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CAPACITY AND MATCHED SIGNAL OF QUANTUM MECHANICAL CHANNELS: I *

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I. INTRODUCTION

The capacity C of a discrete communication channel is defined as the maximum value of the transmission rate R under the variation of the probabilities of input symbols $p_1 \dots p_m$, subject to a set of constraints

$$p_1 \geq 0 \quad p_2 \geq 0 \quad \dots \quad p_m \geq 0 \tag{1}$$

$$\sum_{i=1}^{m} p_i = 0. \tag{2}$$

In case of discrete memoryless channels, R is calculated by Shannon's formula

$$R(p) = \sum_j \sum_i p_i P_{ij} \log \left[\frac{P_{ij}}{\sum_k p_k P_{kj}} \right] \tag{3}$$

where P_{ij} is the probability that the transmitted symbol S_i is received as a symbol S'_j. We do not have to assume that the number of symbols accepted by the receiver is equal to the number of symbols used by the transmitter, i.e., that the matrix $||P_{ij}||$ is a square matrix.

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Shannon[1] has shown that the above maximization problem can be solved in a closed form, and the p_i and the capacity C_o are given respectively by

$$p_i = \sum (p^{-1})_{ji} \exp\left[\sum_h (P^{-1})_{jh} \sum P_{hk} \log P_{hk}\right] \qquad (4)$$

and

$$\exp C = \sum_j \exp\left[\sum_h (P^{-1})_{jh} \sum P_{hk} \log P_{hk}\right] \qquad (5)$$

where $(P^{-1})_{jh}$ is the (j-h) element of the inverse of the matrix $||P_{ij}||$.

It was later noted by several authors[2] that (5) is not always valid since (4) gives in many cases negative values which are of course unrealizable. In such cases, some of the p's must take the smallest possible value allowed by (1), namely zero, and the capacity can be obtained by calculating R for p's satisfying the condition that either

$$\frac{\partial R}{\partial p_i} - \lambda = 0 \text{ and } p_i \geq 0, \qquad (6a)$$

or

$$\frac{\partial R}{\partial p_i} - \lambda < 0 \text{ and } p_i = 0 \quad \text{for } i = 1, 2, \ldots, m. \qquad (6b)$$

Although it can be shown that (6) has a unique solution for p_i's when the matrix $||P_{ij}||$ is a nonsingular square matrix, actual computation of them is somewhat difficult and there seems to be no published work which actually solved (6), except in rather trivial cases.

In this paper we will give a method to solve (6) numerically instead of analytically. The method will then be applied to a certain type of discrete channels which resulted from a quantum theory of communication channels.[3] The matrix P_{ij} for this type of channel is given by

$$P_{ij} = \binom{i}{j} K^j (1 - K)^{i-j} \tag{7}$$

where K is a parameter. It can be shown that (7) gives the probability that i photons being sent out from the transmitter j photons actually come into the receiver, where K is the power attenuation factor of the channel, caused by a linear attenuation due to Ohmic losses and/or radiation into free space. It is assumed that the conductors and external space are at the absolute zero temperature so that no thermal photons enter the channel. It is to be noted that the formula (7) also results from a simple probabilistic model in which each photon has a definite probability of anihilation per unit path traversed. Our result will therefore be applicable to a rather wide class of communication channels where the number of some objects being transmitted is used to represent information.

The present work has been done with two objectives. First, we are interested in the value of the capacity of quantum mechanical channels with given power and attenuation, since it is to be a basic quantity which sets the ultimate limit of the information-carrying capacity of a given channel no matter what system of coding was used. Second, these channels serve as an excellent example of a noisy discrete channel on which we can study the general characteristics of matched input probabilities for noisy channels. The result of our numerical calculation actually showed a quite interesting behavior of such probabilities, which could hardly be anticipated without numerical calculation.

The numerical procedure used in the solution of maximization problem which is actually one of convex programming, is considered as a valuable by-product of this work, and will be described in detail.

II. METHOD OF SOLUTION OF THE MAXIMIZATION PROBLEM

If we ignore the constraints imposed by the inequalities (1), the maximum of (3) would be obtained by putting the partial derivatives of $R' = R + \lambda \sum p_i$, with respect to p_i, equal to zero, where λ is the Lagrange multiplier which is to be determined from the condition

$$\sum p_i = 1 . \tag{8}$$

Hence we have,

$$\frac{\partial R'}{\partial p_i} = \frac{\partial R}{\partial p_i} + \lambda = \sum_j P_{ij} \log \frac{P_{ij}}{\sum p_k P_{kj}}$$

$$- \sum_j \sum p_k P_{kj} \frac{P_{ij}}{\sum p_k P_{kj}} + \lambda = \sum_j P_{ij} \left(\log \frac{P_{ij}}{\sum p_k P_{kj}} - 1\right) + \lambda = 0$$

$$= \sum_j P_{ij} \log P_{ij} - \sum_j P_{ij} \log \sum p_k P_{kj} + \lambda - 1 = 0 . \tag{9}$$

If we solve the system of linear equations

$$\sum_j P_{ij} X_j = \sum_j P_{ij} \log P_{ij}, \tag{10}$$

we have

$$\log \sum_k p_k P_{kj} = X_j + \lambda - 1 \tag{11}$$

or

$$\sum_k p_k P_{kj} = e^{\lambda-1} e^{X_j} \tag{12}$$

where λ is determined by

$$e^{-(\lambda-1)} = \sum_j e^{X_j} , \tag{13}$$

and p_k may be obtained by solving (12).

Actual calculations have shown that for larger K values (12) gives all positive p's, so that this method gives a satisfactory solution to the problem. It is found, however, that for the values of K smaller than a critical value K_c, which depends on m, the maximum number of transmitted photons (cf. Fig. 2 .), some of the p's obtained from (12) become negative, and in that case some of the p's must take the value 0, the smallest physically realizable value. For these vanishing p's, the corresponding equations in (10) should be dropped, so that (10) determines X_j only incompletely. This ambiguity is made use of in solving (12) since there are in (12) less variables than there are equations; and it would be impossible, without such an ambiguity, to solve (12).

However, a solution of such mixed equations is found to be rather difficult since it leads to a transcendental equation.

For this reason, we decided to use a direct step by step method of searching the maximum of $R(x_1 \ldots x_n)$. There are three known algorithms for such a maximum search, namely

1. Primitive regula falsi method: Successive values of the variables determined using only the values of R. (zero-order process)

2. Steepest ascent method: Successive values of variables are determined using the values of the first derivatives of R. (first-order process)

3. Quadratic approximation: Successive values of variables are determined using the values of the second derivatives of R. (second-order process)

For well-behaved functions, the speed of convergence increases with the order of the process, while the slower processes are more adaptable to ill-behaved functions than faster ones.

We actually decided to use method 2 on the following grounds:

1. Our function is sufficiently well-behaved (R is known to be convex) so that we can exclude the very slow process of method 1.

2. The first derivatives of R are obtained as an intermediate product in the calculation of R.

3. Method 3 involves a solution of a system of linear equation in each step, so that it may not be much faster than method 2 even in well-behaved cases.

4. Method 3 is more difficult to incorporate the constraints (1) than method 2.

The steepest ascent method of maximizing a function $F(x_1 \ldots x_n)$ consists of an iterative process.

$$x_i^{(r)} = x_i^{(r-1)} + \epsilon_r F_i (x^{(r-1)}) \tag{14}$$

where F_i denotes the partial derivative of F with respect to x_i. It is well known that the speed of convergence depends on the proper choice of "step size" ϵ. If ϵ is too small, the convergence would be uniform but very slow. If it is too large, x's will be "overcorrected" so that convergence becomes oscillatory or, more often, the process would diverge. The best value of ϵ in general depends on the second derivatives of F, which is not available to us by assumption. We will therefore find some other method whereby we can find the optimum or near-optimum ϵ value in each step from available data.

After trying out several other methods, it was decided to use two successive values of the first derivatives F_i, which were used in (14), as a clue to determine the step size, based on the contention that the relation of the gradient at two points will give some pertinent information as to the curvature of the surface. Actually we calculate ϵ successively from the forgoing ϵ by a formula

$$\epsilon_{r+1} = \frac{\delta_{r-1,r-1}}{\alpha\delta_{r-1,r-2} - \delta_{r-1,r-1}} \epsilon_r \tag{15}$$

where δ_{rs} is defined by

$$\delta_{rs} = \sum_i F_i^{(r)} F_i^{(s)} \tag{16}$$

and α is a positive parameter which is greater than one and which was chosen empirically so as to get the best result. Values between 1.2 and 1.5 were used with satisfactory results. It is important to note that (15) adjusts the value of ϵ only relatively to its previous value, so that our scheme is scale-independent, i.e., if it works successfully in a certain case, it may also work as well when the scale of x's is changed by a constant factor.

The formula (16) does not have a precise theoretical foundation. The following argument, however, will be convincing enough to justify the use of the formula.

Let us assume that F can be approximately described as a quadratic polynomial in $x_1 \ldots x_n$ so that

$$F = F_0 + \sum_i a_i x_i + 1/2 \sum_{ij} b_{ij} x_i x_j. \tag{17}$$

For simplicity, we use a coordinate system whose origin is at the maximum point of F and the axes are directed to the principal axes of the quadratic part of (17). We then have,

$$F = F_{max} - 1/2 \sum_i C_i \; \xi_i^2 \tag{18}$$

where C_i's are non-negative in our problem on account of convexity of the entropy functions, and they are all positive if the matrix $||P_{ij}||$ is a nonsingular square matrix. Then we have

$$F(x^{(r)}) = F_{max} - 1/2 \sum C_i \; \xi_i^{(r)^2} \tag{19}$$

$$F_i(x^{(r)}) = - C_i \; \xi_i^{(r)} \tag{20}$$

so that, from (14), we have

$$\xi_i^{(r+1)} = \xi_i^{(r)} - \epsilon \; C_i \; \xi_i^{(r)} = (1 - \epsilon \; C_i)\xi_i^{(r)} \tag{21}$$

$$\delta_{r-1, r-1} = \sum_i^+ \; C_i^2 \; \xi_i^{(r-1)^2} \tag{22}$$

$$\delta_{r-1, r-2} = \sum^+ \; (1 - \epsilon C_i) \; C_i^2 \; \xi_i^{(r-1)^2} \; . \tag{23}$$

$\sum^+$ means that summation is taken over those i's for which $p_i > 0$. Now, let us make the following assumption on ξ_i and C_i. Assume that the values of the subscript i falls in either of the two groups. For the i in the first group, all C_i's have practically the same value, say C, while corresponding ξ_i's are arbitrary. For the i in the second group, C_i's are arbitrary, but ξ_i's in this group are small compared to the ξ_i's in the first group. Then we can write (22) and (23) as

$$\delta_{r-1, r-1} = C^2 \sum^1 \xi_i^{(r-1)^2} \tag{24}$$

$$\delta_{r-1, r-2} = (1 - C) C \sum^1 \xi_i^{(r-1)^2} \tag{25}$$

so that we have

$$\frac{\delta_{r-1, r-1}}{\delta_{r-1, r-2} - \delta_{r-2, r-2}} = \frac{1}{1 - (1 - \epsilon C)} = \frac{1}{\epsilon C} . \tag{26}$$

Hence, if we put $\alpha = 1$ in (15), we obtain

$$\epsilon' = \frac{\delta_{r-2, r-1}}{\delta_{r-1, r-2} - \delta_{r-2, r-2}} \quad \epsilon = \frac{1}{C} \tag{27}$$

so that, for the i's in the first group

$$\xi_i^{(r)} = (1 - \epsilon' C) \xi_i^{(r-1)} = 0 \tag{28}$$

while for the i's in the second group ξ_i's are small from the outset. That is, under the assumption made above, we get at the correct ξ_i in a single step if we use the rule (15) with $\alpha = 1$.

Actually, the assumption may not be satisfied, and also the functional form is not strictly quadratic, so that the process must be repeated to obtain the maximum. These facts would sometimes result in an overcorrection, expecially when the ratio in (15) becomes very large. It is rather easy, however, to prevent such effects from being catastrophic, since we can program in such a way that, if a trial fails, that is, if it turns out that

$$F\left(x^{(r-1)}\right) \leq \left(x^{(r)}\right) \tag{29}$$

and that there is no improvement, this trial is entirely discarded and a new set of values of $x^{(r)}$ is tried using a value of ϵ which is smaller than the previous unsuccessful one by a fixed factor. Actually ϵ is reduced to 1/4 of the previous value. When this trial again fails, ϵ is reduced again by the same factor, and so on.

The correction factor α in the rule (15) was added for the same purpose of preventing a too drastic change of ϵ. In fact, a larger value of α will be effective in preventing overcorrection, while it tends to slow down the convergence in more favorable cases, that is, in the case that C_i's in (18) are relatively uniform. In our program a value $\alpha = 4/3$ was chosen as the standard, but it can be specified in the data card if desired.

The convergence is fairly rapid in favorable cases, for example, accuracy of 10^{-7} in R and 10^{-3} in p_i is obtained with 7 iterations. In very unfavorable cases, it took more than 200 to get an accuracy of 10^{-5} in R and 10^{-3} in p_i.

III. CONVERGENCE OF THE PROCESS

Actual mode of convergence of the above iterative process is of some interest. Convergence is very fast when the curvature of the F surface is nearly isotropic (i.e., C_i's in (18) have all approximately the same values), as is evident from the forgoing discussion. The worst case occurs when the curvature of F surface is strongly anisotropic (C_i's are spread out), in which case the F surface has an appearance of a deep valley in the case of two dimensions. If we take a small ϵ, small enough to assure stability against large C_i (against sidewise displacement), convergence in the directions with small C_i (speed of rolling down the valley) would be too small. If on the other hand, we take a larger ϵ so as to get a higher speed, sidewise instability would set in. What actually occurred in our process with an automatic adjustment of ϵ according to (15) is a curious oscillatory change of ϵ, which goes on as follows (Table I).

Let us assume that ϵ is small at first. Then the convergence is appreciable only in the steep directions and, after several cycles, these 'steep' variables would become fairly small so that we are at the bottom of the valley. This means that the condition assumed in deriving (15) is now approximately satisfied, and causes a rapid increase of ϵ since ϵ is now determined principally by the curvature in flat directions. This increase of ϵ however, causes instability in steep directions, while it accelerates the convergence in flat directions. This instability is not significant at first since "steep" variables are first very small, but they gradually increase and at last cause a rather catastrophic instability, running up the 'side' of the valley. This instability is detected and an automatic reduction of ϵ is made, and the process is repeated, ϵ going up and down repeatedly.

This oscillatory behavior of ϵ may at first glance seem an undesirable feature. It will be shown, however, that this is exactly the way we can get the best convergence feasible with a steepest ascent process. To show this, let us assume the quadratic relation (18), and denote the value of ϵ in each step by $\epsilon^{(r)}$. Then, the effect of n successive steps of iteration would be written as

$$\xi_i^{(m+n)} = \prod_{r=m+1}^{m+n} (1 - \epsilon^{(r)} C_i) \cdot \xi_i^{(m)} \qquad (30)$$

and we will have to make

$$\gamma_i = \prod_{r=m+1}^{m+n} (1 - \epsilon^{(r)} C_i) \qquad (31)$$

small for every i. We have to solve a kind of minimax problem where $\underset{i}{\mathrm{Max}}(\gamma_i)$ is to be minimized. If we regard (31) as a polynomial in C_i, i.e., if we define a function

$$\gamma(x) = \prod \left[1 - \epsilon^{(r)} x\right]. \qquad (32)$$

Then, the problem can be replaced by a problem of choosing $\epsilon^{(r)}$ so that $\gamma(x)$ shows a minimax behavior over an interval where C_i's are distributed. This problem has a well known solution using a Chebyshev polynomial, and the result is that $\epsilon^{(m+1)}, \epsilon^{(m+2)}, \ldots, \epsilon^{(m+n)}$ must be equal to $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ given by

$$\alpha_r = \frac{a+b}{2} + \frac{a-b}{2} \cos \frac{(2r+1)\pi}{2n}, \qquad (33)$$

taken in any order. In other words, the best convergence can be attained by a successive application of the operation (21) with different values of ϵ, whose distribution is given by (33). Of course we do not know anything about how well our rule of changing the value of ϵ approximates this optimum distribution, but it is almost certain that this process gives better results than any other method in which ϵ is fixed at a certain value.

IV. CONSTRAINTS

Thus far we have considered that all variables $x_1 \ldots x_m$ are independent. In our actual case we must take account of the constraints due to the relation

$$\sum p_i = 1 \tag{34}$$

$$p_i \geq 0. \tag{35}$$

The first constraint can best be taken care of using a Lagrange multiplier, and considering p_i's as if they are independent. We then have

$$p_i^{(r+1)} = p_i^{(r)} + \epsilon \left(\frac{\partial R}{\partial p} - \lambda\right) \tag{36}$$

where λ should be determined so that

$$\sum p_i^{(r+1)} = 1 \,. \tag{37}$$

In fact, λ will be given by

$$\lambda = \frac{1}{m} \sum_{i=1}^{m} \frac{\partial R}{\partial p_i} \,. \tag{38}$$

The second constraint does not cause any serious complication either. In this case we do not have to use Lagrange multipliers since the constraint applies on single variables. When any of the variables p_i turns out to be negative, we simply put it equal to zero, while variables having positive values may be adopted as they are. This however, is a slight oversimplification, and actually, we have to take account of the first constraint (34) together. Since the relation (36) is satisfied only in the case that $p_i^{(r+1)}$ is positive, we cannot use the Lagrange multiplier given by (38). Hence, we instead determine λ so that

$$\sum_i^{+} p_i^{(r+1)} = 1 , \tag{39}$$

the summation being extended over all i with $p_i^{(r+1)} > 0$. Since the value of λ also affects the positiveness of $p_i^{(r+1)}$'s this process is recursive, and must be repeated using new $p_i^{(r+1)}$ and testing the signs until self-consistency is established.

V. TERMINATION

Iteration is terminated if either

$$\delta_{rr} \leq 5.10^{-7} \tag{40}$$

or

$$R^{(r+1)} \leq R^{(r)} \quad \text{with } \epsilon < 1/m^{+} . \tag{41}$$

where m^{+} is the number of positive-valued p_i's. The first condition simply means that the gradient is small enough. The latter condition is added to avoid a situation in which ϵ is indefinitely reduced according to the rule, while $R^{(r+1)} \leq R^{(r)}$ is always satisfied due to round-off errors.

It is easily seen that the p_i's thus obtained actually satisfy conditions (6) for maximizing (3) under the given constraints. The exit condition (40) or (41) establishes that the derivatives of R in the 'free' variables are approximately zero, while for the remaining variables $\partial R/\partial p_i + \lambda < 0$, since they are those satisfying

$$p_i^{(r)} + \epsilon \left(\frac{\partial R}{\partial p_i} + \lambda\right) < 0 \tag{42}$$

and

$$p_i^{(r)} \geq 0. \tag{43}$$

VI. RESULTS AND DISCUSSION

Fig. 1 shows the result of the calculation of C for the quantum-mechanical channel with $m = 2 \sim 11$ and 29. Here the subscript for p_i is supposed to run from 0 to m. Each of these curves shows a fairly simple monotone dependence on K, having the same general pattern, with the exception of the case $m = 2$. The curve first rises almost linearly but with a slightly decreasing slope, until it reaches an inflection point. After the inflection point, it bends upwards and ends up at $C(1) = \log m$ with a rather high but finite slope. The steepness of the curve at $K = 1$ shows that even a slight loss of photons in the channel causes considerable loss of the information capacity. In the case $m = 2$, the curve has no inflection point and is concave upwards everywhere.

Much more interesting is the behavior of the input probabilities p_i that maximize $R(p_0 \cdots p_m)$, shown in Fig. 2a-2j. From these figures we can see the following general characteristics of these quantities as function of K.

1. The probabilities of the lowest and the highest signal levels, p_0 and p_m, are always positive.

2. All probabilities except the above two have a value 0 for a certain interval or intervals of K. In particular, all of these vanish for small values of K. None of these vanish, on the other hand, if K is greater than a critical value K_c, which depends on m. That is, the explicit formula (5) for C is valid within the domain $K_c \leqq K \leqq 1$.

For all K values outside this domain, one or more p_i's have a value 0, which means that some of the $m + 1$ different levels are actually not used in the signal, since use of these levels does not contribute to increase the channel capacity. To express this situation, we will say that the transmitted (input) signal is degenerate. The degeneracy of the transmitted signals becomes higher as K is decreased.

3. The dependence of each p_i on K is rather irregular. It is not even a smooth curve, but has several breaks, i.e., discontinuities of the slope. These breaks are closely related to the degeneracy. More specifically, they occur at the values of K at which one of the curves impinges upon the horizontal axis $p_i = 0$. The slope of this particular curve changes abruptly to zero at this point owing to the constraint $p_i \geqq 0$, and the discontinuity of slope of this particular p_i apparently induces similar discontinuities of the slopes of all other p_i's.

From all these results, it is clear that the constraints imposed by the positiveness of p_i's are really important, and that the explicit formulas (4) and (5), which ignore these constraints, are not valid except in rather special cases.

In order to get a better overall picture of this seemingly complicated situation, maps have been drawn that show the regions of K - i plane where $p_i(K)$ takes a positive value. Figs. 3 and 4 show such maps for m = 2 ~ 11 and 29. Since i is a discrete variable, each value of i is represented by a strip of a finite breadth. These maps, especially those for larger m, will show the general characteristics of the degeneracy of matched signals in a most impressive manner.

The following additional facts are recognized by looking at these maps.

4. Signal levels with positive probabilities (active levels) are not distributed at random, but form one or more coherent bands. That is, several p_i's having consecutive integers as subscripts tend to have positive values, and p_i's in between these bands vanish. More precisely, $p_i(K)$ tends to have more or less continuous dependence on i.

5. As K is varied starting from 0, one such band appears at a certain value of K. It then broadens and splits into two bands, a new band appears between these two, this latter again broadens and splits, and so on. This general behavior of the distribution of

active levels as function of K is somewhat obscured by the discrete nature of the "variable" i, but it is not hard to conjecture from these maps the limit for $m \to \infty$ where Km and i/m remain finite. Fig. 5 shows a possible form of the map in the above limit.

The limit $m \to \infty$ with finite Km is of practical importance because it corresponds to a communication over a channel of very high attenuation so that the transmitted signals are virtually continuous, but received signals are so weak that single photons must be counted by using a photon counter.

It is interesting to interpret the above results in terms of the "quantization" of the transmitted signals. ("Quantization" in quotation marks denotes that we mean quantization in the conventional meaning of the word used in communication systems, rather than in its original meaning in quantum mechanics.) The results in Fig. 5 show that when the received signal is very weak ($Km < 3$), a communication system with the greatest transmission rate is one using only two kinds of symbols: no signal and maximum power signal, i.e., operating in the on-off mode. This confirms the adequacy of our current practice of using on-off signals (telegraph, PCM., etc.) in low signal-to-noise ratio situations.

In the adjacent range of values of Km ($3 < Km < 8$) we have a system with 3 distinct signals. Strictly speaking, the new signal level appearing somewhere between the two extreme levels ($i = 0$ and $i = m$) is not a single level, but a band consisting of a large number of consecutive levels. In other words, we have here a peculiar mixed communication system using both discrete and continuous signals, possibly a new concept in communication systems! While this seems to be an attractive possibility, it might be quite misleading. It can be shown, by numerical calculation, that, in general, the value of the rate function $R(p_0 \cdots p_m)$ is not substantially reduced when the band of levels in the middle is replaced by a single level, having a probability equal to the sum of the probabilities of all levels in the band. From Table I which gives the function

Table I. Comparison of the information rates for optimum and approximately optimum probabilities

m = 29

K	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	R
.19				.0069	.2034					0.77947
					.2104					0.77923
.20				.0434	.1783					0.79788
					.2218					0.79759
.21				.0645	.1681					0.81570
					.2327					0.81536
.22				.0906	.1506					0.83311
					.2413					0.83255
.23				.1120	.1376					0.84974
					.2497					0.84915
.24				.1290	.1282					0.86586
				.2573						0.86521
.25			.0072	.1406	.1160					0.88128
				.2639						0.88074
.26			.0249	.1343	.1027	.0059	.0025			0.89643
				.2706						0.89567
.27		.0660	.0325	.0835	.0196	.0097	.0250	.0497		0.91126
		.1400					.1465			0.91048
				.2865						0.90959
.28		.1522				.0196	.0738	.0540		0.92586
		.1522					.1476			0.92553
				.2999						0.92292
.29	.0226	.1435					.0231	.1221		0.94034
		.1662						.1453		0.93998
.30	.0844	.0829						.1388	.0176	0.95490
	.1673							.1564		0.95451

$R(p_0 \dots p_m)$ for both the optimum signal and a 3-level or a 4-level discrete signal, we see the difference is less than 0.0008, which is quite insignificant. This fact implies that, for all practical purposes, these mixed type signals are equivalent to certain "quantized" (discrete) signals. From Fig. 5 we see that a 3-level quantization gives maximum capacity for $3 < mK < 8$, 4-level quantization for $8 < mK < 11.5$ etc.

These results seem to be of special importance from the information theory point of view. Quantization of signal levels, a well-established technique which is useful in signal transmissions over noisy channels, is here obtained by a purely mechanical procedure of finding a signal distribution that maximizes the theoretical transmission rate. Our results, moreover, give the optimum choice of quantization levels. It is true that these results are valid only for a very specific form (7) of the channel matrix $||P_{ij}||$, and little is known about the validity of our conclusions in more general cases. It is also true that there is no definite relationship between the theoretical transmission rate for a signal ensemble and its usefulness in a practical communication systems. Nevertheless, these results, as well as results of similar calculations on other channels, are believed to provide a valuable guidance in the design of communication systems.

An important question which we will try to anwer is whether the "quantization" of optimum signals is a general property of matched signals of various types of noisy channels, or it is more or less peculiar to our quantum-mechanical channel given by (7), and, if the latter is true, to what factor it depends. We will not be able to give a complete answer to this question, but the following comments will give us at least a partial answer to it.

As the first comment, we can point out that there exists an important class of channels, the cyclic channels, for which nothing like the "quantization" occurs. A cyclic channel is defined as a channel whose matrix elements P_{ij} depend only on $i - j (\mathrm{mod}\ m)$, where m is the number of symbols. It is easy to show that, owing to its high symmetry, the maximum rate is realized by a uniform distribution $p_i = 1/m$, $i = 1, 2, \ldots m$.

Secondly, we consider another class of channels closely related to the cyclic channels: convolutional channels, defined by the matrix elements P_{ij} which only depends on the difference $i - j$. This class of channels is important because in the limit $m \to \infty$ these channels become channels with additive noises. Fig. 6 shows the results of calculation for convolutional channels whose matrix elements are given by binomial coefficients.

$$P_{ij} = \binom{a}{j - i} 2^{-a} \tag{44}$$

This form of P_{ij} was chosen because it is in some sense an approximation for a Gaussian noise, and its limit for $a \to \infty$ is exactly the Gaussian function.

Unlike cyclic channels, convolutional channels can have degenerate input distributions. These data show an oscillatory change of p_i with i, which is damped out as one goes away from the boundary $i = 0$. The amplitude of this oscillation is big enough near the boundary so that some p_i's become zero. This example suggests that degeneracy of input distributions may result from the boundedness of the input signal levels.

There is another factor that can cause degeneracy of input signals in some channels. The functional form of $R(p_1, p_2, \ldots p_m)$ is such that it contains p_i's only in the form of linear combinations $q_j = \sum_i P_{ij} p_i$. This has the consequence that the vector (p_i) maximizing R is not uniquely determined if the rank m_o of the matrix $||P_{ij}||$ is less than m, the number of the independent variables (i.e., if $||P_{ij}||$ is degenerate in case $||P_{ij}||$ is a square matrix). It is further shown that we can find the vector (p_i) so that only m_o of the m components are nonvanishing. Since the vanishing of a certain component p_i occurs over a certain interval of a parameter, such as K, it is most probable that some of the p_i's vanish even when the matrix $||P_{ij}||$ is approximately degenerate. We know that this is the case for our P_{ij} when K is small, because the values of P_{ij}'s for a large j are all very small. This will account for the degeneracy of the input distribution p_i when K is small.

As a conclusion, we may say with a fair certainty that the high degree of degeneracy of the input distribution as seen in our results is neither a very specific property of the P_{ij} for our quantum mechanical channel, nor a general property of any channel, but is a cumulative effect of the boundedness of i and the approximate degeneracy of our $||P_{ij}||$.

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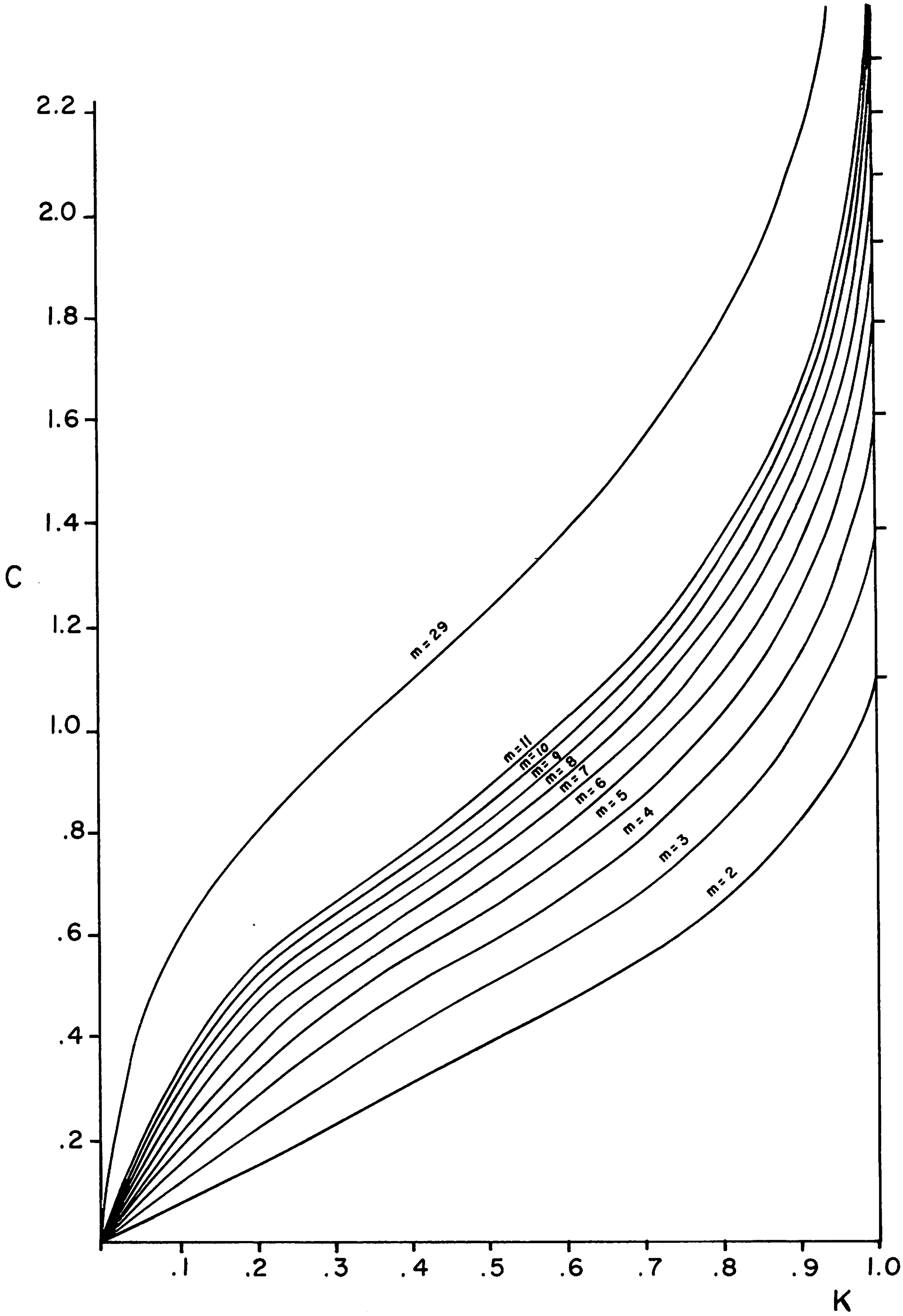

Fig. 1. Channel capacity as function of K.

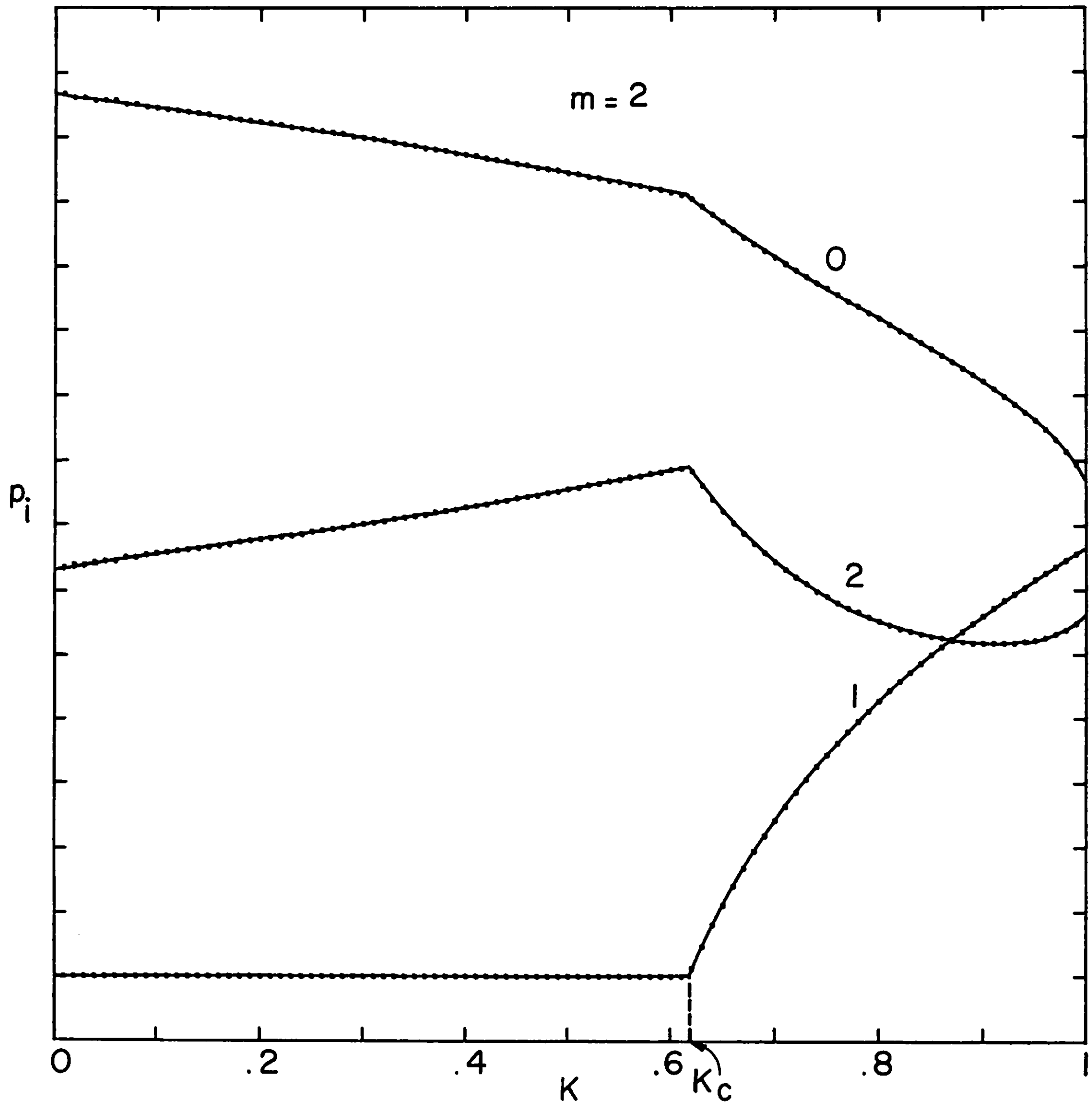

Fig. 2a (b, c, d, e, f, g, h, i, j). Symbol probabilities of matched signals p_i as function of K. Origins are shifted to avoid confusion. Flat bottoms of the curves correspond to $p_i = 0$.

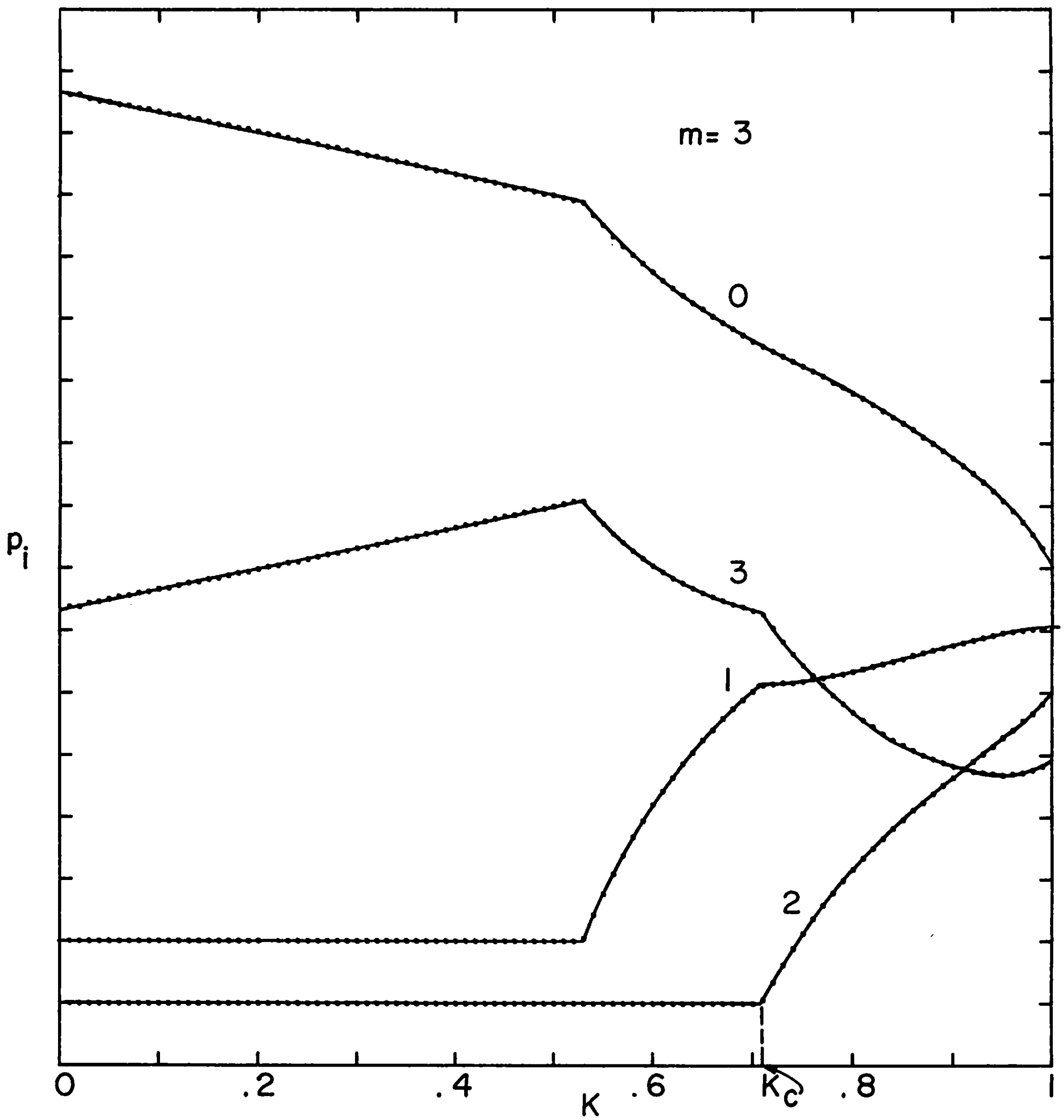

Fig. 2b.

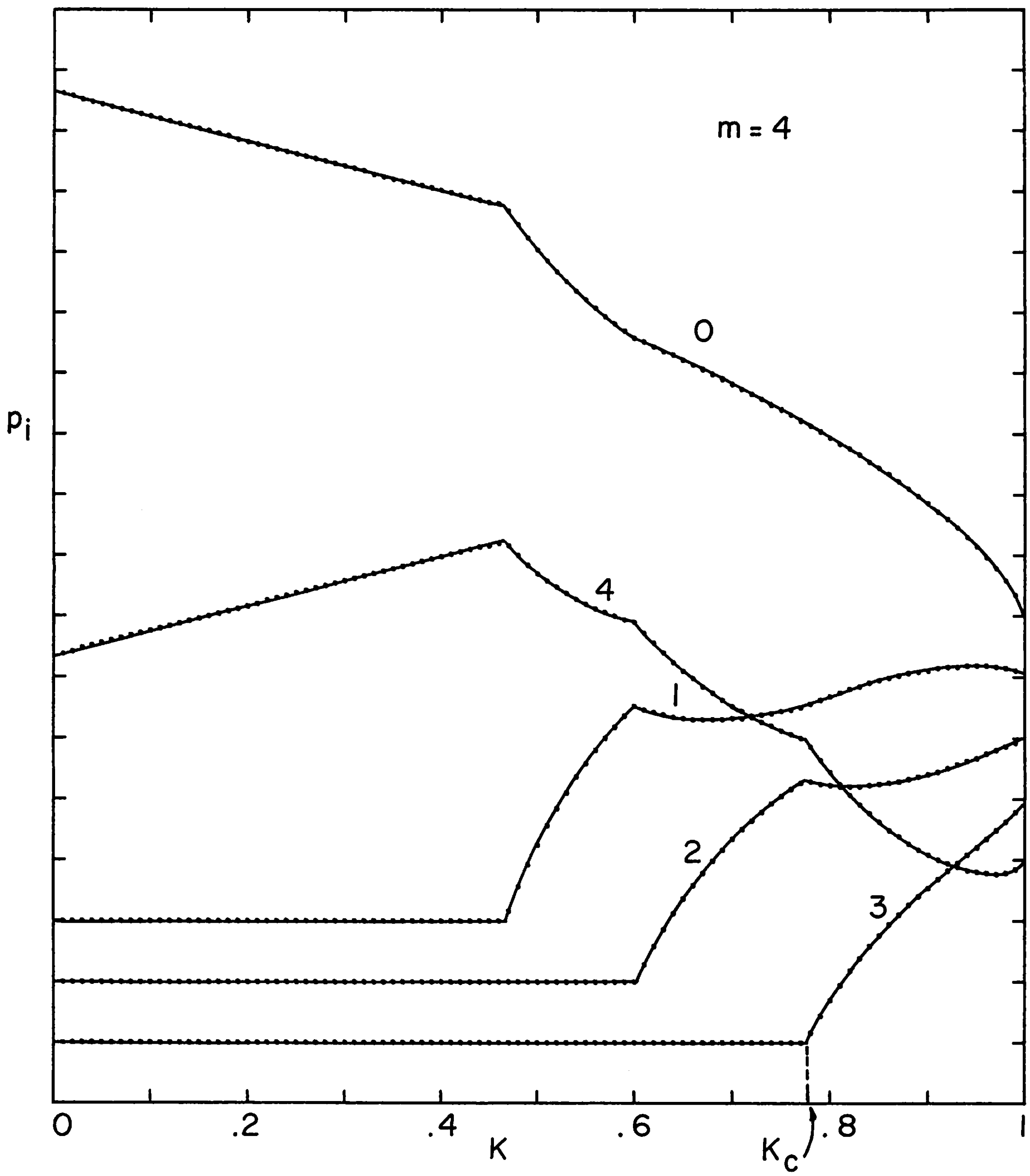

Fig. 2c.

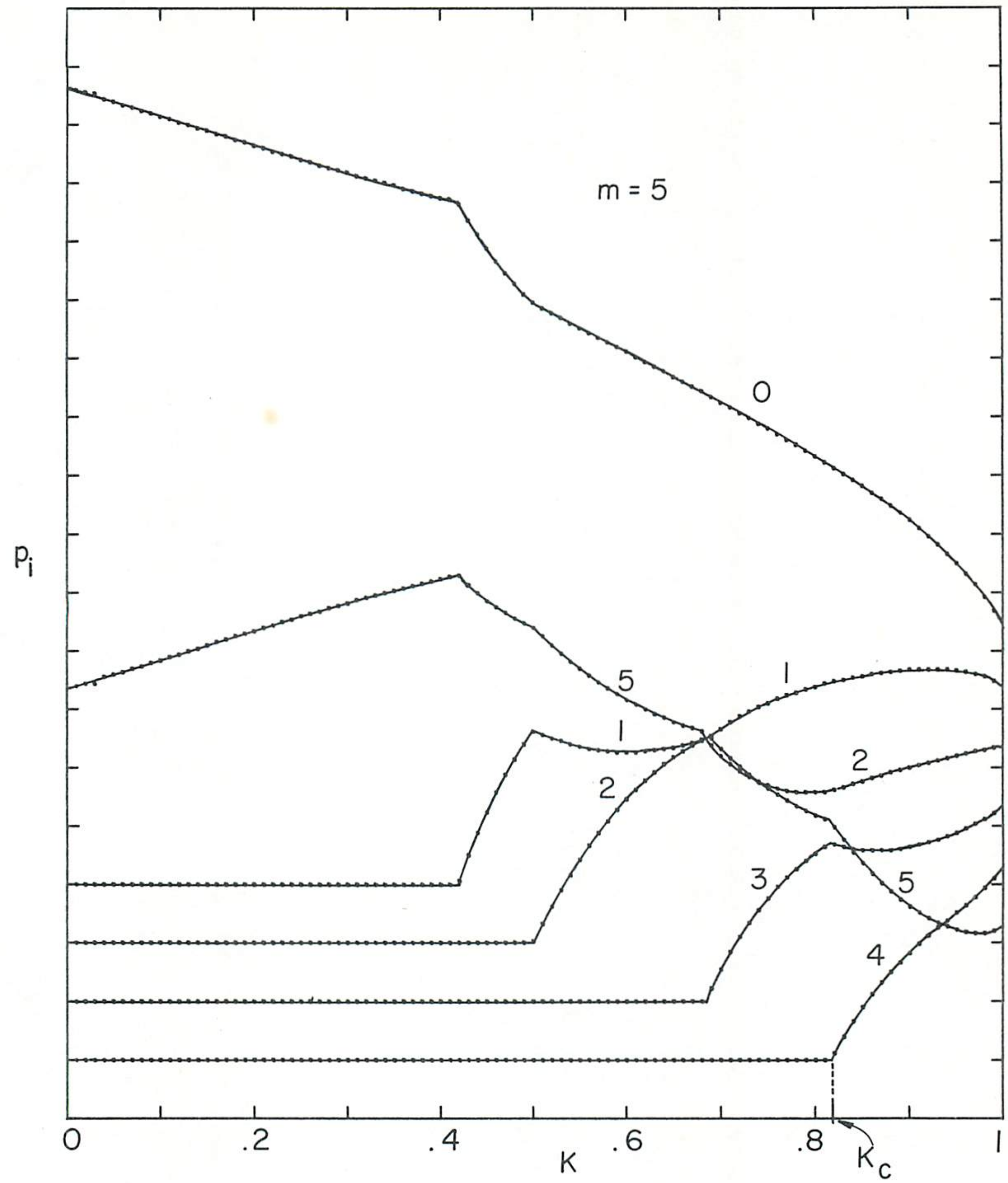

Fig. 2d.

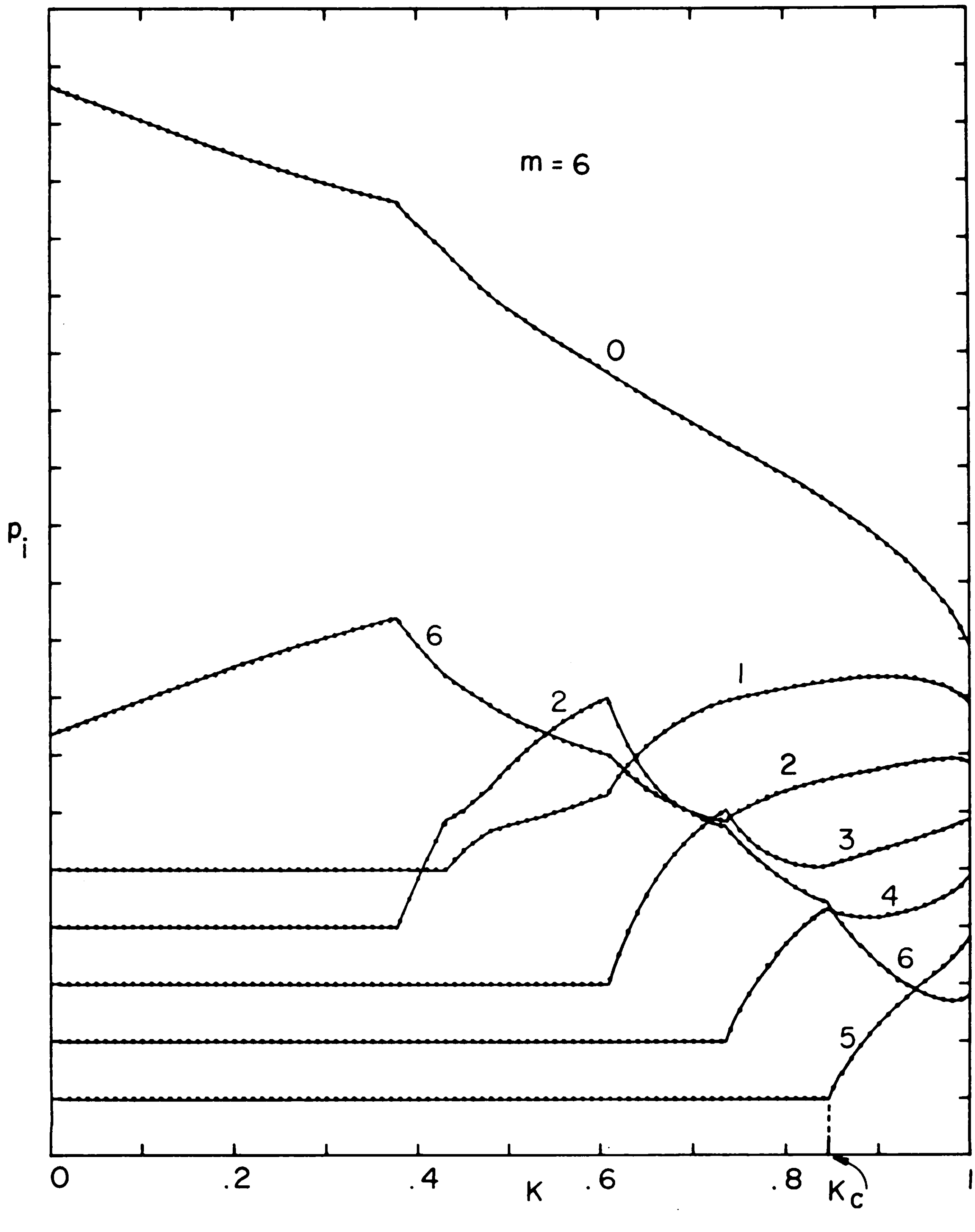

Fig. 2e.

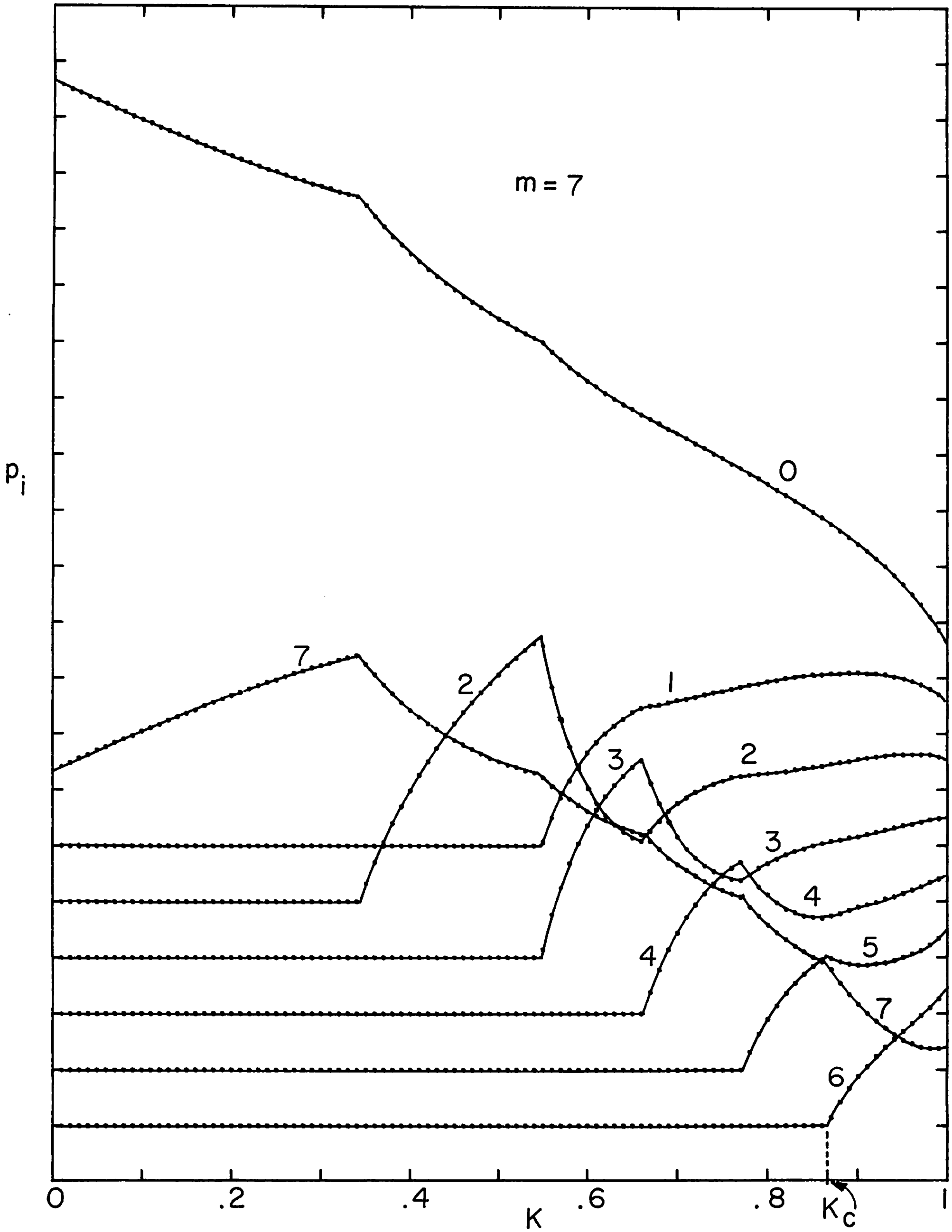

Fig. 2f.

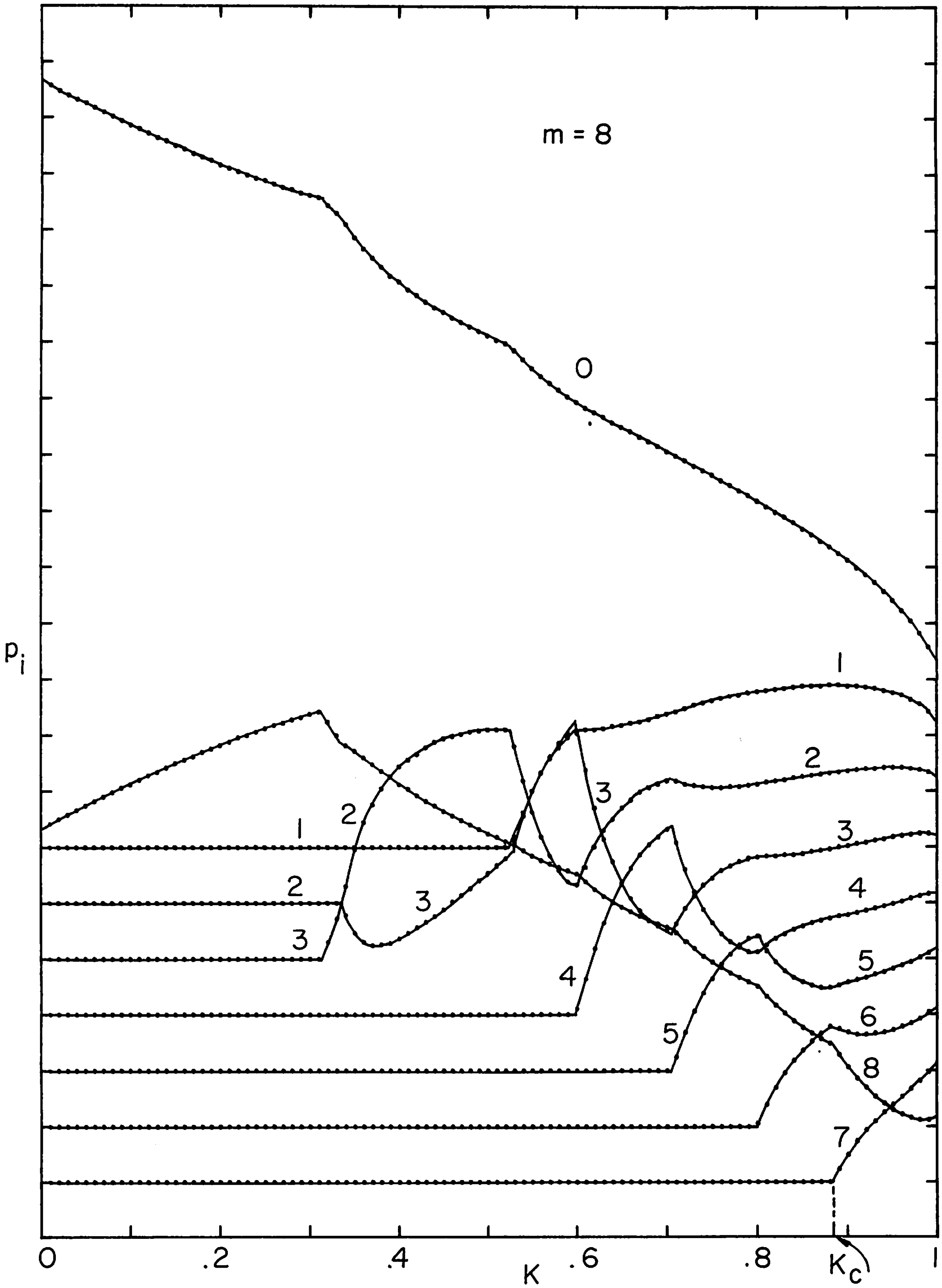

Fig. 2g.

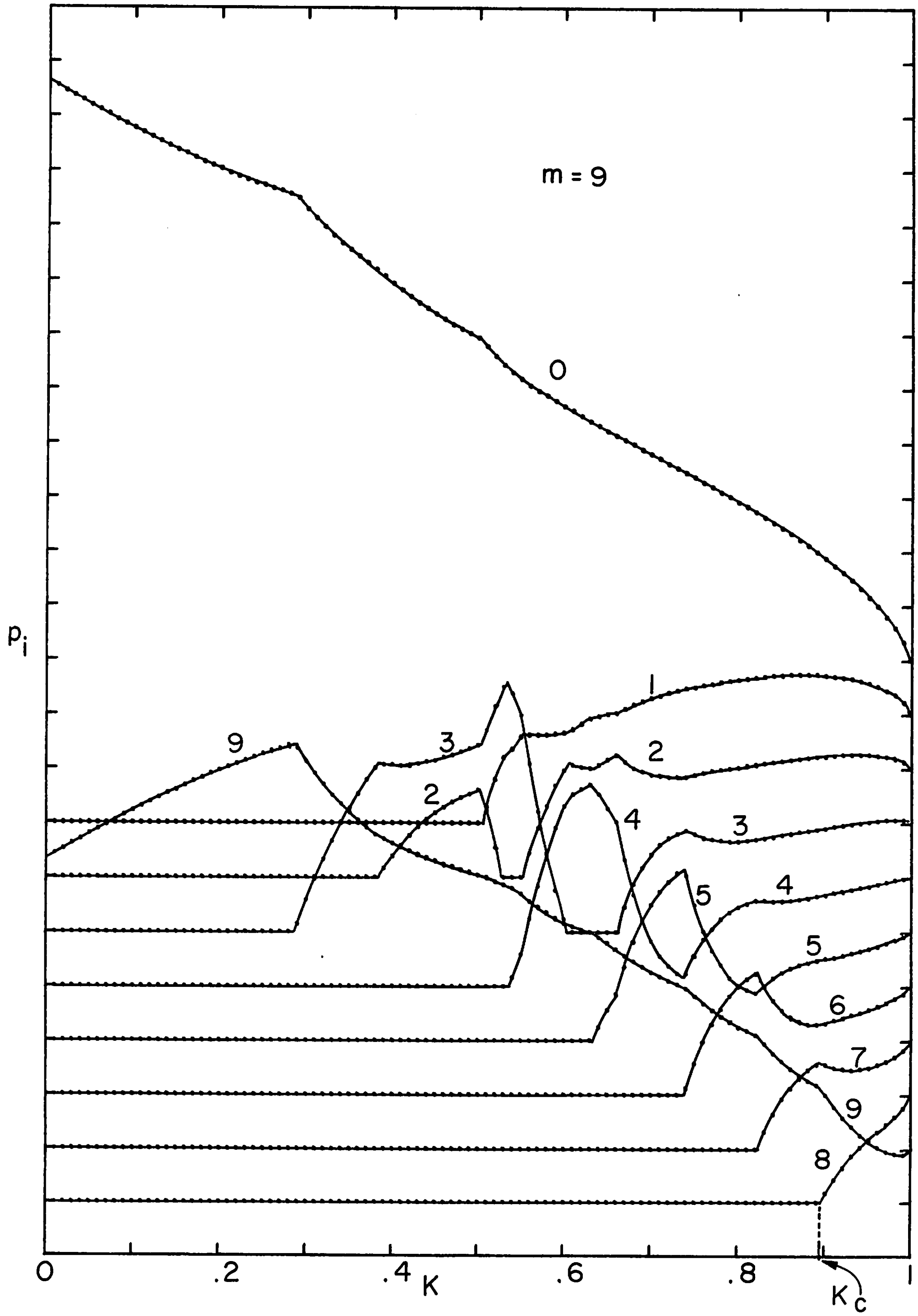

Fig. 2h.

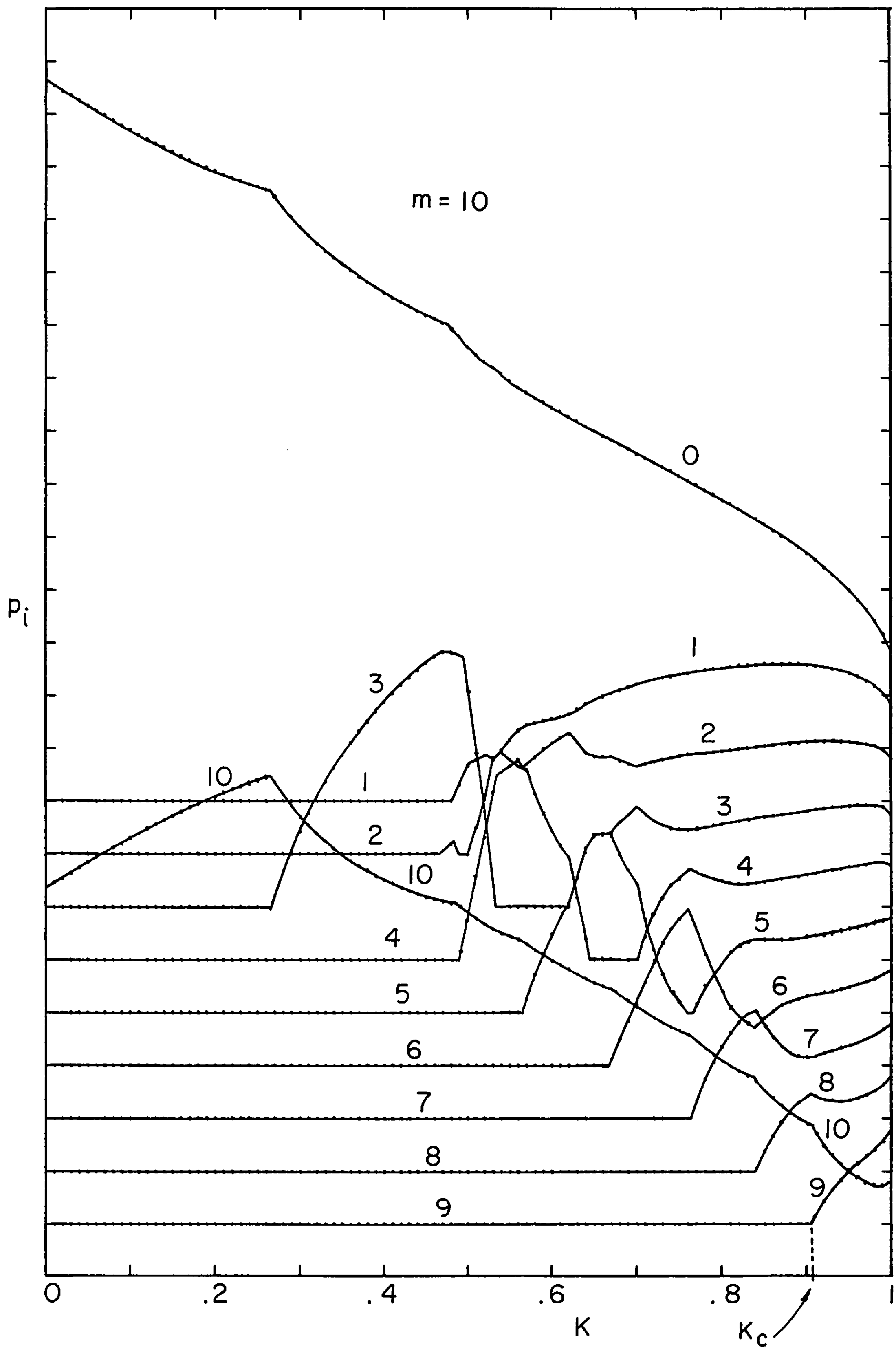

Fig. 2i.

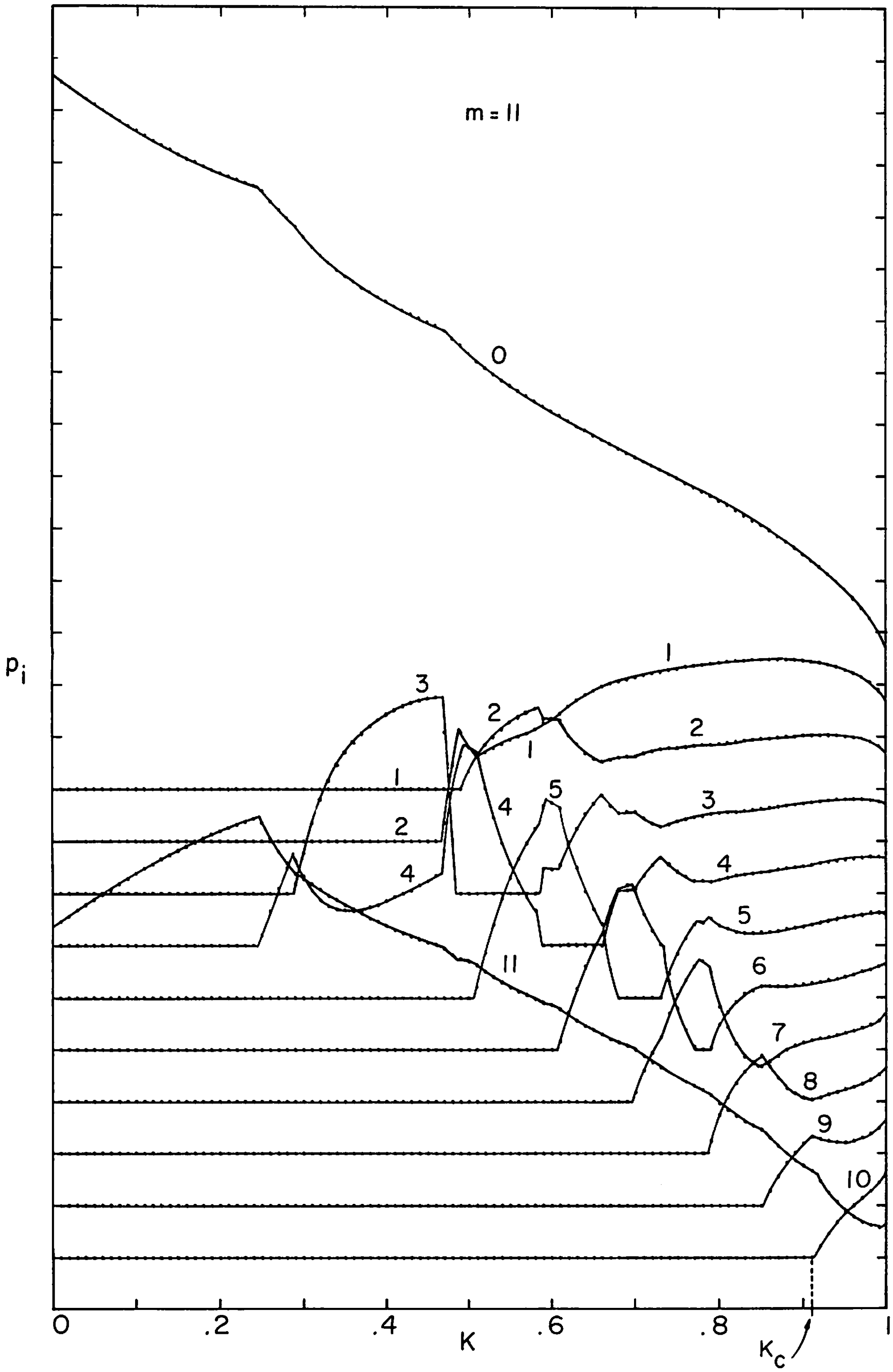

Fig. 2j.

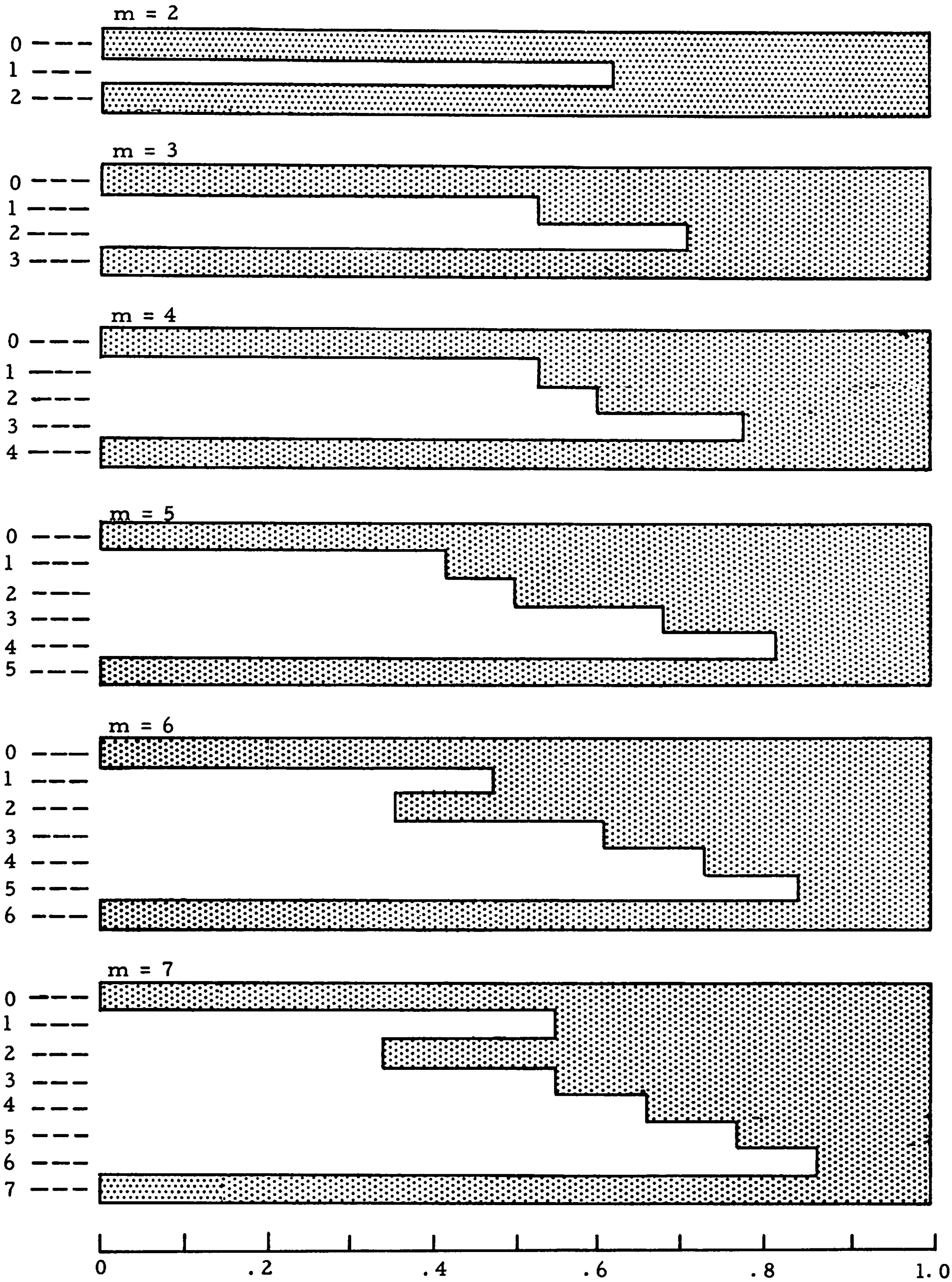

Fig. 3a (b, c). Maps showing positive regions of $p_i(K)$ in i - K plane.

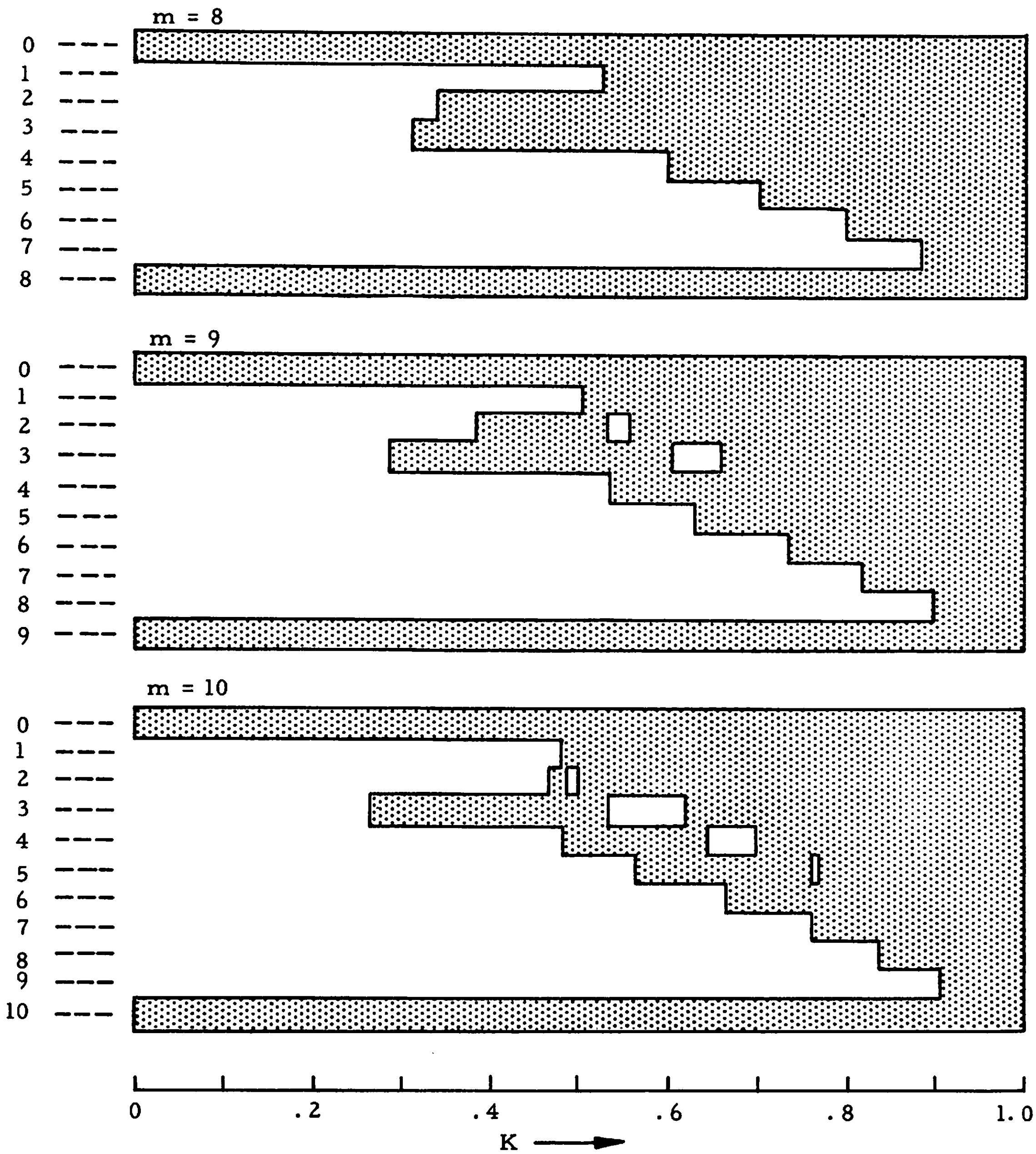

Fig. 3b.

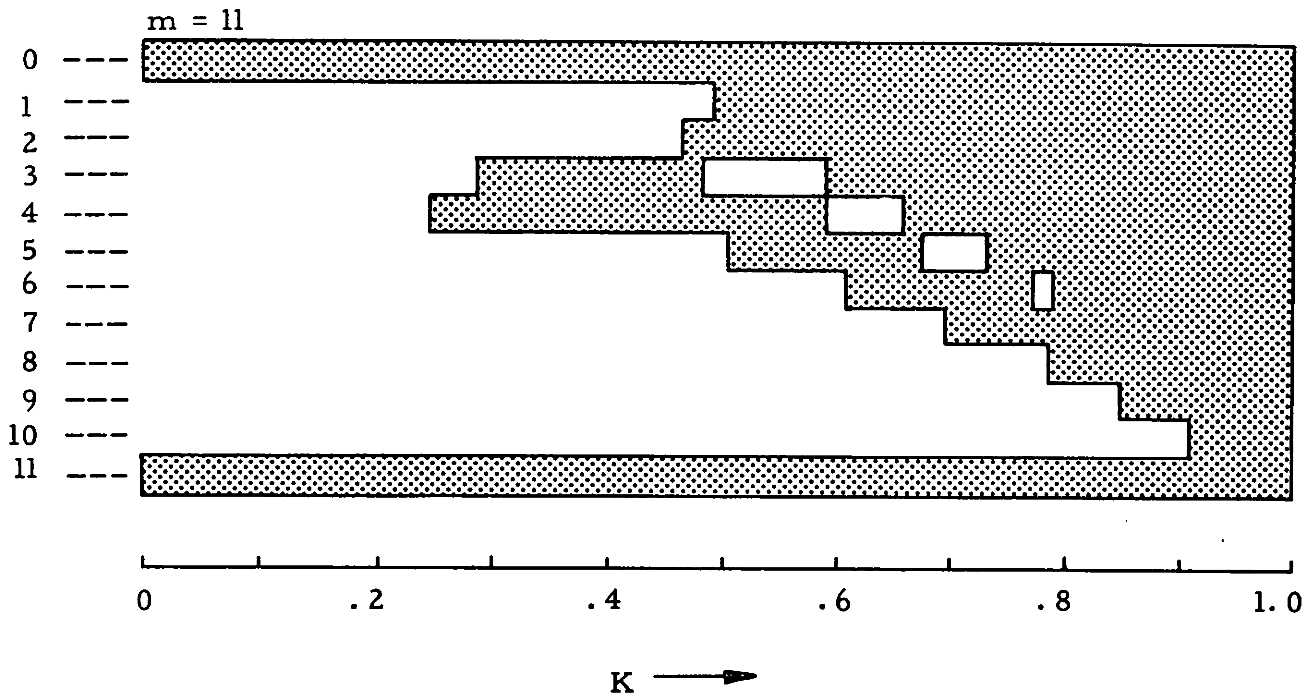

Fig. 3c.

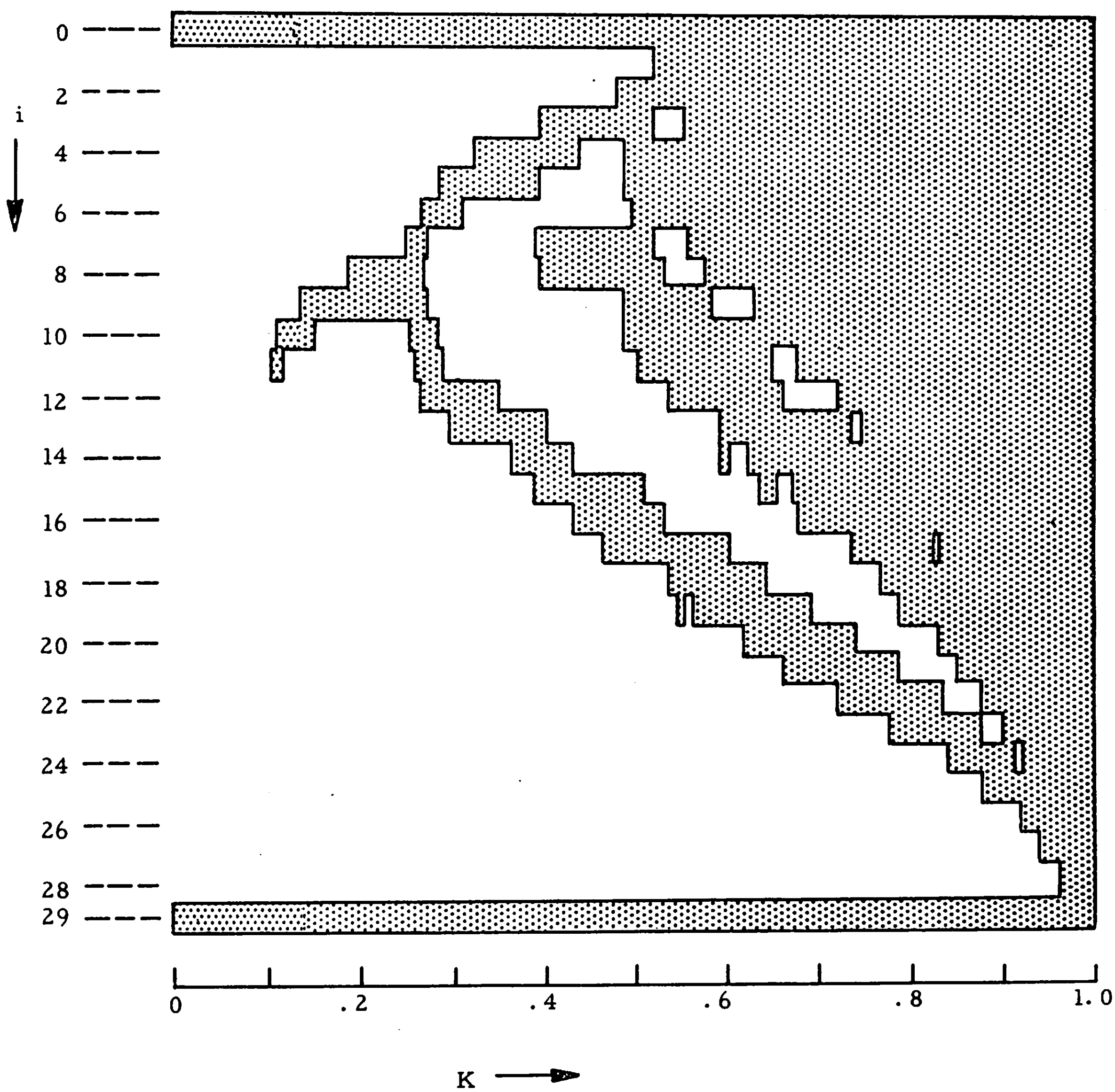

Fig. 4. Map showing positive region of $p_i(K)$ in i - K plane.
(m = 29)

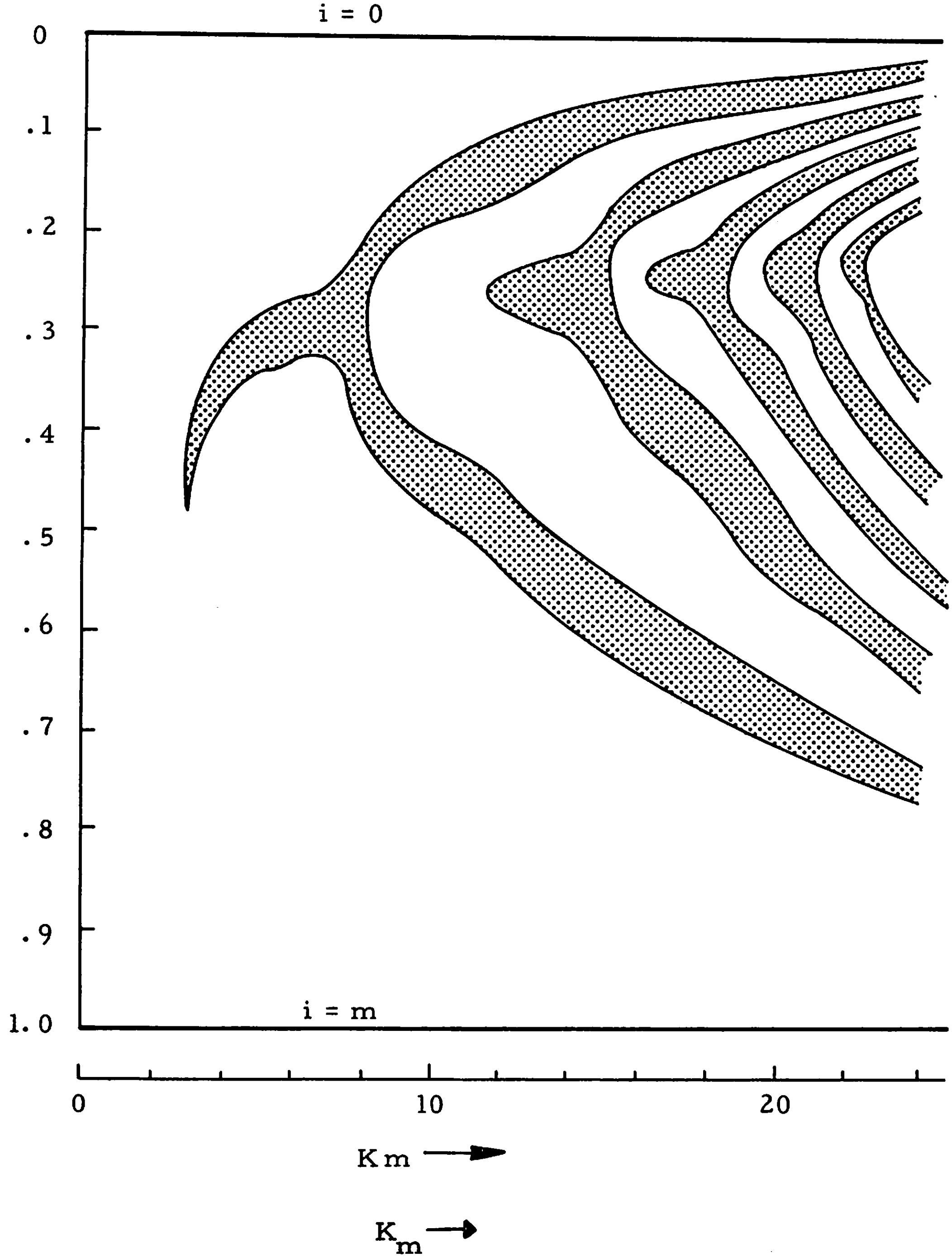

Fig. 5. Probable limiting form of the map showing regions of positive $p_i(K)$ in i/m - Km plane where $m \to \infty$.

$a = 2$

$\left(\frac{1}{4}, \frac{2}{4}, \frac{1}{4}\right)$

$m = 28$

$a = 3$

$\left(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\right)$

$m = 27$

$a = 4$

$\left(\frac{1}{16}, \frac{4}{16}, \frac{6}{16}, \frac{4}{16}, \frac{1}{16}\right)$

$m = 26$

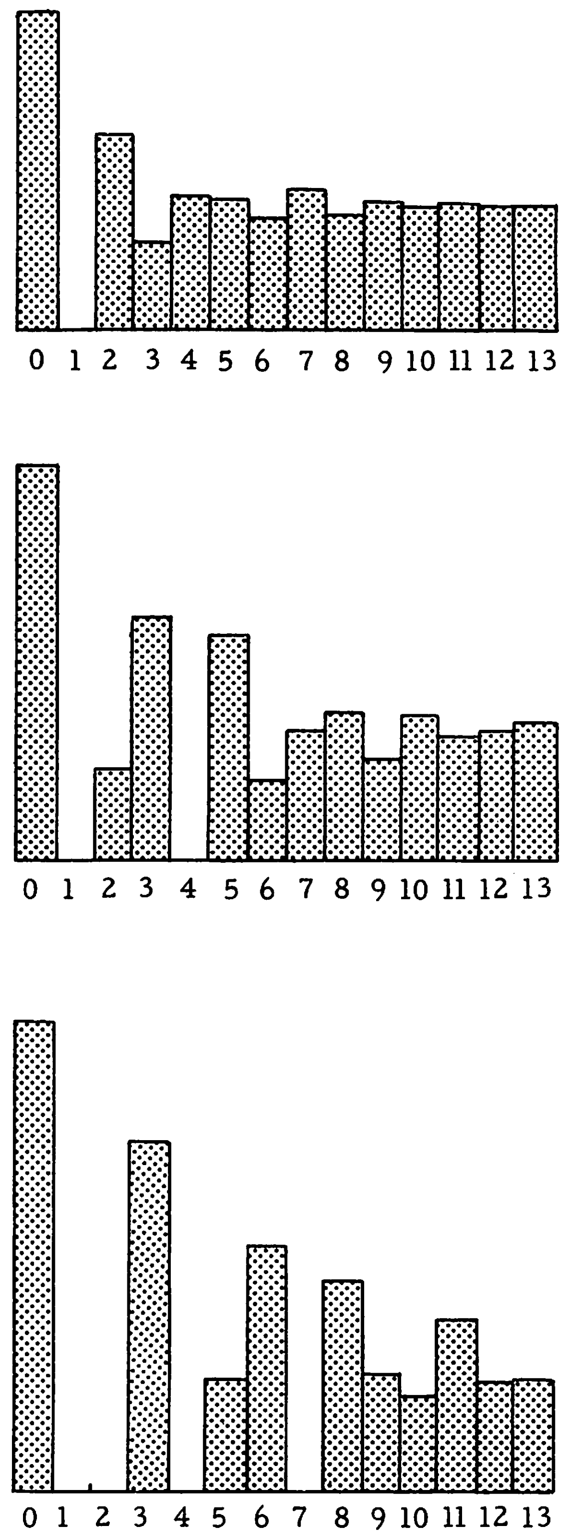

Fig. 6. Input signal probabilities for convolutional channels with binomial noises.

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I. INTRODUCTION

In a previous report[1] (henceforth referred to as Part I), we have developed a numerical procedure to find the capacity of any noisy discrete channel, and applied it to calculate the capacity of quantum-mechanical channels. In this paper, we will use the above method to answer a more fundamental question concerning the quantum-mechanical communication channels, namely, to compare the capacities of quantum-mechanical channels employing two different methods of detection: (1) direct counting of photons, and (2) phase-sensitive detection after a linear amplification by a maser.

The problem at hand is that of comparing two major classes of systems of electrical communication, one which uses only the amplitude of waves and one which uses both amplitude and phase. In ordinary circumstances, in which external noise is the main factor that limits the sensitivity of the receiver, the one which uses both amplitude and phase has a definite advantage over the other in that it virtually doubles the degree of freedom of the signal and hence doubles the capacity, provided both cosine and sine components are utilized. In fact, the system which uses amplitude only is currently used only where simplicity of the receiving equipment is a primary concern (e.g., radio broadcast).

The situation is quite different in quantum-mechanical channels, that is, in a circumstance where external noises are so low that the existence of a finite energy quantum is the main factor that determines the capacity. In such cases, the effect of the receiver in the whole communication system must be interpreted as a quantum-mechanical observation on the incoming signal, which is supposed to represent a quantum-mechanical state of the system. Since amplitude and phase, or equivalently, cosine and sine components, of a wave are physical quantities which are canonically conjugate to each other, simultaneous observation of these two quantities is subject to a restriction imposed by the uncertainty relations; and the values of these quantities can be determined only to a limited accuracy.

However, if only the amplitude is to be observed, we can measure it to any desired accuracy. Actually, the fact that the amplitude, which is proportional to the square root of the energy or photon number, allows only a discrete set of values, limits the amount of information obtained by one observation, but the spacings between these discrete values change in inverse proportion to the amplitude, and hence, become very small when the amplitude is large. This means that the amount of information obtained by amplitude observation, when the phase is not observed at all, can be much larger than the amount of information to be obtained from the amplitude alone when both amplitude and phase are observed. Actually, the information carried by the amplitude alone when amplitude alone was observed is shown to be about equal to the information carried by both amplitude and phase when both were observed.

For the purpose of comparing the above two systems, we will first have to slightly extend our method developed in Part I, so that we can obtain the channel capacity under an <u>average power</u> limitation. Actually, in Part I, we have dealt with the capacity of channels under peak-power limitation although no mention was made of this point there. The reason for making this modification is more technical than fundamental. As will be seen later, quantum-mechanical channels using phase-sensitive detection can be treated in a close analogy to a classical continuous channel with a gaussian noise. Since the calculation of the capacity of a classical channel with a gaussian noise is much easier in the average-power-limited case than in the peak-power-limited case, it was decided to compare the two types of channels in the average-power-limited case.

II. COMPUTATIONAL PROCEDURE

The computational procedure used in Part I for maximizing the the entropy function, $R(p_1 \cdots p_m)$, is modified so that it gives the maximum under an additional constraint

$$\sum i\, p_i = I. \tag{1}$$

Analytically, this would be done by introducing an additional Lagrange multiplier μ, so that we maximize

$$R^*(p) = R(p) + \lambda \sum p_i + \mu \sum i\, p_i \tag{2}$$

with suitable values of λ and μ. It is also possible to get an explicit solution of this problem in a manner similar to that given in the beginning of Sec. II of Part I if we ignore the constraints imposed by the inequalities, i.e.,

$$p_i \geq 0 \qquad\qquad i = 1, 2, \ldots m \tag{3}$$

(see appendix). In most cases, however, the constraints of (3) make an analytical approach impractical; we therefore will take a numerical approach as we did in Part I. The procedure to be used in the power-limited case is a steepest ascent procedure quite similar to that used in Part I, with the sole difference that an additional constraint, Eq. (1), on p_i must be satisfied by the new vector in each step of variation.

Since Δp_i must be chosen to maximize $R(p + \Delta p)$ under the constraints (1) and

$$\sum_i p_i = 1, \tag{4}$$

we have to make use of Lagrange multipliers and get Δp_i from

$$\Delta p_i = \epsilon \frac{\partial R^*}{\partial p_i} = \epsilon \left(\frac{\partial R}{\partial p_i} + \lambda + \mu \ i\right) . \tag{5}$$

λ and μ are determined so that R satisfies the constraints (1) and (4), and, in addition,

$$p_i = 0 , \tag{6}$$

for those i's for which

$$\frac{\partial R^*}{\partial p_i} < 0 . \tag{7}$$

Actual calculation goes as follows. The new trial vector (p_i) is obtained from the previous vector (p_i') in two steps, namely

$$p_i'' = p_i' + \epsilon \frac{\partial R}{\partial p_i} \tag{8}$$

and

$$p_i = p_i'' + \lambda + \mu \ i, \tag{9}$$

where for convenience, we have written λ for $\epsilon\lambda$ and μ for $\epsilon\mu$ in (5). Since

$$\sum_{+} p_i = 1 \tag{10}$$

and

$$\sum_{+} i \ p_i = I \ , \tag{11}$$

we have

$$\left.\begin{aligned} \sum_{+} p_i'' + \lambda \sum_{+} 1 + \mu \sum_{+} i = 1 \\ \sum_{+} i\, p_i'' + \lambda \sum_{+} i + \mu \sum i^2 = I \end{aligned}\right\} \qquad (12)$$

$\sum_{+}$ denotes a summation in i which extends over those i's for which $p_i > 0$.

Solving (12) in λ and μ, we obtain

$$\lambda = \frac{\sum_{+} i^2 \left(\sum_{+} p_i'' - 1\right) - \sum_{+} i \left(\sum_{+} i\, p_i'' - I\right)}{\left(\sum_{+} i\right)^2 - m_{+} \sum_{+} i^2}$$

$$\mu = \frac{\left(\sum_{+} p_i'' - 1\right) \sum_{+} i - m_{+} \left(\sum_{+} i\, p_i'' - I\right)}{\left(\sum_{+} i\right)^2 - m_{+} \sum_{+} i^2} \qquad (13)$$

where $m_{+} = \sum_{+} 1$. Since the definition of $\sum_{+}$ depends on the result of calculation itself, the formula (13) is implicit. However, it can be evaluated iteratively, by successively applying (13), (8), and (9) to get a new set of p_i's using the signs of the former p_i's, until self-consistency is established.

In a general average-power-limited photon channel there is a priori no upper limit of the number of photons transmitted. In fact, the probability distribution for a lossless channel $(K = 1)$ is given by

$$p_i = (1 - u)\, u^i \qquad (14)$$

where

$$u = \frac{I}{I+1} . \qquad (15)$$

For the present case where $K < 1$, we cannot tell a priori how this distribution is changed, but the result of calculation shows that the distribution tends to disperse still more towards higher i. Since actual numerical calculations can only be done with a finite upper limit of i, care should be taken to ensure that the limit is sufficiently high to keep the error reasonably small.

III. RESULTS

Fig. 1 shows the results of calculation of the capacity for several values of I, plotted as function of $K I$, the average received power. For a given $K I$, the capacity is higher for higher K (and lower I), which is as expected, since the loss of photons along the channel is the cause of the fluctuation. As K tends to zero, the capacity tends to a limiting curve, which gives the capacity of channels with a high attenuation and a high power of the transmitter, such that the power of the received signal is in the order of a few photons per degree of freedom. While this is exactly the case which interests us most, it is difficult to make the calculation for a large value of I, since it would require a high upper limit of i (i.e., a large number of variables). Therefore, we will make a compromise and will make calculations for the case $K = 0.2$, which is supposed to be a fair approximation to the limit $K \to 0$.

Fig. 2 gives the transmitter probability distribution for a number of cases. It is interesting to note that we can recognize the formation of several bands in the power-limited case too. Since in this case no sharp boundary of i exists, it must be the result of the approximate degeneracy of the channel matrix.

We also note that the highest i $(i = m)$ always has a nonvanishing probability. Obviously, this is the effect of an artificial cut-off of the i, and shows that the upper limit used here was still not large enough, although acceptable. This probability p_m would be interpreted as an accumulation, due to an artificial boundary at $i = m$, of those probabilities which would otherwise have been distributed over the i's lying beyond the boundary.

IV. THEORY OF LINEAR CHANNELS WITH A PHASE-SENSITIVE DETECTION

Now we will deal with the quantum-mechanical channels which use the other method of detection, namely, amplification by a maser followed by a phase-sensitive detection. Such channels are most conveniently dealt with by the quantum theory of linear systems,[2] as will be outlined below.

The whole communication system in question will be described schematically as in Fig. 3. The transmitter gives off a signal, which is to be represented by a quantum-mechanical state vector, to the channel. The effect of the channel is considered as a linear transformation operating on its input-state vectors to give the output-state vectors. The amplifier is another linear transformation, which results in state vectors corresponding to the amplified output. The detector will make a certain observation on the output-state vector.

The only essential assumption we have to make on the nature of the two boxes "channel" and "amplifier" is that they are linear systems in the classical sense. Let x, y, and z denote electromagnetic quantities (voltages, currents, or any other quantities linearly related to electromagnetic field variables) related to the channel input (transmitted signal), the channel output (amplifier input), and the amplifier output, respectively. Then we can write down the classical relations between these quantities as

$$y = k\ x \tag{16}$$

and

$$z = \mathcal{K}\ y \,. \tag{17}$$

We assume that x, y, and z are normalized in a natural unit so that the quantum-mechanical zero-point fluctuations of these quantities are

$$\langle \Delta x^2 \rangle = \langle \Delta y^2 \rangle = \langle \Delta z^2 \rangle = 1/2 \ . \tag{18}$$

Now it should be noted that relations (16) and (17) are not consistent with (18), which means that (16) and (17) cannot be correct quantum-mechanical equations. The situation will be rectified if we introduce another input variable in (16) and write

$$y = k\ x + k'\ x' \tag{19}$$

where

$$k' = \sqrt{1 - k^2} \ . \tag{20}$$

If x' also is similarly normalized, we have

$$\langle \Delta x'^2 \rangle = 1/2 \tag{21}$$

and (19) is consistent with (18) and (21).

We can interpret x' as a variable which represents the effect of external disturbances, or of coupling with the environment, on the channel output. In classical mechanics, we could ignore the term $k'x'$ in (19) altogether and write it as (16), by just assuming that

$$x' = 0 \ . \tag{22}$$

That is, we assume that the external environment is in the quiescent state, which means that all the conductors, insulators, and the surrounding media are at absolute zero temperature.

However, this assumption is not allowed in quantum mechanics, since the variable x' is subject to the zero-point fluctuation even at absolute zero.

In a similar manner, we must modify (17) to

$$z = \mathcal{K}\, y + \mathcal{K}'\, y' \tag{23}$$

where

$$\mathcal{K}' = \sqrt{\mathcal{K}^2 - 1}\ . \tag{24}$$

The analogy between (19) and (23) is evident; but we can also recognize an essential difference between them which comes from the fact that $\mathcal{K} > 1$. (23) is not compatible with (18) if we assume

$$\langle \Delta y''^2 \rangle = 1/2\ . \tag{25}$$

This means that an amplifier cannot be in a quiescent state even if the input signal is just zero-point fluctuation. Combining (19) and (23), we get

$$z = k\,\mathcal{K}\, x + k'\,\mathcal{K}\, x' + \mathcal{K}\, y'\ . \tag{26}$$

One of the important properties of linear systems is that any linear relation, such as (26), remains valid in quantum mechanics if if is interpreted as a stochastic equation, provided that the variables appearing in the right-hand side are dynamically independent (noninteracting) of each other. Then, these variables can be regarded as stochastic variables having gaussian distributions and (26) can be used to obtain various moment relations among these variables.

Let us now make a few more assumptions in order to get a simple result for our channel.

a. $k \ll 1$, that is, the channel attenuation is very high, so that the zero-point fluctuation or any other noise coming from the transmitter is negligible, i.e., the transmitter can be treated classically.

b. $\mathcal{K} \gg 1$, that is, the amplifier gain is so high that the phase-sensitive detector can be treated classically and hence the output is very large.

c. The channel is noiseless, i.e., x' consists of zero-point fluctuation only. Hence,

$$\langle x'^2 \rangle = 1/2 . \tag{27}$$

d. The amplifier is an ideal maser amplifier, which operates with molecules having a perfect population inversion, that is, zero population in the lower energy level. Then it can be shown that

$$\langle y'^2 \rangle = 1/2 . \tag{28}$$

Using these assumptions and the certainly valid assumption that x, x', and y are statistically independent, we can easily obtain the formulas

$$\langle z^2 \rangle = k^2 \mathcal{K}^2 \langle x^2 \rangle + \mathcal{K}^2 (1 - k^2) \langle x'^2 \rangle + (\mathcal{K}^2 - 1) \langle y'^2 \rangle$$

$$= \mathcal{K}^2 (k^2 \langle x^2 \rangle + 1) , \tag{29}$$

$$\langle (z - \langle z \rangle)^2 \rangle = k^2 \mathcal{K}^2 \langle (x - \langle x \rangle)^2 \rangle + \mathcal{K}^2 (1 - k^2) \langle x'^2 \rangle$$

$$+ (\mathcal{K}^2 - 1) \langle y'^2 \rangle$$

$$\doteq \mathcal{K}^2 . \tag{30}$$

Since we know that the output z has a gaussian distribution, and that both x and z can be regarded as classical quantities, we can forget the quantum-mechanical nature of the system and apply Shannon's theory of gaussian channels, and for the capacity per one degree of freedom we get

$$C = \log \frac{S+N}{N} = \log \frac{\langle (z - \langle z \rangle)^2 \rangle}{\langle z^2 \rangle} = \log (k^2 \langle x^2 \rangle + 1). \tag{31}$$

Changing the symbols to the ones used in the foregoing sections, i.e.,

$$K = k^2 , \tag{32}$$

$$I = \langle x^2 \rangle , \tag{33}$$

we get the final result

$$C = \log (K I + 1) . \tag{34}$$

It is interesting to see the contribution of each of x' and y' in the total noise separately. From (30) we find that exactly one half of the noise comes from $\langle x'^2 \rangle$ (the channel) and the other half comes from $\langle y'^2 \rangle$ (the amplifier).

V. CONCLUSION AND COMMENTS

The values of C computed from (34) are plotted in Fig. 1 together with the curves for the photon channels in the same scale. The broken curve is to be compared with the limit for $K \rightarrow 0$ of the full curves. Although this limit is not exactly known, the curve for $K = 0.2$ can be a good approximation, and we finally obtain the following conclusion.

The capacity curves for the two detection schemes cross over at a point corresponding to the average received power

$$K I \doteq 0.7 . \tag{35}$$

If the received power level is below this crossover, the photon-counting mode gives higher capacity, but if the received power level is above it, maser detection followed by a phase-sensitive detector gives higher capacity.

This result seems rather remarkable, and it would be of some interest to look for a somewhat more intuitive explanation of this result.

First, let us consider the low power-level case. It is well known that in ordinary communication, when the signal-to-noise ratio is very low and the power limited in average-power basis, a quantized or on-off system is more advantageous than continuous systems. In on-off systems, significant improvement in the error rate can often be obtained by accumulating the available power for a certain time, and releasing it in a big burst. In other words, a high pulse power with a low duty ratio is a good measure for combatting noise when the power is limited only on the average. Our result (Fig. 2) also seems to show that such low duty ratio gives the highest transmission rate when $K I$ is small. For this reason the capacity of discrete channels behaves approximately as

$$C = K I \log \left[1/(K I) \right] \tag{36}$$

in the very low-signal range and has a high slope at small I. In continuous signal schemes, a standard way of reducing noise is to take a long-time average, which is mathematically equivalent to reducing the bandwidth by a narrow filter. This evidently leads to a reduction of the transmission rate in proportion to the bandwidth. In fact, if we use an approximate form for Shannon's formula which is valid if $S \ll N$, namely

$$C = WS/N = S/(N/W), \tag{37}$$

we find that the capacity is just proportional to the signal power and independent of the bandwidth if the noise is evenly distributed over the whole frequency band. That is to say, the continuous system is not suitable for an efficient use of a given bandwidth if the available power is too small. At any rate, the logarithmic factor in (36) clearly indicates that the discrete system is suited to the low power-level situations.

Next consider the high power-level case. Here we will use a conventional representation of a signal as a plane vector (x_c, x_s), where

$$x = x_c \cos \omega t + x_s \sin \omega t. \tag{38}$$

In the classical theory, any signal can be represented by a vector, or a single point in the (x_c, x_s) plane. In quantum theory, however, this representation of x would be altogether meaningless since x_c and x_s are canonically conjugate variables which cannot be observed simultaneously. Nevertheless, it is possible to represent a signal, in a somehwat unrigorous way, by means of a finite domain in the (x_c, x_s) plane instead of a point. The area of such a domain cannot be smaller than a certain minimum, which is determined by the uncertainty relation

$$\Delta x_c \ \Delta x_s \geqq 1/2 . \tag{39}$$

Now, the transmitted signal in the system using both amplitude and phase will be represented by a small circle, as shown by A in Fig. 4(a), because it is natural to define x_c and x_s with the same degree of uncertainty while satisfying the uncertainty relation (39).

In the system using amplitude only, the signal will be represented by a very thin circular ring having the same area as A, shown by B in the same figure.

If these two different signals are passed through the same channel, the output signal y would be represented by A and B, respectively, in Fig. 4(b) (drawn in a different scale).

Now the width of ring B is about the same as the diameter of the circle A, owing to the zero-point noise which affected both signals additively. We see that the area of B is now much larger than the area of A.

In the system using both amplitude and phase, this signal is further passed through a maser amplifier, and the amplified output has again a circular shape, having, however, an area twice as large as that before amplification (measured relative to the signal). In the system using amplitude only, the amplitude, or energy, is measured directly by means of a counter so that there is no such deterioration. The situation may be visualized by Fig. 4c. The effect of the amplifier is still not enough to affect the situation. Since a larger area of the signal domain may correspond to a smaller amount of information, we can see that the system which uses amplitude only gives less information than the other one, if the signal-power level is high.

APPENDIX

ANALYTIC SOLUTION OF THE CAPACITY EQUATION IN THE AVERAGE POWER LIMITED CASE

The maximization problem of (2) can be solved analytically in a similar manner as in the case without the power condition (1). Differentiating (2) with respect to p_i, we get

$$\frac{\partial R}{\partial p_i} + \lambda + \mu i = \sum_j P_{ij} \log (P_{ij} / \sum_k p_k P_{kj}) + \lambda + \mu i - 1 = 0 . \quad \text{(A-1)}$$

Hence,

$$C = R = \sum_i p_i \sum_j P_{ij} \log (P_{ij} / \sum_k p_k P_{kj})$$

$$= -\sum (\lambda + \mu i - 1) p_i = - \lambda - \mu \; I + 1. \tag{A-2}$$

Let q_j^o be the solution of the system of linear equations

$$\sum_j P_{ij} \; \log q_j^o = \sum_j P_{ij} \log P_{ij} \; , \; (i = 1, 2, \ldots m) \tag{A-3}$$

and put

$$q_j = q_j^o \; \exp (\alpha + \beta j) \; . \tag{A-4}$$

Then we have

$$\sum_j P_{ij} \log q_j = \sum_j P_{ij} \log P_{ij} + \alpha + \beta K i, \tag{A-5}$$

where we have used the relations

$$\sum_j P_{ij} = 1 \tag{A-6}$$

$$\sum_j P_{ij} \; j = K i \tag{A-7}$$

which come from the conditions of conservation of probability and conservation of power, respectively.

If we further put

$$\alpha = \lambda - 1 \; , \tag{A-8}$$

$$\beta = \mu / K , \tag{A-9}$$

(A-1) will be satisfied if

$$q_j = \sum_i P_{ij} \, p_i , \tag{A-10}$$

that is, q_j's are the probabilities of received signals.

From the relations

$$\sum_j q_j = 1 \tag{A-11}$$

$$\sum_j j \, q_j = K I \tag{A-12}$$

we get the equations

$$e^{\lambda - 1} \sum_j (e^{\mu/K})^j \, q_j^o = 1 \tag{A-13}$$

$$e^{\lambda - 1} \sum_j j \, (e^{\mu/K})^j \, q_j^o = K I . \tag{A-14}$$

Since q_j^o's are known, these equations can be used to determine the two parameters λ and μ, and hence C by (A-2). (A-13) and (A-14) are algebraic equations of a high degree and may be difficult to solve, but still the method will be practical when it gives reasonable results (i.e., real non-negative probabilities).

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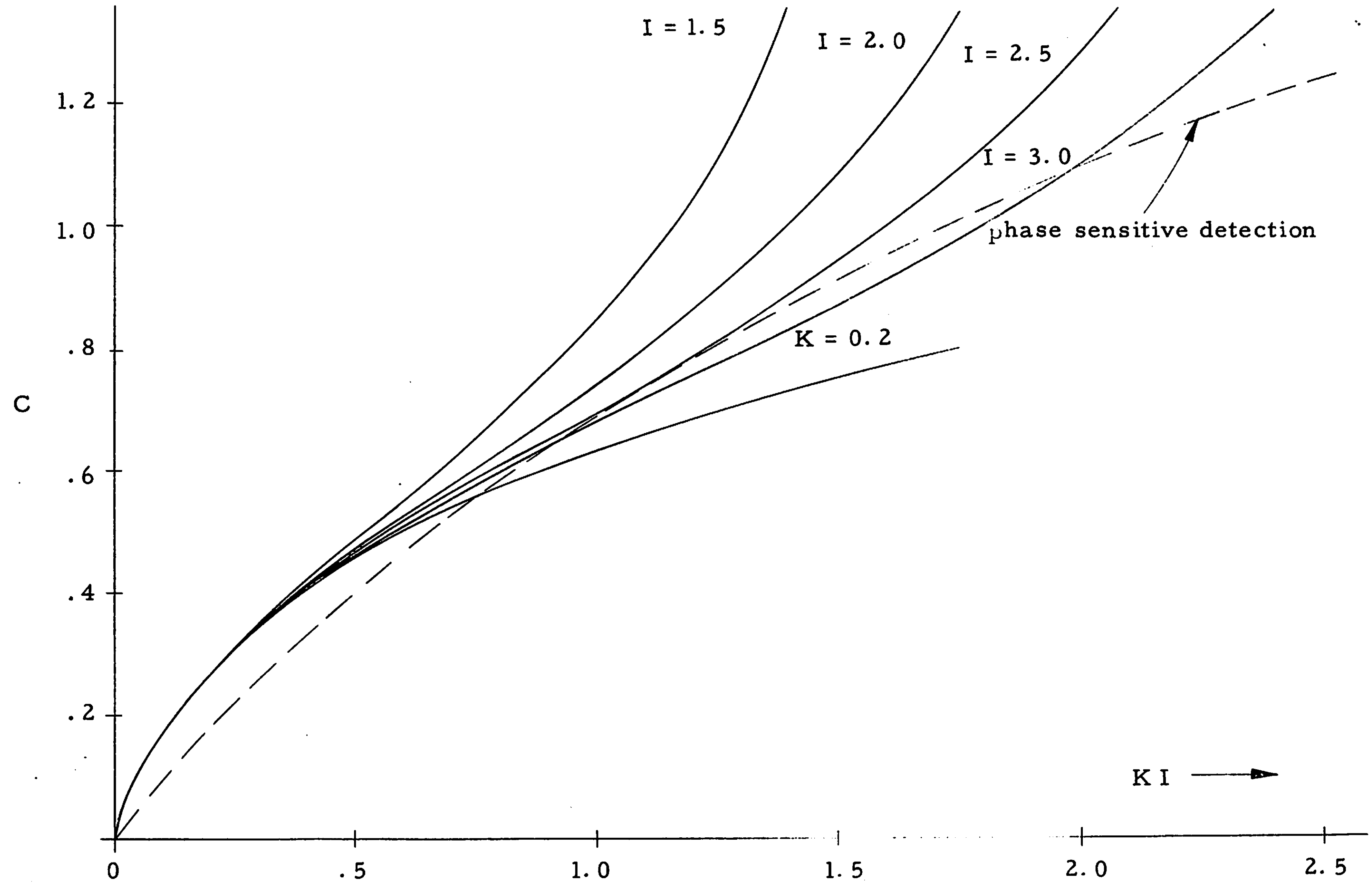


Fig. 1. Channel capacity as function of K I, the average received power. Numerical values of C are based on the logarithm to base e. They must be multiplied by $\log_2 e$ to convert to the number of bits.

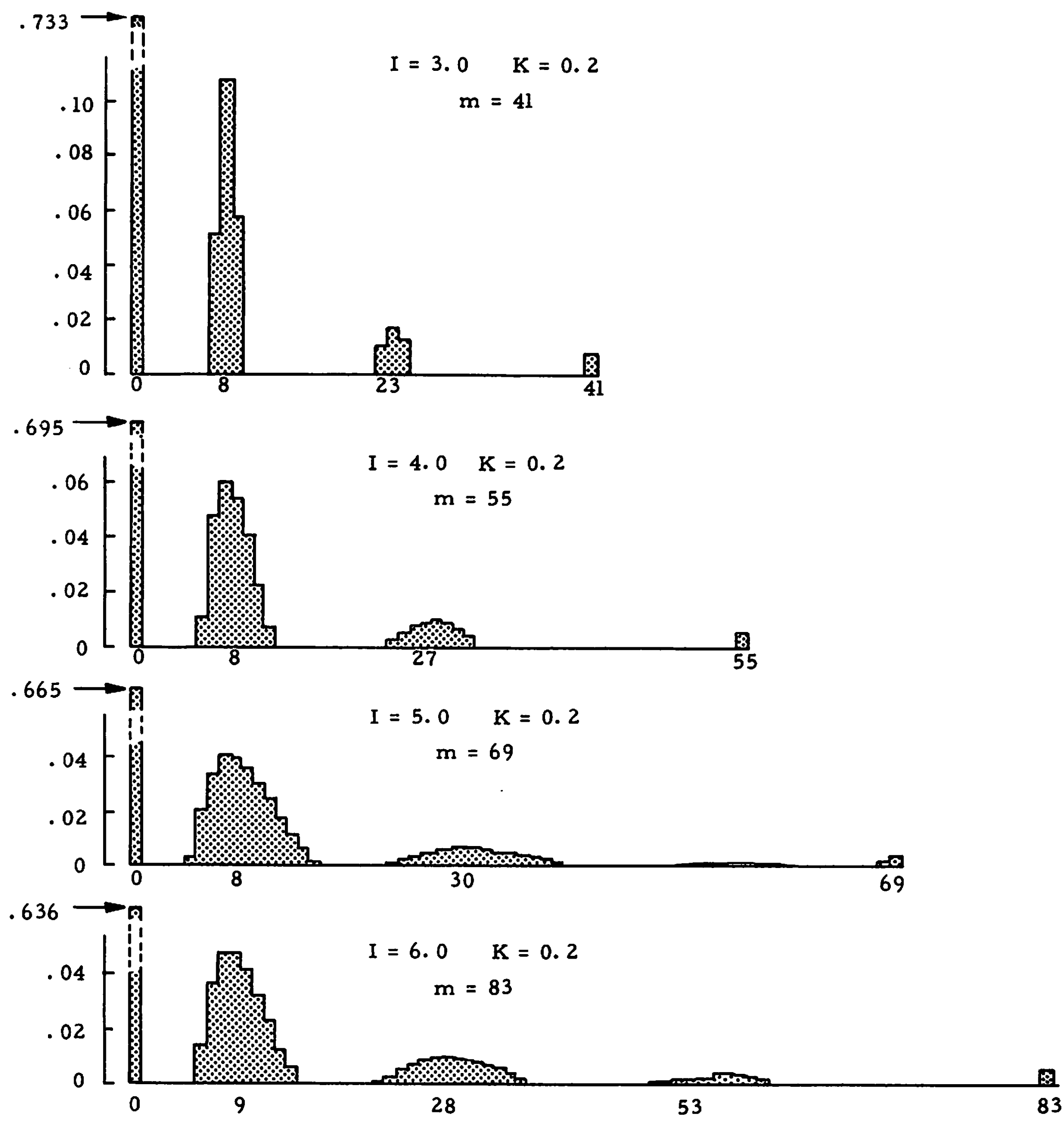


Fig. 2. Probability distribution for average-power-limited photon channels.

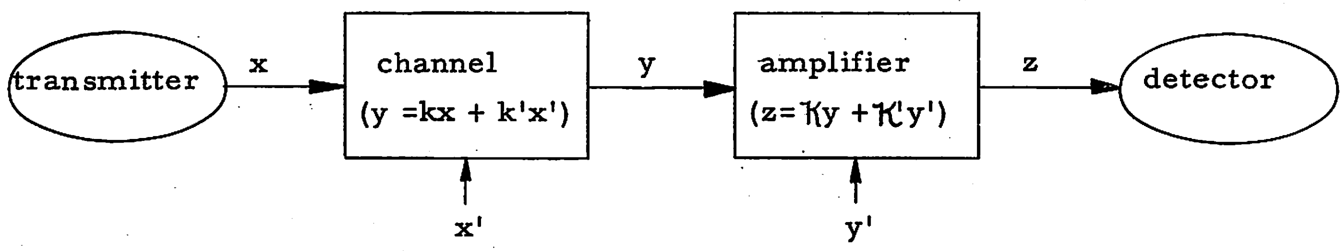
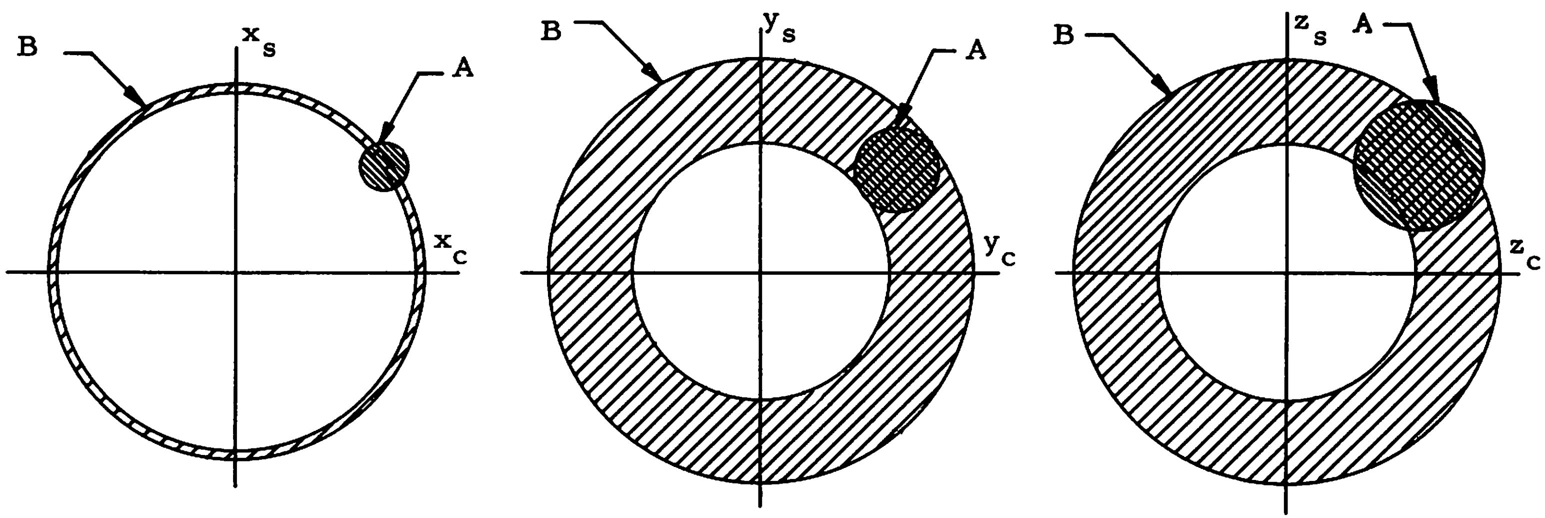


Fig. 3. Block diagram of a communication system using an amplifier followed by a phase-sensitive detector.



(a.) Transmitted signal (b.) Received signal (c.) Amplified signal

Fig. 4. Graphical representation of two kinds of signals on the vector plane.