \omega_c$. If the full set of coupled equations had been used taking the limit $a \longrightarrow \infty$ would have led to the right- and left-hand polarized cyclotron modes. Since these modes are transverse, the quasi-static analysis could not be expected to give an accurate result in this limit. For the longitudinal mode on the other hand, $\nabla \times \underline{E} = 0$, identically, and the quasi-static approximation is no longer an approximation.

For small D we can perform an asymptotic expansion in ζ, and keeping only terms to order $1/\zeta$ we obtain the cold plasma dispersion relation,

$$\frac{p_{n\nu}^2}{a^2} = \frac{-k_z^2\left(1 - \sum_i \frac{\omega_{pi}^2}{(\omega - k_z\bar{u}_i)^2}\right)}{1 + \sum_i \frac{\omega_{pi}^2}{\omega_{ci}^2 - (\omega - k_z\bar{u}_i)^2}} \tag{25}$$

which valid to first order in D.

III. PROPAGATION IN A STATIONARY PLASMA

A stationary plasma will now be considered with ions of infinite mass. The summation in Eq. (22) is thus taken over only one species. With the aid of the IBM 7090 the dispersion relation was solved for several sets of parameters. In Figs. 1 and 2 this relation is plotted in bold lines, for $\omega_c/\omega_p = 2.0$ and $.5$ along with plots, in light lines, of the dispersion relations for longitudinal oscillations in a warm unbounded plasma (from Eq. (24)), and a bounded cold plasma (from Eq. (25)). For low values of k_z where the guide wavelength λ_g is much greater than D, the temperature effects are unimportant compared to the effects of the guide walls and the dispersion relation approximates the cold plasma relation. With larger values of k_z where λ_g is small compared to the guide dimensions, the finite geometry is of little influence and the dispersion relation approaches that of the infinite medium. It should be noted that, for finite temperature plasma columns propagation exists for frequencies that are forbidden regions in the zero temperature approximation.

An expression for small damping can be derived from Eq. (22) by expanding $Z(\zeta)$ to first order in terms of ζ_{im},

$$Z(\zeta) = Z(\zeta_R) + j\, \zeta_{im}\, Z'(\zeta_R)$$

$$\zeta = \zeta_R + j\, \zeta_{im} \tag{26}$$

The following expressions are utilized in obtaining the result,

$$\begin{aligned} Z'(\zeta) &= -2(1 + \zeta\, Z(\zeta)) \\ Z(\zeta_R) &= j\pi^{1/2} \exp(-\zeta_R^{\,2}) - 2\, \zeta_R\, Y(\zeta_R) \\ Y(\zeta_R) &= \exp(-\zeta_R^{\,2})/\zeta_R \int_0^{\zeta_R} \exp(t^2)\, dt \end{aligned} \tag{27}$$

Upon separating real and imaginary terms we solve for ζ_{im} and obtain,

$$\zeta_{im} = \frac{\omega_{im}}{k_z v_T \sqrt{2}} = \frac{-\pi^{\frac{1}{2}}\left[\zeta_{3R} \exp(-\zeta_{3R}^2) + \frac{D^2 p_{n\nu}^2}{a^2\, 2\sqrt{2}}\, \frac{1}{k_z D(\frac{\omega_c}{\omega_p})} \exp(-\zeta_{2R}^2) - \exp(-\zeta_{1R}^2)\right]}{2\zeta_{3R}\left[(2\zeta_{3R}^2 - 1)Y(\zeta_{3R}) - 1\right] + 4\frac{D^2 p_{n\nu}^2}{a^2\, 2\sqrt{2}}\, \frac{1}{k_z D(\frac{\omega_c}{\omega_p})}\left[\zeta_{2R}^2\, Y(\zeta_{2R}) - \zeta_{1R}^2\, Y(\zeta_{1R})\right]} \tag{28}$$

If $a \longrightarrow \infty$ and $Y(\zeta_{3R})$ is expanded in an asymptotic series in $1/\zeta$ the familiar Landau damping expression is recovered.

$$\omega_{im} \cong -\frac{\pi^{1/2}\, \omega^4}{2\sqrt{2}\, v_T^3\, k_z^3} \exp\left(-\frac{\omega^2}{2k_z^{\,2}\, v_T^{\,2}}\right) \tag{29}$$

For the parameters used in Figs. 1 and 2 the damping associated with the space-charge mode (lower branch for $\omega_c > \omega_p$ and upper branch for $\omega_c < \omega_p$) corresponds closely to the Landau damping.

IV. INSTABILITIES

If, instead of one stream, we consider the situation of two streams with a relative drift velocity, there is then a possibility of growing waves or instability. (The question of whether these instabilities are convective or nonconvective has been determined elsewhere and will not be considered here).[20, 21] If Eq. (22) has a solution with positive imaginary ω the plasma is unstable. In searching for such solutions we follow the approach used by several authors for unbounded plasmas.[13-15] A contour is traced over a semi-circle in the upper half of the ω-plane. Defining $H(\zeta)$ to be the RHS of Eq. (22) we trace the corresponding contour in the $H(\zeta)$ plane. This contour bounds all values of $H(\zeta)$ corresponding to positive imaginary ω. Since the LHS of Eq. (22) is always positive real an unstable solution will exist only if the curve in the $H(\zeta)$ plane intersects the positive real axis. If only two streams are present graphical means may be used to locate the limiting values of $k_z D$ for which instability exists and also the limiting ratios of relative drift velocity to thermal velocity. Eq. (22) is written in the form,

$$\frac{D_1^{\,2} p_{n\nu}^{\,2}}{a^2} + k_z^{\,2} D_1^{\,2} - H_1(\zeta^{(1)}) = H_2(\zeta^{(2)}) \tag{30}$$

where $\zeta^{(1)}$ and $\zeta^{(2)}$ are assumed to vary independently in the ζ-plane. Contours of the RHS and LHS of Eq. (30) corresponding to the contour in the ω-plane are mapped in the complex H-plane. If the two curves intersect there is a solution and by requiring that $\zeta^{(1)}$ and $\zeta^{(2)}$ have the same value of ω we can solve for the limiting values of the drift velocity. The value of k_z at which the two curves first intersect is the maximum value of k_z for which instability is possible. Fig. 3 is a typical plot. Before discussing these plots in greater detail, certain limiting cases will be examined.

For a plasma with infinite transverse dimensions the dispersion relation reduces to Eq. (24). In this case the contours retain the same size and shape for all values of k_z, and for two streams with equal Debye lengths, D, the criteria for instability can be written[13]

$$k_z^2 D^2 < .570 \tag{31}$$

In the limit of infinite magnetic field Eq. (22) reduces to,

$$D_1^2 k_z^2 + \frac{D_1^2 p_{n\nu}^2}{a^2} = - \sum_i \frac{D_1^2}{D_i^2} (1 + \zeta_{3i} Z(\zeta_{3i})) \tag{32}$$

which for equal Debye lengths results in the limit for instability,

$$D^2 k_z^2 + \frac{D^2 p_{n\nu}^2}{a^2} < .570 \tag{33}$$

the result obtained by Harrison.[8] From this result we see the possibility of attaining a stable plasma for all k_z if the condition,

$$\frac{D^2 p_{n\nu}^2}{a^2} > .570 \tag{34}$$

is satisfied. Hence, we investigate the limit $k_z \longrightarrow 0$ in the general case, when ω_c is finite, to determine whether a similar condition exists. In the limit $k_z \longrightarrow 0$, $Z(\zeta_{1i})$ and $Z(\zeta_{2i})$ can be expanded in an asymptotic series when $|\omega - k_z \bar{u}_i| \neq \omega_c$. For identical streams the dispersion relation reduces to,

$$\frac{p_{n\nu}^2}{a^2} (D^2 + \frac{2v_T^2}{\omega_c^2}) = - \sum_{i=1}^{2} (1 + \zeta_{3i} Z(\zeta_{3i})) \tag{35}$$

and comparing this expression with Eq. (24) and Eq. (32) it follows that the condition,

$$\frac{p_{n\nu}^2}{a^2} (D^2 + \frac{2v_T^2}{\omega_c^2}) < .570 \tag{36}$$

must be satisfied for instability. The additional term $2v_T^2/\omega_c^2$ acts to enhance the Debye wavelength.

In the vicinity of $|\omega - k_z\bar{u}_i| = \omega_c$, the asymptotic expansion is not valid; the term in $H(\zeta)$ that is dependent upon ω_c leads to another loop in the contour. This loop appears in Fig. 3 but does not lead to any further interactions in that particular case. The ω_c dependent terms are inversely proportional to k_z as examination of Eq. (22) reveals. For small k_z the contours are as shown in Fig. 4. We now have two additional interactions. The stability previously considered was the interaction between space-charge waves, and the limit of stability found in Eq. (36) referred to this interaction. The stability limits for the cyclotron cyclotron and the cyclotron space-charge interactions are found, respectively, from the intersection of the additional loops with each other or from the intersection of the additional loop of one stream with the contour that exists at infinite magnetic field of the other stream.

In Figs. 5 and 6 the limiting values of k_zD for instability are plotted against the limiting values of $\bar{u}_o/2\sqrt{2}v_T$ where $\bar{u}_o$ is the relative drift velocity, for values of ω_c/ω_p of 2, and 1/2 respectively. Unstable regions lie below the curve in question and within the cross-hatched areas. The plots display some results known from the cold plasma theory; the space-charge interaction is weakened as the radius of the beam decreases, while the cyclotron interactions are strengthened. For $\omega_c/\omega_p = 2$ points A and B in Fig. 3 correspond to points A and B in Fig. 5 and represent the minimum and maximum values of $\bar{u}_o$ for instability. Similarly points C, D and E, in Fig. 4 give the minimum values of $\bar{u}_o$ for the space-charge, cyclotron, and cyclotron space-charge instabilities, respectively.

As an example we consider the dispersion characteristic presented in Fig. 7a for two cold counterstreaming beams.[5] The parameters for the cold plasma were selected to give the critical condition (limit of stability) for coupling of the space-charge waves. The cyclotron cyclotron and space-charge cyclotron interactions lead to the presence of growing waves. The counterpart of the two growing modes with temperature included is presented in Fig. 7b. With the addition of temperature the slow cyclotron wave that previously was a growing wave is now a decaying wave. These results are further illustrated in Fig. 8 where the limiting values for instability are plotted. The vertical dashed line indicates the value of $\bar{u}_o/2\sqrt{2}\,v_T$ in Fig. 7b. (We note that the dashed line intersects only the cyclotron space-charge region of instability.) Typical values of the physical quantities corresponding to this example are $\omega_p \approx 4 \times 10^9$ rad/sec, $a \approx 1$ cm, $B_o \approx 450$ gauss, V_T corresponds to a temperature of 4 eV, and $\bar{u}_o$ corresponds to a voltage of 1000 volts.

The limits of instability, found above, correspond to a filled waveguide. For this configuration both the cyclotron and the space-charge modes disappear when $\omega \rightarrow 0$ and hence the critical dependence of Eq. (36) on ω_c. In an unfilled waveguide the space-charge wave becomes a surface wave when ω_c goes to zero and the presence or absence of instabilities would not be expected to be strongly dependent on ω_c. To verify this postulate we consider two beams of radius a in a waveguide of radius b. In the limit of $k_z b << 1$, Eq. (17) becomes,

$$\epsilon_{rr} Ta \frac{J_1(Ta)}{J_0(Ta)} = \frac{1}{\ln \frac{b}{a}} \tag{37}$$

(for the axially symmetrical mode). If $b >> a$ the LHS may be expanded and we obtain,

$$\epsilon_{zz}(Ta)^2 = \frac{2}{\ln \frac{b}{a}} \tag{38}$$

and from Eq. (12) and Eq. (18), for identical streams

$$\frac{2D^2}{a^2 \ln \frac{b}{a}} + k_z D^2 = - \sum_i \left(1 + \zeta_{3i} Z(\zeta_{3i})\right) \tag{39}$$

with k_z small.

Hence,

$$\frac{2\,D^2}{a^2 \ln \frac{b}{a}} > .570 \tag{40}$$

is the condition for stability as $k_z \longrightarrow 0$. This result has been obtained for an infinite magnetic field;[8] here we find that it is valid independently of the value of ω_c and in particular for $\omega_c \longrightarrow 0$.

CONCLUSIONS

The propagation of space-charge waves in a warm bounded plasma were shown to have a smooth transition from the characteristics of a bounded zero temperature plasma to an unbounded finite temperature plasma as k_z increases. The inclusion of temperature leads to the possibility of propagation between the plasma frequency and the cyclotron frequency, as well as above the upper frequency limit, $(\omega_p^{\,2} + \omega_c^{\,2})^{1/2}$, of the cold plasma theory. Experimental results indicate that the dispersion relation follows the cold plasma theoretical curve[3, 22, 23] closely to as large a k_z as points have been obtained, but these measurements did not include the transition region. Presumably data is difficult to obtain where the curve deviates from the cold plasma curve because it is in this region that attenuation becomes significant. The cyclotron mode is especially difficult to measure experimentally as has been noted in the literature.[3] The high attenuation which we have calculated for this mode, even at comparably small values of the wave number, can well account for the experimental difficulties.

The decrease in the strength of cyclotron space-charge interaction in experiments,[5] from that predicted from the zero temperature theory, can at least partly be explained by introducing a finite temperature. More striking is the complete suppression of the cyclotron cyclotron instability for our representative example. This interaction has not yet been experimentally observed. It has also been pointed out by several investigators[24] that oscillations have been observed for frequencies at which waves would not propagate in the absence of temperature effects. Despite the fact that attenuation for propagating waves is usually high in these regions it is possible to have instabilities if the interaction is strong enough.

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REFERENCES

1. W. C. Hahn, G. E. Review 42: 258 (1939).
2. V. Bevc and T. E. Everhart, J. Electronics and Control 13:185 (1962).
3. A. W. Trivelpiece and R. W. Gould, J. A. P. 30:1784 (1959).
4. L. D. Smullin and P. Chorney, Proc. Symp. on Electronic Wave-guides, Brooklyn Polytechnic Press, N. Y., p. 229 (1958).
5. B. J. Maxum and A. W. Trivelpiece, J. of App. Phys. (to be published).
6. B. Vural, RCA Rev. 22: 753 (1961).
7. R. J. Briggs and A. Bers, MIT Res. Lab. of Elec., Quarterly Progress Reports 70: 129 and 71: 131 (1963).
8. E. R. Harrison, Proc. Phys. Soc., 79, Part 2: 317 (1962)
9. J. D. Jackson, J. Nucl. Energy Part C: Plasma Physics, 1: 171 (1960).
10. T. H. Stix, The Theory of Plasma Waves, McGraw-Hill, (1962)
11. J. E. Drummond, Phys. Rev., 110: 293 (1958).
12. K. N. Stepanev, S. P., JETP, 7: 892 (1958).
13. O. Buneman, Phys. Rev., 115: 503 (1959).
14. O. Penrose, Phys. Fluids, 3: 258 (1960).
15. E. A. Jackson, Phys. Fluids, 3: 786 (1960.
16. A. J. Lichtenberg, IEEE Trans. of PTG on Elec. Devices, ED-11: 62 (1963).
17. E. G. Harris, J. Nuclear Energy Part C: Plasma Physics, 2: 138 (1961).
18. L. D. Landau, Journal of Physics U. S. S. R., 10: 25 (1946).

19. B. D. Fried and S. D. Conte "The Plasma Dispersion Function" Academic Press, N. Y., (1961).

20. P. A. Sturrock, Phys. Rev., 112: 1488 (1958)

21. R. J. Briggs and A. Bers, MIT Res. Lab. of Elec. Quarterly Progress Report 71: 122 (1963)

22. G. D. Boyd, R. W. Gould, L. M. Field, Proc. IRE, 49: 1906 (1961).

23. R. N. Carlile, University of California, Elec. Res. Lab., TM-30 (1963).

24. C. Etievant, G. Kino, L. D. Smullin, private communications.

LIST OF ILLUSTRATIONS

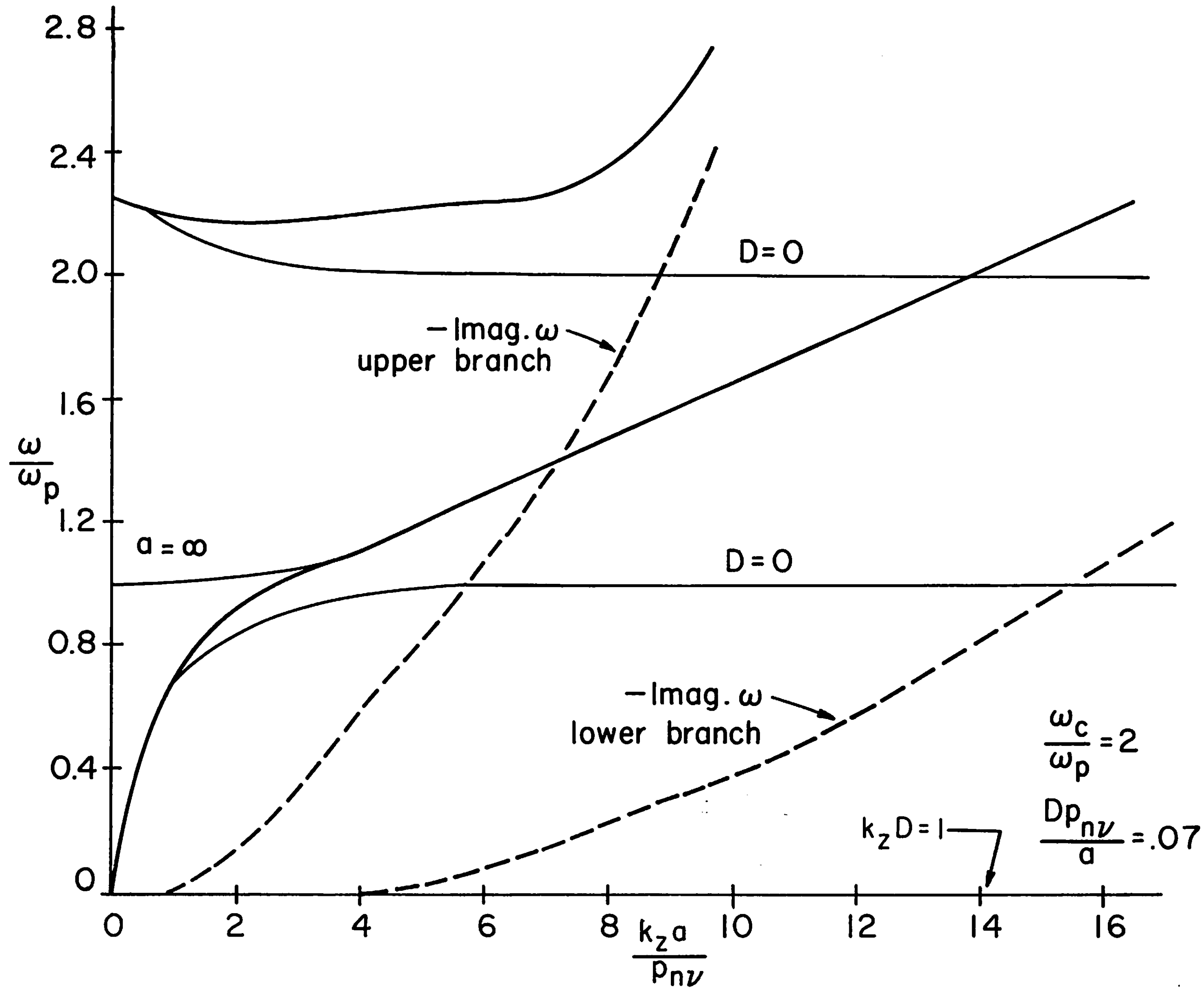

Fig. 1. Dispersion relation for filled waveguide comparing finite and zero temperature with $\omega_c/\omega_p = 2$.

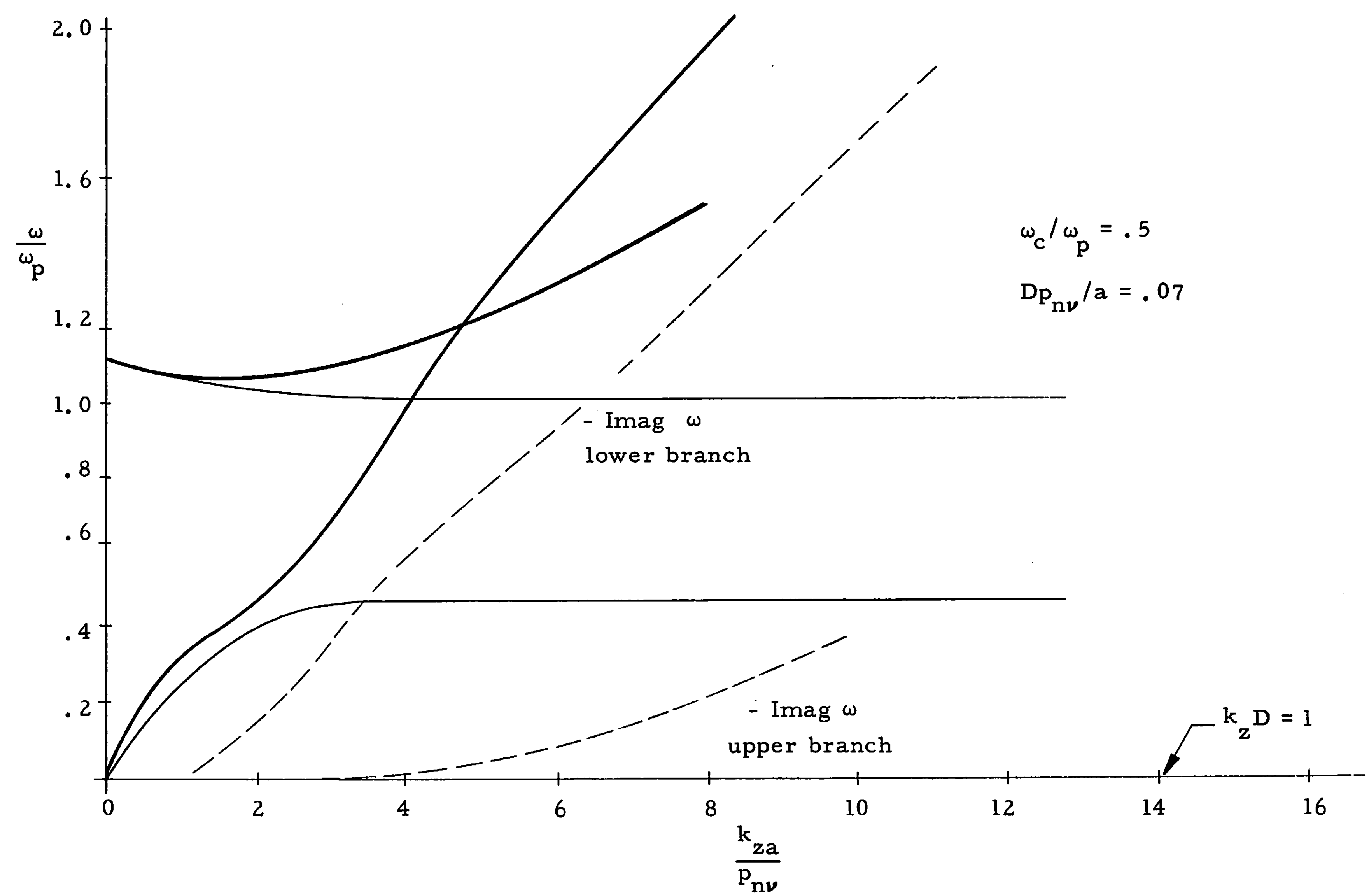

Fig. 2. Dispersion relation as in Fig. 1 with $\omega_c/\omega_p = .5$.

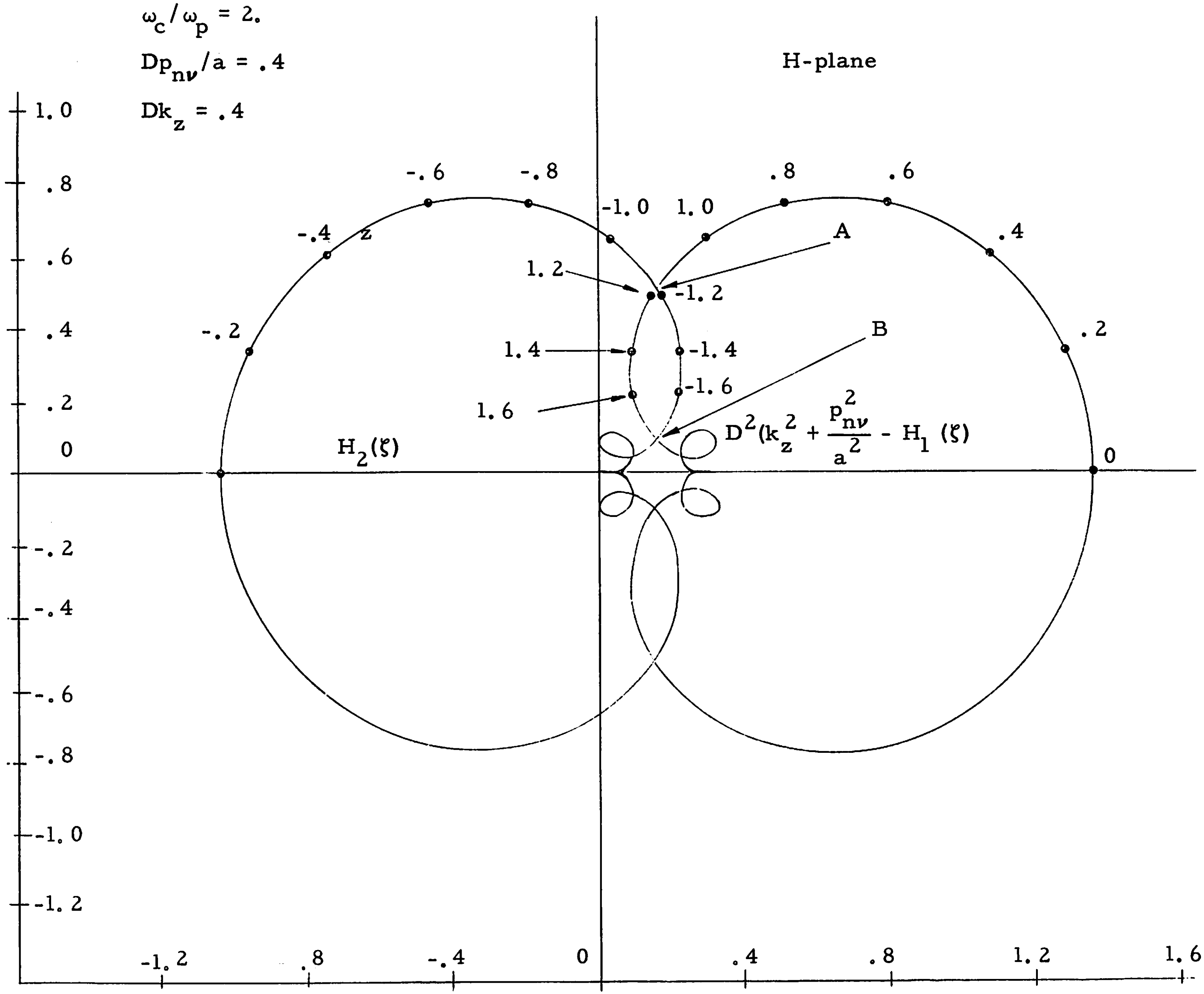

Fig. 3. Solution to Eq. (30) for real ω.

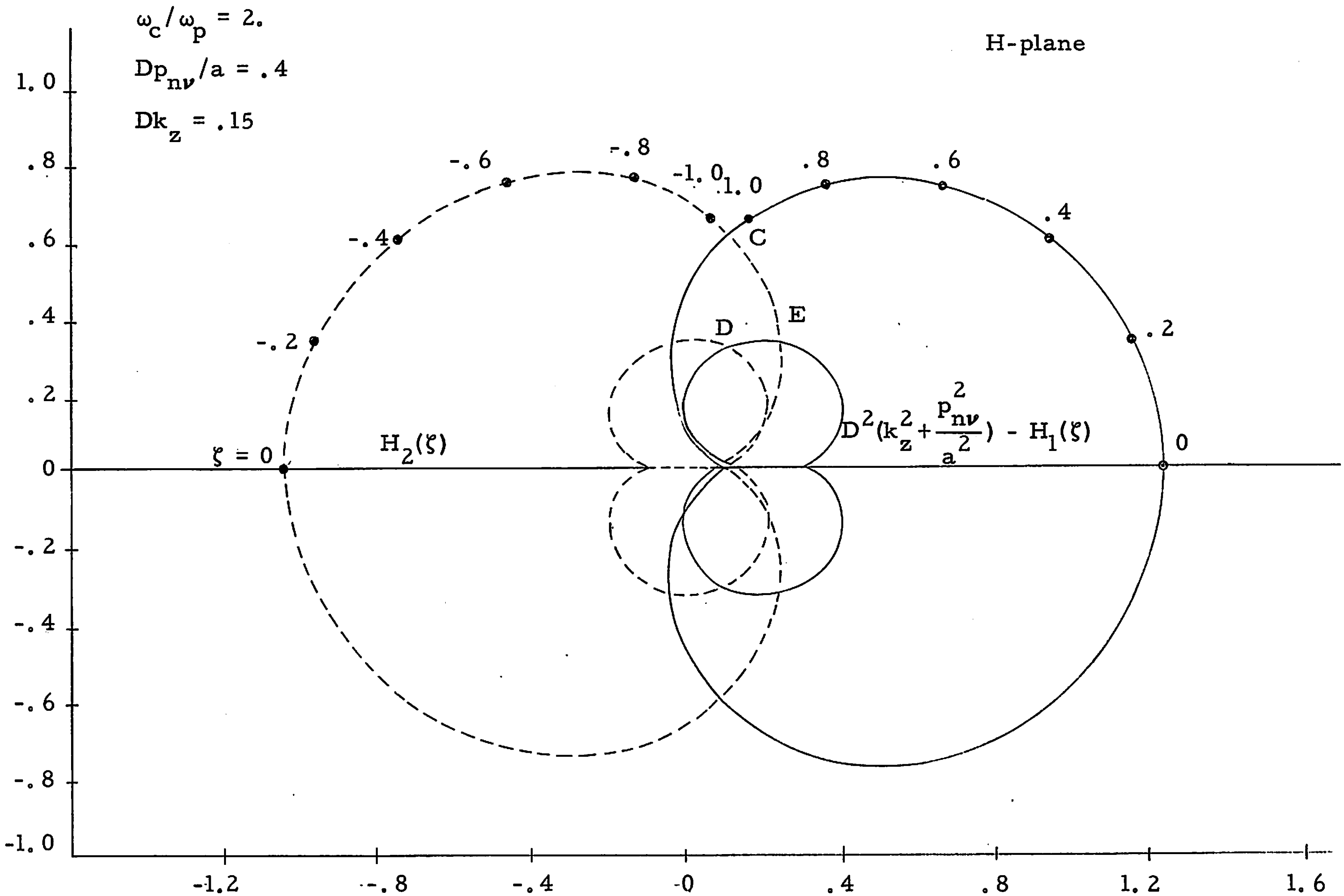

Fig. 4. Solution as in Fig. 3 for smaller value of k_z.

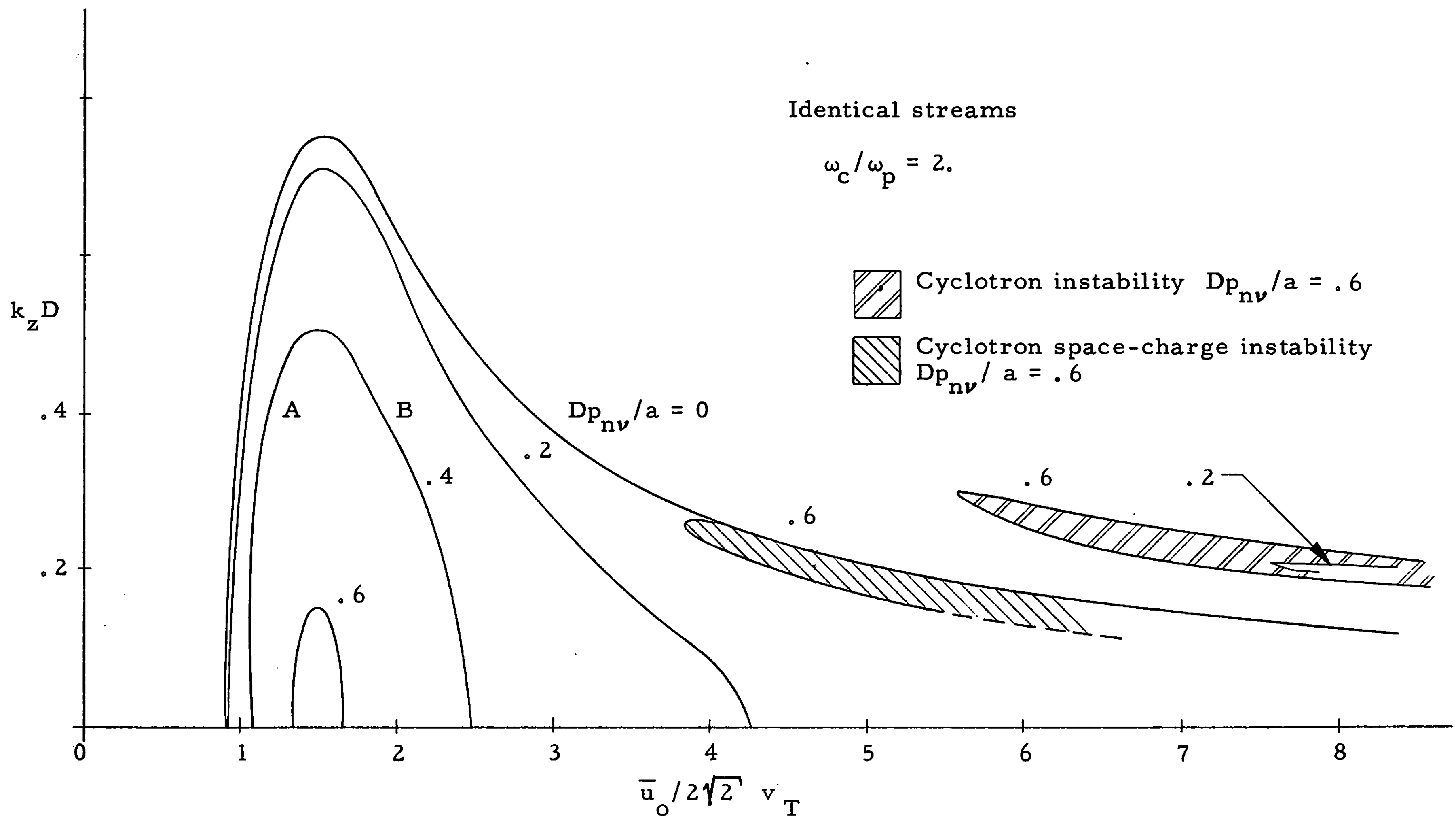

Fig. 5. Stable and unstable regions as functions of the parameter $Dp_{n\nu}/a$ $\omega_c/\omega_p = 2.$

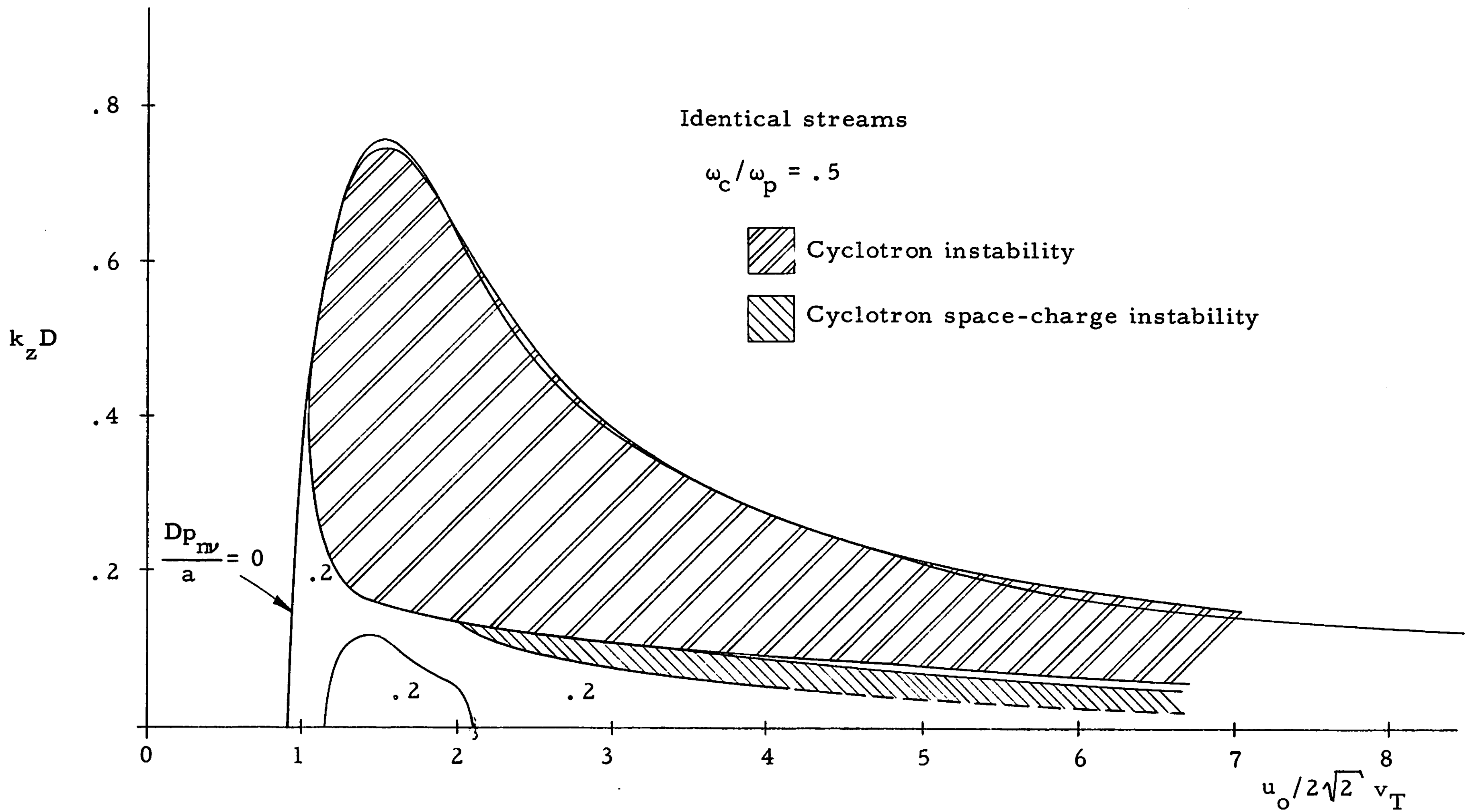

Fig. 6. Stable and unstable regions as in Fig. 5 with $\omega_c/\omega_p = .5$.

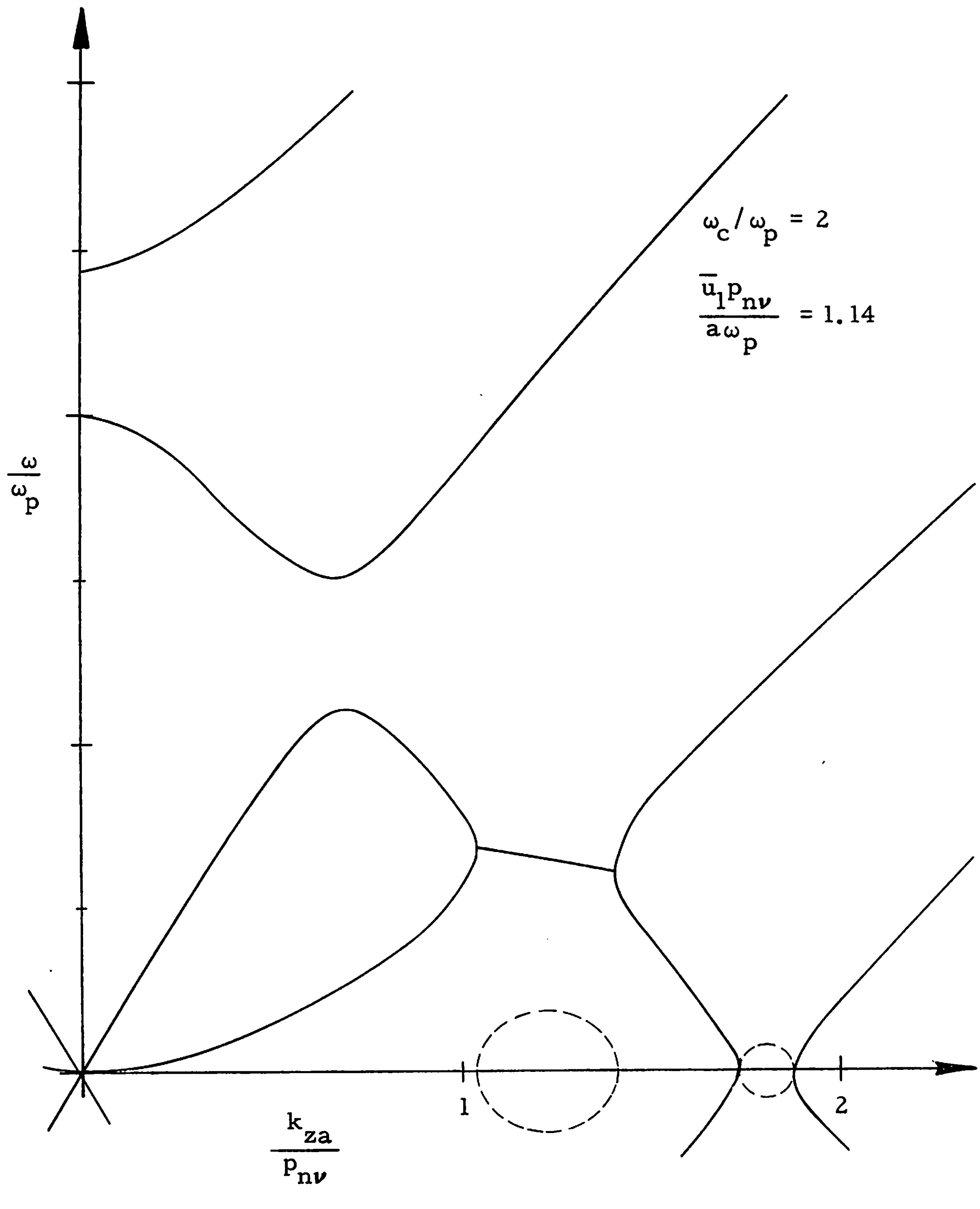

Fig. 7a. The dispersion curve for two identical counter-streaming beams in a filled waveguide with zero temperature.

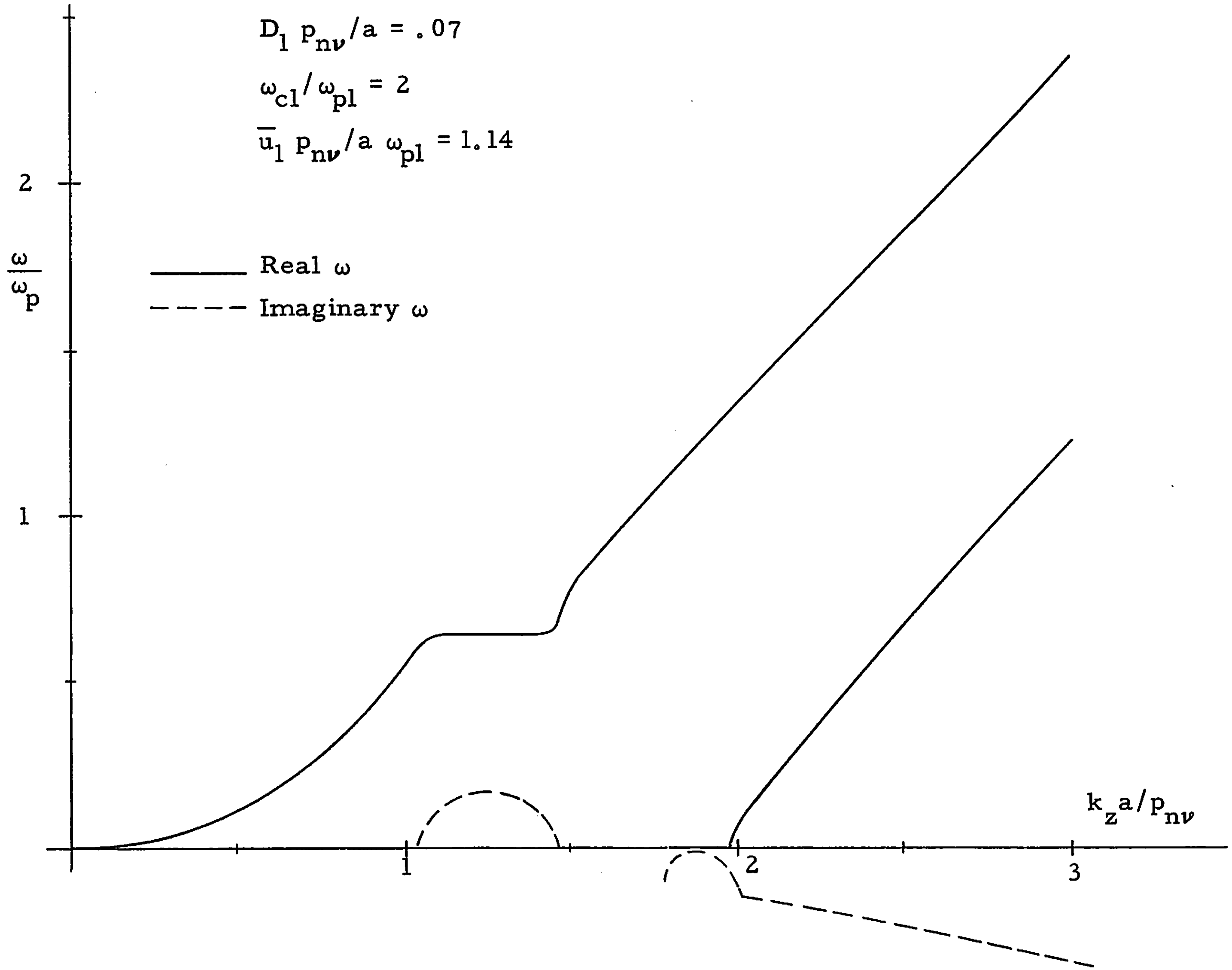

Fig. 7b. The dispersion curve for two identical counter-streaming beams in a filled waveguide with finite temperature for modes which in the absence of temperature have regions of growth.

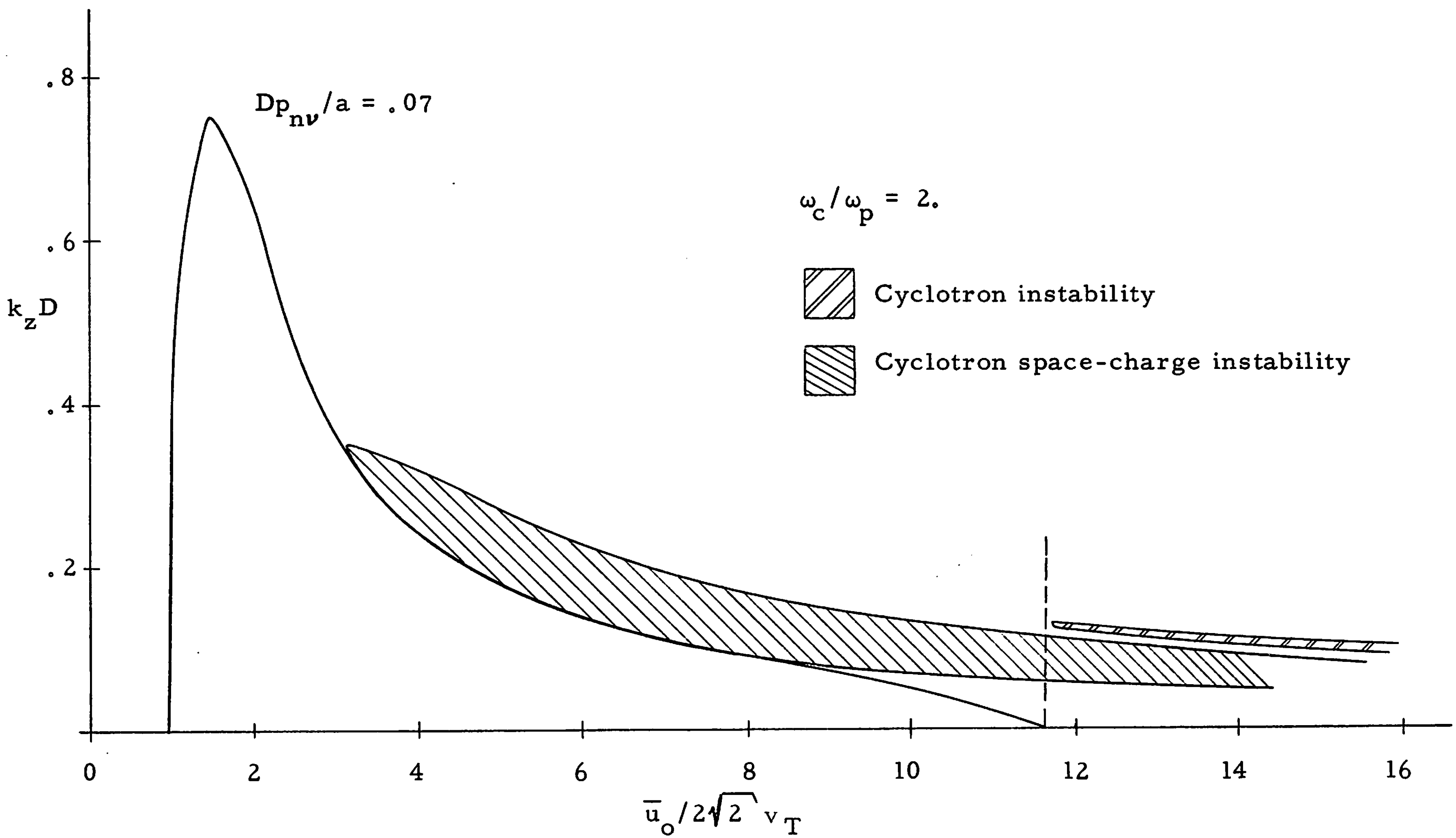

Fig. 8. Stable and unstable regions corresponding to Fig. 7b.