A NOTE ON THE FINITE z-TRANSFORM*

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In recent publications[1,2] the finite z-transform method was introduced and discussed. This method represents a certain modification of the ordinary (or infinite) z-transform which is suitably applicable to boundary value discrete problems. Such cases occur in linear array antenna and finite termination of RC ladder network or in voltage distribution in an insulated string and in two-plate electrostatic capacitor. With the availability of tables of finite z-transforms these and other problems can be conveniently solved.

The purpose of this note is to show recurrence relationship whereby the finite z-transform of the real function n^r can be calculated and to present a table of finite z-transform for values of r up to ten. The value of this table lies in the ready solution of some of the mentioned problems and in obtaining of a finite sum of series which are often encountered in digital communication systems.[3,4]

We will review first the definition of the z-transform and later establish the recurrence relationship for the finite z-transforms of n^r.

The finite z-transform of a real fraction "f" is destined as:

$$\mathcal{J}_f\,[f] = \sum_{n=0}^{N-1} f(n)\, z^{-n} \tag{1}$$

The above equation is related to the z-transform of "f" in the following:

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$$\mathcal{Z}_f\left[f\right] = \mathcal{Z}\left[f(n)\left\{u(n) - u(n - N)\right\}\right] \tag{2}$$

where $u(n)$ is a unit-step function defined at integer values of "t".

One can evaluate (2) directly from the definition of the z-transform or more conveniently by using the convolution z-transform as follows.[5]

$$\mathcal{Z}_f\left[f\right] = \frac{1}{2\pi i}\int_\Gamma \frac{\mathcal{F}(p)}{p}\,\frac{z/p}{(z/p - 1)}\left[1 - (z/p)^{-N}\right] dp \tag{3}$$

where the contour "Γ" encloses in a positive sense the poles of $\mathcal{F}(p)$ in the p-plane. Using the relationship (3), one can obtain and tabulate the finite z-transform of any of the encountered functions. However, for the function n^r an easy procedure can be obtained using a certain recurrence relationship. This procedure is based on utilizing the recurrence relationship and the table obtained previously[5, 6] for the z-transform of n^r.

From the definition of (1), one can write

$$\mathcal{Z}_f\left[n^r\right] = \sum_{n=0}^{N-1} n^r z^{-n} = \sum_{n=0}^{\infty} n^r z^{-n} - \sum_{n=N}^{\infty} n^r z^{-n} = \mathcal{Z}\left[n^r\right] - \sum_{n=N}^{\infty} n^r z^{-n} \tag{4}$$

Let $y = n - N$, in the last term of (4), to obtain

$$\sum_{n=N}^{\infty} n^r z^{-n} = \sum_{y=0}^{\infty} (y+N)^r z^{-y-N} = z^{-N}\left[\sum_{y=0}^{\infty} y^r z^{-y} + {}_rC_1 \sum_{y=0}^{\infty} y^{r-1} N z^{-y}\right.$$

$$\left. + {}_rC_2 \sum_{y=0}^{\infty} y^{r-2} N^2 z^{-y} + \ldots + {}_rC_i \sum_{y=0}^{\infty} y^{r-j} N^j z^{-y} + \ldots + \sum_{y=0}^{\infty} N^r z^{-y}\right] \tag{5}$$

The above expression can be written as:

$$\sum_{n=N}^{\infty} n^{r} z^{-n} = z^{-N} \sum_{j=0}^{r} \binom{r}{j} N^{r-j} \mathcal{Z}(n^{j}) \tag{6}$$

Substituting (6) in (4) we obtain the following basic relationship,

$$\mathcal{Z}_f \left[n^{r}\right] = \mathcal{Z}\left[n^{r}\right] - z^{-N} \sum_{j=0}^{r} \binom{r}{j} N^{r-j} \mathcal{Z}\left[n^{j}\right] \tag{7}$$

From a preceding work[5,6] the z-transform of n^r can be calculated from a certain recurrence relationship from which a table of r up to ten had been presented. Therefore, by use of Eq. (7) a similar table is constructed from the finite z-transform of $[n^r]$ utilizing the previously published table.[5] This tabulation is given in the Appendix. Furthermore, one may note that finite summation of $\sum_{n=1}^{N-1} n^r/k^r$ can be readily obtained from the finite z-transform by replacing z by k. In particular, when $k = 1$, the finite sum is given in terms of Bernoulli numbers.[6] Other associated series can be similarly obtained.

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APPENDIX: TABLE OF FINITE z-TRANSFORM OF n^r

r	$\mathcal{Z}_f\left[n^r\right] = \sum_{n=0}^{N-1} n^r z^{-n}$
0	$\dfrac{z - z^{-N+1}}{z - 1}$
1	$\dfrac{z - Nz^{-N+2} + (N-1)z^{-N+1}}{(z-1)^2}$
2	$\dfrac{z^2 + z - N^2 z^{-N+3} + (2N^2 - 2N - 1)z^{-N+2} - (N-1)^2 z^{-N+1}}{(z-1)^3}$
3	$\dfrac{z^3+4z^2+z-N^3z^{-N+4}+(3N^3-3N^2-3N-1)z^{-N+3}-(3N^3-6N^2+4)z^{-N+2}+(N-1)^3z^{-N+1}}{(z-1)^4}$
4	$\dfrac{z^4+11z^3+z-N^4z^{-N+5}+(4N^4-4N^3-6N^2-4N-1)z^{-N+4}}{(z-1)^5}$
	$\underline{-(6N^4-12N^3-6N^2+12N+11)z^{-N+3}+(4N^4-12N^3+6N^2+12N-11)z^{-N+2}-(N-1)^4\ z^{-N+1}}$
5	$\dfrac{z^5+26z^4+66z^3+26z^2+z-N^5z^{-N+6}+(5N^5-5N^4-10N^3-10N^2-5N-1)z^{-N+5}}{(z-1)^6}$
	$\underline{-(10N^5-20N^4-20N^3+20N^2+50N+26)z^{-N+4}+(10N^5-30N^4+60N^2-66)z^{-N+3}}$
	$\underline{-(5N^5-20N^4+20N^3+20N^2-50N+26)z^{-N+2}+(N-1)^5z^{-N+1}}$

TABLE (Cont.)

r	
6	$\dfrac{z^6 + 57z^5 + 302z^4 + 302z^3 + 57z^2 + z - N^6 z^{-N+7}}{(z-1)^7}$
	$\underline{+(6N^6 - 6N^5 - 15N^4 - 20N^3 - 15N^2 - 6N - 1)z^{-N+6}}$
	$\underline{-(15N^6 - 30N^5 - 45N^4 + 20N^3 + 135N^2 + 150N + 57)z^{-N+5}}$
	$\underline{+(20N^6 - 60N^5 - 30N^4 + 160N^3 + 150N^2 - 240N - 302)z^{-N+4}}$
	$\underline{-(15N^6 - 60N^5 + 30N^4 + 160N^3 - 150N^2 - 240N + 302)z^{-N+3}}$
	$\underline{+(6N^6 - 30N^5 + 45N^4 + 20N^3 - 135N^2 + 150N - 57)z^{-N+2}}$
	$\underline{-(N-1)^6 z^{-N+1}}$
7	$\dfrac{z^7 + 120z^6 + 1191z^5 + 2416z^4 + 1191z^3 + 120z^2 + z - N^7 z^{-N+8}}{(z-1)^8}$
	$\underline{+(7N^7 - 7N^6 - 21N^5 - 35N^4 - 35N^3 - 21N^2 - 7N - 1)z^{-N+7}}$
	$\underline{-(21N^7 - 42N^6 - 84N^5 + 280N^3 + 504N^2 + 392N + 120)z^{-N+6}}$
	$\underline{+(35N^7 - 105N^6 - 105N^5 + 315N^4 + 665N^3 - 315N^2 - 1715N - 1191)z^{-N+5}}$

TABLE (Cont.)

r	
	$\underline{-(35N^7-140N^6+560N^4-1680N^2+2416)z^{-N+4}}$
	$\underline{+(21N^7-105N^6+105N^5+315N^4-665N^3-315N^2+1715N-1191)z^{-N+3}}$
	$\underline{-(7N^7-42N^6+84N^5-280N^3+540N^2-392N+120)z^{-N+2}}$
	$\underline{+(N-1)^7 z^{-N+1}}$
8	$\frac{z^8+247z^7+4293z^6+15619z^5+15619z^4+42932z^3+247z^2+z-N^8z^{-N+9}}{(z-1)^9}$
	$\underline{+(8N^8-8N^7-28N^6-56N^5-70N^4-56N^3-28N^2-8N-1)z^{-N+8}}$
	$\underline{-(28N^8-56N^7-140N^6-56N^5+490N^4+1288N^3+1540N^2+952N+247)z^{-N+7}}$
	$\underline{+(56N^8-168N^7-252N^6+504N^5+1890N^4+504N^3+5292N^2+8568N+4293)z^{-N+6}}$
	$\underline{-(70N^8-280N^7-140N^6+1400N^5+1330N^4-5320N^3-6860N^2+9800N+15619)z^{-N+5}}$
	$\underline{+(56N^8-280N^7+140N^6+1400N^5-1330N^4-5320N^3+6860N^2+9800N-15619)z^{-N+4}}$
	$\underline{-(28N^8-168N^7+252N^6+504N^5-1890N^4+504N^3+5292N^2-8568N+4293)z^{-N+3}}$

TABLE (Cont.)

r	
	$+(8N^8-56N^7+140N^6-56N^5-490N^4+1288N^3-1540N^2+952N-247)z^{-N+2}$
	$-(N-1)^8 z^{-N+1}$.
9	$\dfrac{z^9+502z^8+14608z^7+88234z^6+156190z^5+88234z^4+14608z^3+5022z^2+z}{(z-1)^{10}}$ *
	$-z^{-N+10}$ [following bracketed terms.]
	$\Big[N^9(1-9z^{-1}+36z^{-2}-84z^{-3}+126z^{-4}-126z^{-5}+84z^{-6}-36z^{-7}+9z^{-8}-z^{-9})$
	$+9N^8(z^{-1}-8z^{-2}+28z^{-3}-56z^{-4}+70z^{-5}-56z^{-6}+28z^{-7}-8z^{-8}+z^{-9})$
	$+36N^7(z^{-1}-6z^{-2}+14z^{-3}-14z^{-4}+14z^{-6}-14z^{-7}+6z^{-8}-z^{-9})$
	$+84N^6(z^{-1}-2z^{-2}-8z^{-3}+34z^{-4}-50z^{-5}+34z^{-6}-8z^{-7}-2z^{-8}+z^{-9})$
	$+126N^5(z^{-1}+6z^{-2}-34z^{-3}+46z^{-4}-46z^{-6}+34z^{-7}-6z^{-8}-z^{-9})$
	$+126N^4(z^{-1}+22z^{-2}-32z^{-3}-86z^{-4}+190z^{-5}-86z^{-6}-32z^{-7}+22z^{-8}+z^{-9})$
	$+84N^3(z^{-1}+54z^{-2}+134z^{-3}-434z^{-4}+434z^{-6}-134z^{-7}-54z^{-8}-z^{-9})\Big]$

*These two polynomials are symmetric with the same coefficients.

TABLE (Cont.)

r	
	$+36N^{2}(z^{-1}+118z^{-2}+952z^{-3}+154z^{-4}-2450z^{-5}+154z^{-6}+952z^{-7}+118z^{-8}+z^{-9})$
	$+9N(z^{-1}+246z^{-2}+4046z^{-3}+11326z^{-4}-11326z^{-6}-4045z^{-7}-246z^{-8}-z^{-9})$
	$+(z^{-1}+502z^{-2}+14608z^{-3}+88234z^{-4}+156190z^{-5}+88234z^{-6}+14608z^{-7}+502z^{-8}+z^{-9})$ *
10	$\dfrac{z^{10}+1013z^{9}+47840z^{8}+455192z^{7}+1310354z^{6}+\ldots+z^{*}}{(z-1)^{11}}$
	$-z^{-N+11}$ [following bracketed terms.]
	$\Big[N^{10}(1-10z^{-1}+45z^{-2}-120z^{-3}+210z^{-4}-252z^{-5}+210z^{-6}-120z^{-7}+45z^{-8}-10z^{-9}+z^{-10})$
	$+10N^{9}(z^{-1}-9z^{-2}+36z^{-3}-84z^{-4}+126z^{-5}-126z^{-6}+84z^{-7}-36z^{-8}+9z^{-9}-z^{-10})$
	$+45N^{8}(z^{-1}-7z^{-2}+20z^{-3}-28z^{-4}+24z^{-5}-24z^{-6}-28z^{-7}-20z^{-8}-7z^{-9}+z^{-10})$
	$+120N^{7}(z^{-1}-3z^{-2}-6z^{-3}+42z^{-4}-84z^{-5}+84z^{-6}-42z^{-7}+6z^{-8}+3z^{-9}-z^{-10})$
	$+210N^{6}(z^{-1}+5z^{-2}-40z^{-3}+80z^{-4}-46z^{-5}-46z^{-6}+80z^{-7}-40z^{-8}+5z^{-9}+z^{-10})$
	$+252N^{5}(z^{-1}+21z^{-2}-54z^{-3}-54z^{-4}+276z^{-5}-276z^{-6}+54z^{-7}+54z^{-8}-21z^{-9}-z^{-10})$
	$+210N^{4}(z^{-1}+53z^{-2}+80z^{-3}-568z^{-4}+434z^{-5}+434z^{-6}-568z^{-7}+80z^{-8}+53z^{-9}+z^{-10})$

TABLE (Cont.)

r	
	$+120N^3(z^{-1}+117z^{-2}+834z^{-3}-798z^{-4}-2604z^{-5}+2604z^{-6}+798z^{-7}-834z^{-8}-117z^{-9}-z^{-10})$
	$+45N^2(z^{-1}+245z^{-2}+3800z^{-3}+7280z^{-4}-11326z^{-5}-11326z^{-6}+7280z^{-7}+3800z^{-8}$ $+245z^{-9}+z^{-10})$
	$+10N(z^{-1}+501z^{-2}+14106z^{-3}+73626z^{-4}+67956z^{-5}-67956z^{-6}-73626z^{-7}-14106z^{-8}$ $-501z^{-9}-z^{-10})$
	$+(z^{-1}+1013z^{-2}+47840z^{-3}+455192z^{-4}+1310354z^{-5}+\ldots+z^{-10})^{*}\big]$

*The asterisks on the previous and on this page denote two polynomials which are symmetric with the same coefficients.