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## A GENERAL FORMULATION OF THE TOTAL SQUARE INTEGRALS FOR CONTINUOUS SYSTEMS

by

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# A GENERAL FORMULATION OF THE TOTAL SQUARE INTEGRALS FOR CONTINUOUS SYSTEMS * 

E. I. Jury and A. G. Dewey

The problem of evaluating the Total Square Integral arises in the analysis and design of feedback control systems, both for deterministic and for stochastic inputs. The first approach to this problem was made by James, et al. ${ }^{1}$ using the solution of a recurrence formula. Integrals up to the seventh-order were tabulated. A few years later, Newton et al. ${ }^{2}$ presented a method based on the solution of a matrix equation dependent on whether $n$, the order of the system, were even or odd. The earlier table was extended up to the tenth-order. Kalman and Bertram ${ }^{3}$ recently mentioned an approach from the point of view of Lyapunov functions. However, this leads to the solution of a system of algebraic equations and the method has no computational advantage over that of Newton.

The contributions of this note are: (1) The formulation of a general expression for any order $n$ as the ratio of two determinants. This could easily be programmed on a digital computer. (2) It is shown that the denominator determinant is precisely equal to $\Delta_{n}$, the $n$-th Hurwitz determinant. This equivalence guarantees the existence of the Total Square Integral for a stable system.

[^0](3) Finally a scheme is shown whereby the amount of computation is minimized. This scheme involves the expansion of only ( $\mathrm{n}-2$ ) determinants of order ( $n-3$ ).

It is of interest to note that this research was motivated by the authors'work onobtaining a similar relationship for the Total Square Integral in the discrete case. 4,5

The Total Square Integral for an $n$-th -order continuous system is given by Parseval's Theorem, as

$$
\begin{equation*}
I_{n}=\frac{1}{2 \pi j} \int_{-j \infty}^{j \infty} G(s) G(-s) d s, \tag{1}
\end{equation*}
$$

where it is assumed that

$$
\begin{align*}
G(s) & =\frac{B(s)}{A(s)}, \\
B(s) & =\sum_{i=0}^{n-1} b_{i} s^{i}, \\
A(s) & =\sum_{i=0}^{n} a_{i} s^{i}, a_{0} \neq 0, a_{n}>0 \\
& =a_{n} \prod_{i=0}^{n}\left(s-s_{i}\right), \tag{2}
\end{align*}
$$

and $A(s)$ has zeros $s_{i}$ in the left half-plane only. Using the method of Newton et al. ${ }^{2}$ we find that

$$
\begin{equation*}
I_{n}=\frac{P_{n-1}}{2 a_{n}} \tag{3}
\end{equation*}
$$

where $P_{n-1}$ is the solution of the $n$ equations for $P_{i}$, $i=0,1, \ldots n-1$;

$$
\begin{array}{ll}
\sum_{i=0}^{m} a_{m-i}(-1)^{i} P_{i}=d_{m} & \text { for } 0 \leq 2 m \leq n-1, \\
\sum_{i=m-n}^{n-1} a_{m-i}(-1)^{i} P_{i}=d_{m} & \text { for } n \leq 2 m \leq 2 n-1 \tag{4}
\end{array}
$$

and the $d_{m}, m=0,1, \ldots n-1$ are given by

$$
d_{m}=\left\{\begin{array}{ll}
\sum_{i=0}^{m}(-1)^{i} b_{i} b_{m-i} & \text { for } 0 \leq 2 m \leq n-1  \tag{5}\\
\sum_{i=m-n+1}^{n-1}(-1)^{i} b_{i} b_{m-i} \text { for } n \leq 2 m \leq 2 n-2
\end{array} .\right.
$$

This expression can be written more compactly as

$$
\begin{equation*}
d_{m}=\sum_{i, j=0}^{n-1}(-1)^{i} b_{i} b_{j}, \quad \text { where } i+j=2 m \tag{6}
\end{equation*}
$$

If we let $d=\operatorname{col}\left(d_{0}, d_{1}, \ldots, d_{n-1}\right)$,

$$
\begin{align*}
& p_{i}=(-1)^{i} P_{i}  \tag{7}\\
& \underline{p}=\operatorname{col}\left(p_{0}, p_{i}, \ldots, p_{n-1}\right),
\end{align*}
$$

and
then we can write Eq. (4) as the matrix equation

$$
\begin{equation*}
\underline{\Omega} \underline{p}=\underline{d} \tag{8}
\end{equation*}
$$

where $\underline{\Omega}$ is the matrix


Solving Eq. (8) for $p_{n-1}$ using Cramer's rule we have

$$
\begin{equation*}
p_{n-1}=\frac{|\underline{\Omega}|}{|\underline{\Omega}|} \tag{9}
\end{equation*}
$$

where $\underline{\Omega}_{1}$ is the matrix formed from $\underline{\Omega}$ by replacing its last column by the vector d.

So using Eqs. (3), (7), and (9) we have

$$
\begin{equation*}
I_{n}=\frac{(-1)^{n-1}}{2 a_{n}} \frac{\left|\underline{\Omega}_{1}\right|}{|\underline{\Omega}|} \tag{10}
\end{equation*}
$$

We will now investigate the form of $|\underline{\Omega}|$.
Perform the following manipulations on the matrix $\Omega$ Take the transpose

Interchange columns 1 and $n, 2$ and $n-1, \ldots$
Interchange rows 1 and $n, 2$ and $n-1, \ldots$
This will not introduce a change of sign in $|\underline{\Omega}|$, so we have


This is seen to be precisely $\Delta_{n}$, the $n$-th Hurwitz determinant for the polynomial $A(s)$. Hence,

$$
\begin{equation*}
|\underline{\Omega}|=\Delta_{n} \tag{ll}
\end{equation*}
$$

For a stable system $\Delta_{n}$ is strictly positive; that is, it never vanishes. So $|\underline{\Omega}|$ never vanishes, and from Eq. (10) this guarantees the existence of the integral.

Alternatively, using Orlando's formulas ${ }^{6}$ we have

$$
\begin{align*}
|\underline{\Omega}|=\Delta_{n} & =a_{0} \Delta_{n-1} \\
& =(-1)^{\frac{n(n-1)}{2}} a_{0} a_{n}^{n-1} \prod_{i=1}^{n} \prod_{j=1}^{n}\left(s_{i}+s_{j}\right),  \tag{12}\\
& i<j
\end{align*}
$$

where the $s_{i}$ are given by Eq. (2).
Our assumptions guarantee that $|\underline{\Omega}|$ be strictly positive.
In order to demonstrate the procedure for calculating $I_{n}$ we will first illustrate it by means of a sixth-order example, and then the general result for any order $n$ will be stated.

Given

$$
\begin{equation*}
G(s)=\frac{B(s)}{A(s)}=\frac{b_{0}+b_{1} s+b_{2} s^{2}+b_{3} s^{3}+b_{4} s^{4}+b_{5} s^{5}}{a_{0}+a_{1} s+a_{2} s^{2}+a_{3} s^{3}+a_{4} s^{4}+a_{5} s^{5}+a_{6} s^{6}}, \tag{13}
\end{equation*}
$$

then

$$
\begin{equation*}
I_{6}=\frac{(-1)^{5}}{2 a_{6}} \frac{\left|\underline{\Omega}_{1}\right|}{|\underline{\Omega}|} \tag{14}
\end{equation*}
$$

where

$$
\begin{aligned}
& |\underline{\Omega}|=\left|\begin{array}{llllll}
a_{0} & 0 & 0 & 0 & 0 & 0 \\
a_{2} & a_{1} & a_{0} & 0 & 0 & 0 \\
a_{4} & a_{3} & a_{2} & a_{1} & a_{0} & 0 \\
a_{6} & a_{5} & a_{4} & a_{3} & a_{2} & a_{1} \\
0 & 0 & a_{6} & a_{5} & a_{4} & a_{3} \\
0 & 0 & 0 & 0 & a_{6} & a_{5}
\end{array}\right| \\
& \left|\underline{\Omega}{ }_{1}\right|=\left|\begin{array}{lllll}
a_{0} & 0 & 0 & 0 & 0 \\
a_{2} & a_{1} & a_{0} & 0 & 0 \\
a_{4} & a_{3} & a_{2} & a_{1} & a_{0} \\
a_{6} & a_{5} & a_{4} & a_{3} & a_{2} \\
0 & 0 & a_{6} & a_{5} & a_{3} \\
0 & 0 & 0 & 0 & a_{6} \\
a_{4} \\
a_{4} \\
d_{5}
\end{array}\right|
\end{aligned}
$$

and

$$
\underline{\mathrm{a}}=\left[\begin{array}{r}
\mathrm{b}_{0}{ }^{2} \\
2 \mathrm{~b}_{0} b_{2}-b_{1}{ }^{2} \\
2 b_{0} b_{4}-2 b_{1} b_{3}+b_{2}{ }^{2} \\
-2 b_{1} b_{5}+2 b_{2} b_{4}-b_{3}{ }^{2} \\
-2 b_{3} b_{5}+b_{4}{ }^{2} \\
-b_{5}{ }^{2}
\end{array}\right]
$$

The numerator determinant can be expanded as follows:

$$
\begin{equation*}
\left|\Omega_{1}\right|=a_{0} d_{5} \Omega_{5}-a_{0} a_{6}\left(d_{4} \Omega_{4}-d_{3} Q_{3}+d_{2} \Omega_{2}-d_{1} \Omega_{1}\right)-a_{6} d_{0} \Omega_{0} \tag{15}
\end{equation*}
$$

where the $Q_{r}$ 's are given by

$$
\begin{array}{ll}
Q_{5}=\left|\begin{array}{llll}
a_{1} & a_{0} & 0 & 0 \\
a_{3} & a_{2} & a_{1} & a_{0} \\
a_{5} & a_{4} & a_{3} & a_{2} \\
0 & a_{6} & a_{5} & a_{4}
\end{array}\right| & Q_{4}=\left|\begin{array}{lll}
a_{1} & a_{0} & 0 \\
a_{3} & a_{2} & a_{1} \\
a_{5} & a_{4} & a_{3}
\end{array}\right| \\
Q_{3}=\left|\begin{array}{lll}
a_{1} & a_{0} & 0 \\
a_{3} & a_{2} & a_{1} \\
0 & a_{6} & a_{5}
\end{array}\right| \\
Q_{1}=\left|\begin{array}{llll}
a_{3} & a_{2} & a_{1} \\
a_{5} & a_{4} & a_{3} \\
0 & a_{6} & a_{5}
\end{array}\right|
\end{array}
$$

We notice that $Q_{5}$ and $Q_{0}$ are fourth-order determinants; however, by by inspection, we see that the following simple relationship exists.

$$
\begin{aligned}
& Q_{5}=a_{4} Q_{4}-a_{2} Q_{3}+a_{0} Q_{2} \\
& Q_{0}=a_{2} Q_{1}-a_{4} Q_{2}+a_{6} Q_{3}
\end{aligned}
$$

So we only have to expand the four third-order determinants $Q_{1}, Q_{2}$, $Q_{3}$ and $Q_{4}$.

By replacing the $d_{i}$ by the corresponding entries in $|\underline{\Omega}|$ we have

$$
\begin{align*}
|\underline{\Omega}| & =a_{0} a_{5} Q_{5}-a_{0} a_{6}\left(a_{3} Q_{4}-a_{1} Q_{3}\right) \\
& =a_{0}\left[a_{5} Q_{5}-a_{6}\left(a_{3} Q_{4}-a_{1} Q_{3}\right)\right] \tag{16}
\end{align*}
$$

So

$$
\begin{equation*}
I_{6}=\frac{a_{0} d_{5} Q_{5}-a_{0} a_{6}\left(d_{4} Q_{4}-d_{3} Q_{3}+d_{2} Q_{2}-d_{1} Q_{1}\right)-a_{6} d_{0} Q_{0}}{(-1)^{5} 2 a_{6} a_{0}\left[a_{5} Q_{5}-a_{6}\left(a_{3} Q_{4}-a_{1} Q_{3}\right)\right]} \tag{17}
\end{equation*}
$$

Generalizing this procedure for any $n$ we have

$$
\begin{equation*}
I_{n}=\frac{a_{0} d_{n-1} Q_{n-1}-a_{0} a_{n}\left[d_{n-2} Q_{n-2}-d_{n-3} Q_{n-3}+\ldots+(-1)^{n-1} d_{1} Q_{1}\right]+(-1)^{n-1} a_{n} d_{0} Q_{0}}{(-1)^{n-1} 2 a_{n} a_{0}\left[a_{n-1} Q_{n-1}-a_{n}\left(a_{n-3} Q_{n-2}-a_{n-5} Q_{n-3}+a_{n-7} Q_{n-4}+\ldots\right)\right]} \tag{18}
\end{equation*}
$$

where the $d_{i}, i=0,1, \ldots n-1$ are given by Eq. (6), the $Q_{i}, i=1,2, \ldots n-2$ are formed from $|\underline{\Omega}|$
by deleting the first, $(n-1)$ th and $n-$ th columns and the first $(r+1)$ th and n-th rows,
and finally $Q_{0}$ and $Q_{n-1}$ are given by

$$
\begin{gather*}
Q_{0}=a_{2} Q_{1}-a_{4} Q_{2}+a_{6} Q_{3}-a_{8} Q_{4}+\cdots  \tag{19}\\
Q_{n-1}=a_{n-2} Q_{n-2}-a_{n-4} Q_{n-3}+a_{n-6} Q_{n-4}-\cdots \tag{.20}
\end{gather*}
$$

So to calculate $I_{n}$ we only have to expand the $n-2$ determinants $Q_{i}$, $i=1,2, \ldots n-2$ of order n-3.

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