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DECISION MAKING IN INCOMPLETELY KNOWN STOCHASTIC SYSTEMS*

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ABSTRACT

This paper is a study of decision making in a discrete-state discrete-time system whose state transitions constitute a Markov chain with unknown stationary transition matrix $\underline{P}$. The states of the system cannot be observed. The decision at each stage is based on observables whose conditional probability distribution given the state of the system is known.

We consider a class of problems in which the successive observations can be employed to form estimates of $\underline{P}$, with the estimate at time n, $n = 0, 1, 2, \ldots$, then used as a basis for making a decision at time n. The estimates and the corresponding decisions must have the property that as $n \to \infty$, the decision based on the estimate of $\underline{P}$ tends to the optimal decision rule which would be used throughout if $\underline{P}$ were known.

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INTRODUCTION AND SUMMARY

This paper is a study of decision making in a discrete-state discrete-time system whose state transitions constitute a Markov chain with unknown stationary transition matrix $\underline{P}$. The states of the system cannot be observed. The decision at each stage is based on observables whose conditional probability distribution given the state of the system is known.

We consider a class of problems in which the successive observations can be employed to form estimates of $\underline{P}$, with the estimate at time n, n = 0, 1, 2, . . . , then used as a basis for making a decision at time n. The estimates and the corresponding decisions must have the property that as $n \to \infty$, the decision based on the estimate of $\underline{P}$ tends to the optimal decision rule which would be used throughout if $\underline{P}$ were known.

In Sec. I, the formulation of the problem is presented, in detail, and the optimal decision procedure that would be adopted if one knew the transition matrix $\underline{P}$ is given. In Sec. II, sequences of estimates are derived. These estimates are based at each stage on the observations up to that point, and they converge almost surely, when $n \to \infty$, to the unique stationary distribution, and to the transition probabilities of the Markov chain (when it is regular). More generally, if the chain is a k-th order multiple Markov chain, a sequence of estimates which converges to any k-th order distribution can be obtained. A necessary and sufficient condition for the existence of such estimates is given. In Sec. III, using these estimates, an "adaptive" decision procedure is developed which does as well asymptotically, in a well defined sense, as the optimal procedure that one would adopt if one knew the transition matrix $\underline{P}$. Finally, an example is worked out. This example demon-

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strates that the well-studied problem (when $\underline{P}$ is known) of decision making on a Markov signal with additive gaussian noise is a special case of the theory developed above.

This paper is a generalization, to the Markov dependence case, of some of the ideas presented in H. Robbins' paper.[1] As in the Robbins paper, the decision rule is not Bayesian, since no assumptions are made concerning an a priori probability distribution on the space of all possible transition matrices $\underline{P}$, or more generally, on the space of couples (S, $\underline{P}$) where S is the initial probability distribution on the state space of the Markov chain.

I. FORMULATION OF THE PROBLEM

A. NOTATION AND TERMINOLOGY

Let us consider a discrete state discrete time system S whose state transitions constitute a regular Markov chain* with stationary transition matrix $\underline{P}$. If we denote by $\Lambda = \{1, \ldots, r\}$ the state space of system S then,

$$\underline{P} = \begin{bmatrix} p_{11} & \cdots & p_{1r} \\ \vdots & & \vdots \\ p_{r1} & \cdots & p_{rr} \end{bmatrix}$$

where the p_{ij}'s denote the one step transition probabilities, i.e., if at any step (time) the system is in state i, then it moves on the next step to state j with probability p_{ij}. For this reason, in the following discussion we shall use the state of S and the state of the Markov chain interchangeably.

For simplicity we shall restrict ourselves to finite state simple Markov chains. However, all our results can be applied, as we shall indicate later, to k-th order multiple Markov chains and to Markov chains with a countable number of states.** The state transitions of S are assumed to occur at times $n = 1, 2, \ldots$. We suppose that we cannot observe the state of the system, however we observe at each time n, a real-valued random variable x, whose probability distribution depends on the state of S at time n. We further suppose that the conditional probability distribution of the observable random variable x given the state of S is known to us. Thus, at each time n, the

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* A regular Markov chain is one that has no transient states, has only one ergodic class, and this class contains no cyclically moving subclasses. See Doob Ref. 3, page 182.

** Ibid., page 185.

observable random variable x is known to have one of a finite number of specified probability distributions $P_1, \ldots, P_r$, with P_i being the distribution in question if S is in state i, $i = 1, \ldots, r$.

In what follows, subscripts will denote "state" and superscripts will denote "time of observation." The model introduced above is illustrated in Fig. 1. Let $P(\lambda^n = i)$ denote the probability that the Markov chain is in state i at time n. By $P(\lambda^n = i/x^1, \ldots, x^n)$ we shall denote the conditional probability that the Markov chain is in state i at time n. given that the values $x^1, \ldots, x^n$ have been observed. Finally let $\underline{x}^n$ denote the vector $(x^1, \ldots, x^n)$.

We are observing a sequence $x^1, x^2, \ldots,$ of random variables. Without loss of generality let us define these random variables on the sample space $\Omega = \Pi_{n=1}^{\infty} z^n$ where $z^n = R^1$ (the real line) for all n, and denote by $\mathcal{Q}$ the Borel σ-field on this space. Thus the sample space Ω is the coordinate space of all sequences of real numbers $\xi = (\xi^1, \xi^2, \ldots)$, the random variable x^n is defined as the n-th coordinate variable of Ω, so that $x^n(\xi) = \xi^n$. We assume that the random variables $x^1, x^2, \ldots$ are conditionally independent given the states of S.* To simplify the notation we shall use $x^1, x^2, \ldots$ to indicate both the random variables and the values which they take, and it shall always be clear from the context which one we mean.

*This condition implies for example that $P(x^n \epsilon A, x^{n+1} \epsilon B/\lambda^n = i, \lambda^{n+1} = j) = P(x^n \epsilon A/\lambda^n = i) \cdot P(x^{n+1} \epsilon B/\lambda^n = j)$ where A, B are Borel sets and $P(x^n \epsilon A/\lambda^n = i)$ denotes the conditional probability of the set $\{\xi/x^n(\xi) \epsilon A\}$ given $\lambda^n = i$.

Let us now suppose that we have a finite action space $A = \{a_0, \ldots, a_s\}$,* a typical action a_i might be to guess that λ^n, the state of S at time n, is i.** Let a loss function L be defined on $A x \Lambda$ such that $L(a_i, \lambda) \geq 0$ for all $i = 1, \ldots, r$ and all $\lambda \epsilon \Lambda$, $L(a_i, j)$ representing the loss we incur in taking action a_i when the state of the system is j.

Since we cannot observe the state of the system, our problem is to choose, at each time n, a decision function t^n which depends on $x^1, \ldots, x^n$, the observations up to this point, such that when we observe $\underline{x}^n$ we shall take the action $t^n(\underline{x}^n)$. More precisely if $R^n = R^1 x R^1 x \ldots x R^1$ (n times) and $\mathcal{B}^n$ is the Borel σ-field on R^n then we shall denote by t^n any Borel Measurable function from the measureable space $(R^n, \mathcal{B}^n)$ to the action space A, and by T^n the class of all such t^n.

In choosing $t^n(\underline{x}^n) \epsilon A$ at time n, we incur the loss $L(t^n(\underline{x}^n), \lambda^n)$. We want to choose a sequence of decision functions t^n for all $n = 1, 2, \ldots$ in such a way as to minimize the expected loss, i.e., choose t^n such that $E[L(t^n(\underline{x}^n), \lambda^n)]$ is a minimum, where E denotes the expectation with respect to all the random variables $x^1, x^2, \ldots, x^n, \lambda^n$. When no confusion can arise, the superscript n will be omitted; i.e., $t(\underline{x}^n)$ should be interpreted as $t^n(\underline{x}^n)$.

B. THE CASE WHERE $\underline{P}$ IS KNOWN

Suppose we are faced with the above problem and we know the transition matrix $\underline{P}$ of the system S. Let us further suppose that the initial probability distribution on the state space of S is the unique stationary probability distribution. Then the sequence of optimal decision functions $\{t_p^n\}$ will be chosen as follows.

If we denote by $R_n(t^n, \underline{P})$ the expected loss at time n when we use the decision function t^n and the transition matrix is $\underline{P}$, then we have

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* Note that the number of actions is not necessarily equal to the number of states, see example page 31.

** We shall not consider randomized decisions.

$$R_n(t^n, \underline{P}) = E[L(t(\underline{x}^n), \lambda^n)] = E\{E[L(t(\underline{x}^n), \lambda^n)/\underline{x}^n]\} \tag{1}$$

and if we set

$$\emptyset_{\underline{P}}(a, \underline{x}^n) = E[L(a, \lambda^n)/\underline{x}^n] = \sum_{i=1}^{r} L(a, i)\, P(\lambda^n = i/\underline{x}^n) \tag{2}$$

then we get

$$R_n(t^n, \underline{P}) = E[\emptyset_{\underline{P}}(t(\underline{x}^n), \underline{x}^n)] . \tag{3}$$

Let us choose $t_{\underline{P}}(\underline{x}^n)$ such that for almost every $(\mu)\underline{x}^n$, where μ is the measure defined on Ω, we have

$$\emptyset_{\underline{P}}(t_{\underline{P}}(\underline{x}^n), \underline{x}^n) = \min_{a_i \epsilon A} \emptyset_{\underline{P}}(a_i, \underline{x}^n)^{*} . \tag{4}$$

Then for any decision function t^n

$$R_n(t^n_{\underline{P}}, \underline{P}) = E[\min_{a_i \epsilon A} \emptyset_{\underline{P}}(a_i, \underline{x}^n)] \leq R_n(t^n, \underline{P}) \tag{5}$$

so that, defining

$$R_n(\underline{P}) = R_n(t^n_{\underline{P}}, \underline{P}) = E[\emptyset_{\underline{P}}(t_{\underline{P}}(\underline{x}^n), \underline{x}^n)] \tag{6}$$

we have

$$R_n(\underline{P}) = \min_{t^n \epsilon T^n} R_n(t(\underline{x}^n), \underline{P}) . \tag{7}$$

Let us now consider the question of how well we can hope to do in the case where the transition matrix $\underline{P}$ is not known to us.

Since we have assumed that the initial probability distribution on the state space of S is the stationary probability distribution, the Markov chain is a stationary process.** By Lemma 2.3, page (18)it follows that the process $\{x^n\}$ is also stationary. It is well known*** that if $x^1, x^2, \ldots$ is a stationary sequence of random variables, there exists a stationary sequence $\ldots y^{-1}, y^0, y^1, \ldots$ such that the laws of $(y^1, y^2, \ldots)$ and of

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* $t_{\underline{P}}(\underline{x}^n) = a_k$ where k is any integer $0 \leq k \leq s$ such that $\emptyset_{\underline{P}}(a_k, \underline{x}^n) = \min[\emptyset_{\underline{P}}(a_0, x^n), \ldots, \emptyset_{\underline{P}}(a_s, \underline{x}^n)]$.

** Ref. 3, page 459.

*** Ibid., page 456, Loève, M., Ref. 5, page 452.

$(x^1, x^2, \ldots)$ are the same. (Take, for every finite subfamily of y's,

$$\mathcal{L}(y^{k_1}, \ldots, y^{k_m}) = \mathcal{L}(x^{k_1+h}, \ldots, x^{k_m+h})$$

where h is so large that the superscripts of the x's are positive, and apply the consistency theorem).*

Thus, if only questions involving the distributions of $x^1, \ldots, x^n$ are to be considered, we can use the $\{y^n\}$ process instead of the $\{x^n\}$ process.

<u>Lemma 1.1:</u> The sequence of real numbers $\{R_n(\underline{P})\}$ is monotonically non-increasing, i.e., $R_n(\underline{P}) \geq R_{n+1}(\underline{P}) \geq \ldots$.

<u>Proof:</u> By hypothesis, as stated above, the initial probability distribution of the Markov chain is the stationary one; therefore the Markov chain is a stationary process. Now

$$\begin{aligned} R_{n+1}(\underline{P}) &= \min_{t^{n+1} \epsilon T^{n+1}} E\,[L(t(x^1, \ldots, x^n, x^{n+1}), \lambda^{n+1})] \\ &= \min_{t^{n+1} \epsilon T^{n+1}} E[L(t(y^1, \ldots, y^n, y^{n+1}), \lambda^{n+1})] \,. \end{aligned} \tag{8}$$

By stationarity we have

$$R_{n+1}(\underline{P}) = \min_{t^{n+1} \epsilon T^{n+1}} E[L(t(y^0, y^1, \ldots, y^n), \lambda^n)] \tag{9}$$

$$R_n(\underline{P}) = \min_{t^n \epsilon T^n} E[L(t(y^1, \ldots, y^n), \lambda^n)] \,. \tag{10}$$

Let

$$\tilde{T}^{n+1} = \{t^{n+1} / t^{n+1}(y^0, y^1, \ldots, y^n) \text{ depends only on } y^1, \ldots, y^n\}$$

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* M. Loève, Ref. 5, page 93, see also Sec. 3 of this paper, page 23.

and denote each member of $\tilde{T}^{n+1}$ by $\tilde{t}^{n+1}$. Then for any $t^n \epsilon T^n$, we have

$$t^n(y^1, \ldots, y^n) = \tilde{t}^{n+1}(y^0, y^1, \ldots, y^n) \quad \forall y^0. \tag{11}$$

Therefore, we finally get

$$R_{n+1}(\underline{P}) = \min_{t^{n+1} \epsilon T^{n+1}} E[L(t(y^0, y^1, \ldots, y^n), \lambda^n)]$$

$$\leq \min_{t^{n+1} \epsilon \tilde{T}^{n+1}} E[L(t(y^0, y^1, \ldots, y^n), \lambda^n)]$$

$$= \min_{t^n \epsilon T^n} E[L(t(y^1, \ldots, y^n), \lambda^n)] = R_n(\underline{P})$$

and we conclude that

$$R_n(\underline{P}) \geq R_{n+1}(\underline{P}) \qquad \text{Q. E. D.}$$

The sequence $\{R_n(\underline{P})\}$ is bounded by zero, since we have assumed $L(a, \lambda) \geq 0 \quad \forall a, \quad \forall \lambda$, so it is convergent.

Let us denote

$$\lim_{n \to \infty} R_n(\underline{P}) = R(\underline{P}) \tag{12}$$

Note that by stationarity

$$R(\underline{P}) = \min_{t^\infty \epsilon T^\infty} E[L(t(y^{-\infty}, \ldots, y^0, \ldots, y^n), \lambda^n)] = \min_{t^\infty \epsilon T^\infty} E[L(t(x^1, \ldots, x^\infty, \lambda^n)] \tag{13}$$

Let us now consider the case when at each stage n we make our decision on the basis of a finite number, say $m < n$, of x's in the past, i.e., on the basis of $x^n, \ldots, x^{n-m}$.

Let us denote

$$R_{n-m}(\underline{P}) = \min_{t^{m+1} \epsilon T^{m+1}} E\,[L(t(x^{n-m}, \ldots, x^n), \lambda^n)] \tag{14}$$

this definition makes sense only for $n > m$.

We notice that by stationarity $R_{n-m}(\underline{P})$ does not depend on n, so we shall denote

$$W_m = R_{n-m}(\underline{P}) \tag{15}$$

Lemma 1.2: The sequence of real numbers $\{W_m\}$ is monotonically nonincreasing.

Proof:

$$W_m = R_{n-m}(\underline{P}) = \min_{t^{m+1} \epsilon T^{m+1}} E\,[L(t(y^{n-m}, y^{n-m+1}, \ldots, y^n), \lambda^n)]$$

$$W_{m+1} = R_{n-m-1}(\underline{P}) = \min_{t^{m+2} \epsilon T^{m+2}} E\,[L(t(y^{n-m-1}, y^{n-m}, \ldots, y^n), \lambda^n)].$$

By the same argument as in the proof of Lemma 1.1 we have

$$W_m \geq W_{m+1} \qquad \text{Q.E.D.}$$

This sequence is also bounded and thus convergent and we have

$$\lim_{m \to \infty} W_m = \min_{t^\infty \epsilon T^\infty} E\,[L(t(y^{-\infty}, \ldots, y^0, \ldots y^n), \lambda^n)]$$

$$= \min_{t^{\infty} \epsilon T^{\infty}} E[L(t(x^1, \ldots, x^{\infty}), \lambda^{\infty})] = R(\underline{P}) . \tag{16}$$

Therefore we see that

$$\lim_{m \to \infty} W_m = \lim_{n \to \infty} R_n(\underline{P}). \tag{17}$$

C. THE CASE WHERE $\underline{P}$ IS UNKNOWN

We shall consider now the same problem of decision making as above but in the case where $\underline{P}$ is unknown. If in this case we use, at time n, a decision function t^n other than the optimal $t^n_{\underline{P}}$, our expected loss will be

$$R_n(t^n, \underline{P}) = R_n(\underline{P}) + [R_n(t^n, \underline{P}) - R_n(\underline{P})] \tag{18}$$

and the term $[R_n(t^n, \underline{P}) - R_n(\underline{P})] \geq 0$ will be that part of the expected loss due to our ignorance of the true value of $\underline{P}$. Clearly the observed values $x^1, \ldots, x^n$ contain "information" about $\underline{P}$. We hope for large n to be able to extract some information about $\underline{P}$ from the values $\underline{x}^n$ which have been observed. And we further hope to be able to make decisions at each stage, based on the information about $\underline{P}$, which was extracted in such a way that $t^*(\underline{x}^n)$, the decision rule which we would adopt in this case, is in some sense close to the optimal but unknown $t_{\underline{P}}(\underline{x}^n)$ which we would use throughout if we knew $\underline{P}$.

If such a sequence of functions exists, then we shall refer to the sequence $\{t^{*n}\}$ as an adaptive decision procedure. Correspondingly,

$$R_n(t^{*n}, \underline{P}) = E[L(t^*(\underline{x}^n), \lambda^n)] \tag{19}$$

and we know from (7) that

$$R_n(t^{*n}, \underline{P}) \geq R_n(\underline{P}) . \tag{20}$$

Definition 1: If $\lim_{n\to\infty} R_n(t^{*n}, \underline{P}) = \lim_{n\to\infty} R_n(\underline{P}) = R(\underline{P})$ we say that t^{*n} is asymptotically optimal relative to $\underline{P}$, i.e., if the expected loss in using an adaptive decision procedure, when $\underline{P}$ is unknown, converges as $n\to\infty$ to the same limit as the expected loss when $\underline{P}$ is known and the optimal decision procedure is used, then this adaptive procedure is called asymptotically optimal.

Definition 2: If $\lim_{n\to\infty} R_n(t^{*n}, \underline{P}) \leq R(\underline{P}) + \epsilon$ we say that t^* is ϵ-asymptotically optimal relative to $\underline{P}$.

Since $\lim_{m\to\infty} W_m = R(\underline{P})$ then given any ϵ there exists an m_0 such that $W_{m_0} - R(\underline{P}) \leq \epsilon$ or $W_{m_0} \leq R(\underline{P}) + \epsilon$.

Thus we see that if the expected loss in using t^{*n} converges when $n\to\infty$ to the expected loss when $\underline{P}$ is known and we use an optimal decision procedure which is based on a large, but finite, number m_o of observation in the past, then t^{*n} is ϵ-asymptotically optimal.

II. THE ESTIMATION PROBLEM

A. DERIVATION OF ESTIMATES

Let us denote the probability that at time n the Markov chain is in state i by

$$P(\lambda^n = i) = g_i^n \tag{21}$$

Thus we have at each time n a probability distribution on the state space $\Lambda = \{1, \ldots, r\}$.

$$\underline{g}^n = (g_1^n, \ldots, g_r^n) \qquad g_i^n \geq 0 \qquad \forall i, \forall n \tag{22}$$

and

$$\sum_{i=1}^{r} g_i^n = 1 \quad \forall n \, .$$

Since our system is described by a regular Markov chain we have

$$\lim_{n \to \infty} g_i^n = \pi_i \quad (i = 1, \ldots, r) \tag{23}$$

where $\underline{\pi} = (\pi_1, \ldots, \pi_r)$ is the unique stationary probability distribution.

We are observing a sequence $x^1, x^2, \ldots$ of random variables. Denote

$$P\{x^n \epsilon B / \lambda^n = j\} = P_j(B) \, . \tag{24}$$

$B \epsilon \mathcal{Q}$, the σ-field of events.

The global distribution of x^n is

$$P\{x^n \epsilon B\} = \sum_{j=1}^{r} g_j^n \; P_j(B) \, . \tag{25}$$

We want to construct functions

$$\hat{g}_i^n = \hat{g}_i^n (x^1, \dots, x^n) \tag{26}$$

such that

$$\hat{g}_i^n \geq 0, \qquad \sum_{i=1}^{r} \hat{g}_i^n = 1$$

and whatever be $\underline{\pi}$

$$P\,[\lim_{n \to \infty} \hat{g}_i^n = \pi_i (i = 1, \dots, r)] = 1 \tag{27}$$

and functions

$$\hat{g}_{ij}^n = \hat{g}_{ij}^n (x^1, \dots, x^n) \tag{28}$$

such that

$$\hat{g}_{ij}^n \geq 0 \qquad \sum_{j=1}^{r} \hat{g}_{ij}^n = 1$$

and whatever be $\underline{P}$

$$P\,[\lim_{n \to \infty} \hat{g}_{ij}^n = p_{ij} \ (i,\ j = 1, \dots, r)] = 1\,. \tag{29}$$

In general if the Markov chain is a k-th order multiple Markov chain we want a sequence of estimates that will converge almost surely to the k-th order distribution of the Markov chain.

Theorem 2.1:* A necessary and sufficient condition for the existence of such sequences is the following

(A) $\{$If $G = \{g_1, \dots, g_r\}$ and $G' = \{g_1', \dots, g_r'\}$ are any two probability vectors such that

$$\sum_{i=1}^{r} g_i P_i(B) = \sum_{i=1}^{r} g_i' P_i(B) \qquad \forall B \epsilon \mathcal{A}\}$$

then $G = G'$.

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* The proof given on page 14 parallels the proof in Ref. 1. The condition of Theorem 1 is equivalent to the condition for identifiability of finite mixtures given in Ref. 6.

Proof: It is clear that the above condition is necessary since suppose that $\underline{\pi}_1$ and $\underline{\pi}_2$ were different stationary probability distribution but the global distribution of our observable random variable x would be the same under the two for all $B \in \mathcal{Q}$. We could not hope to be able to distinguish between them and find two different sequences which would converge to the true stationary probability. The sufficiency proof is going to be constructive, before proceeding with the proof let us restate condition (A), the necessary and sufficient conditions of the theorem.

Denote by μ any σ-finite measure on R^1 with respect to which all the P_i are absolutely continuous and such that their densities $f_i = dP_i/d\mu$ are square integrable:

$$\int_{R^1} f_i^2(x)\, d\mu(x) < \infty \qquad (i = 1, \ldots, r). \tag{30}$$

For example if we set $\mu = P_1 + \ldots + P_r$ we have $0 \leq f_i(x) \leq 1$ and hence

$$f_i^2(x) \leq f_i(x)$$

$$\int_{R^1} f_i^2(x)\, d\mu(x) \leq \int_{R^1} f_i(x)\, d\mu(x) = 1 < \infty \quad . \tag{31}$$

The functions f_i are elements of the Hilbert space H over the measure space (R^1, μ). The proof of the following Lemma can be found in Ref. 2.

Lemma 2.2: Condition (A) is equivalent to the following condition (B) $f_1, \ldots, f_r$ are linearly independent.

We shall now proceed with the sufficiency proof of the theorem. If $f_1, \ldots, f_r$ are linearly independent we shall show how to construct the sequences (estimates) desired.

Let L_i denote the linear manifold spanned by the r-1 functions $f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_r$. Then we can write uniquely

$$f_i = f_i' + f_i'' \qquad (i = 1, \ldots, r) \tag{32}$$

with

$$f_i' \in L_i \quad \text{and} \quad f_i'' \perp L_i \tag{33}$$

and $f_i'' \neq 0$ because of linear independence.

If we now set

$$h_i(x) = \frac{f_i''(x)}{\int_{R^1} [f_i''(x)]^2 \, d\mu(x)} \tag{34}$$

we have

$$\int_{R^1} h_i(x) f_j(x) \, d\mu(x) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{35}$$

$h_1, \ldots, h_r$ are a reciprocal basis.
Notice now that

$$E[h_i(x^k)] = \sum_{j=1}^{r} g_j^k \int_{R^1} h_i(x^k) f_j(x^k) d\mu(x^k) = \sum_{j=1}^{r} g_j^k \delta_{ij} = g_i^k \tag{36}$$

and $\lim_{k \to \infty} g_i^k = \pi_i$

Also

$$E[h_i(x^k) h_j(x^{k+1})] =$$

$$= \sum_{s,m} g_s^k \, p_{sm} \int_{R^1 x R^1} h_i(x^k) h_j(x^{k+1}) f_s(x^k) f_m(x^{k+1}) d\mu(x^k) d\mu(x^{k+1})$$

$$= \sum_{s,m} g_s^k \, p_{sm} \int_{R^1} h_i(x^k) f_s(x^k) d\mu(x^k) \int_{R^1} h_j(x^{k+1}) f_m(x^{k+1}) d\mu(x^{k+1}) \tag{37}$$

$$= \sum_{s,m} g_s^k \, p_{sm} \, \delta_{is} \, \delta_{jm} = g_i^k p_{ij}$$

thus we have

$$E\,[h_i(x^k)\, h_j(x^{k+1})] = g_i^k \; p_{ij} \tag{38}$$

$$\lim_{k\to\infty} \; g_i^k \, p_{ij} = \pi_i \, p_{ij} \tag{39}$$

In general for any finite w

$$E\,[h_i(x^k)\, h_j(x^{k+1}) \ldots h_c(x^{k+w-1})] = P[\lambda^k = i, \lambda^{(k+1)} = j, \ldots, \lambda^{(k+w-1)} = c] \; .$$

Now let us set

$$\bar{\hat{g}}_i^n = \frac{1}{n} \sum_{k=1}^{n} h_i(x^k) \tag{40}$$

$$\hat{g}_i^n = \frac{\left[\bar{\hat{g}}_i^n\right]^+}{\sum_{j=1}^{r} \left[\bar{\hat{g}}_j^n\right]^+} \; . \tag{41}$$

where $[a]^+$ denotes max $(a, 0)$.

We shall prove that

$$P\,[\lim_{n\to\infty} \bar{\hat{g}}_i^n = \pi_i \quad (i = 1, \ldots, r)\,] \quad = 1. \tag{42}$$

It is clear that this implies that Eq. (27) holds, i.e.,

$$P\left[\lim_{n\to\infty} \hat{g}_i^n = \pi_i \ (i = 1, \ldots, r)\right] = 1$$

but $\hat{g}_i^n$ has the desired properties (26) $\sum_{i=1}^{r} \hat{g}_i^n = 1$ and $\hat{g}_i^n \geq 0$ $(i = 1, \ldots, r)$. Also set

$$\bar{\hat{g}}_{ij}^n = \frac{1}{n} \sum_{k=1}^{n} [h_i(x^k)\, h_j(x^{k+1})] \tag{43}$$

$$\hat{g}_{ij}^n = \frac{\left[\frac{1}{n} \sum_{k=1}^{n} h_i(x^k)\, h_j(x^{k+1})\right]^+}{\sum_{j=1}^{r} \left[\frac{1}{n} \sum_{k=1}^{n} h_i(x^k) h_j(x^{k+1})\right]^+} \tag{44}$$

we shall prove

$$P\left[\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} [h_i(x^k)\, h_j(x^{k+1})] = \pi_i\, p_{ij} \ (i, j=1, \ldots, r)\right] = 1 \tag{45}$$

which clearly implies that Eq. (29) holds, i.e.,

$$P\left[\lim_{n\to\infty} \hat{g}_{ij}^n = p_{ij} \ (i, j = 1, \ldots, r)\right] = 1$$

and $\hat{g}_{ij}^n$ has the desired properties (28) $\hat{g}_{ij}^n \geq 0$ and $\sum_{j=1}^{r} \hat{g}_{ij}^n = 1.$

The generalization to k-th order multiple Markov chain is clear. In order to prove (42) and (45) we shall study the stochastic process $\{x^n\}$.

Lemma 2.3: The process $\{x^n\}$ can be represented as

$$x^n = \emptyset(\lambda^n, U^n)$$

where U^n is a sequence of uniformly distributed random variables on the unit interval, and the sequences $\{U^n\}$ and $\{\lambda^n\}$ are independent.

Proof: $\{x^n\}$ is a sequence of random variables whose conditional distribution given $\lambda^n = i$ is $F_i(\alpha)$.

Let

$$U^n = F_{\lambda^n}(x^n). \tag{46}$$

Then we have

$$x^n = \tilde{F}_{\lambda^n}(U^n) \tag{47}$$

where

$$\tilde{F}(x) = \inf \{y/F(y) > x\}.$$

We want to show that $\{U^n\}$ is a sequence of uniformly distributed random variables on the unit interval, and the sequences $\{U^n\}$ and $\{\lambda^n\}$ are independent.

Since

$$\begin{aligned}
& P(U^1 \leq a_1, \ldots, U^n \leq a_n / \lambda^1 = i, \lambda^2 = j, \ldots, \lambda^n = \ell) \\
&= P(F_{\lambda^1}(x^1) \leq a_1, \ldots, F_{\lambda^n}(x^n) \leq a_n / \lambda^1 = i, \lambda^2 = j, \ldots, \lambda^n = \ell) \\
&= P(x^1 \leq \tilde{F}_i(a_1), \ldots, x^n \leq \tilde{F}_\ell(a_n) / \lambda^1 = i, \lambda^2 = j, \ldots, \lambda^n = \ell) \\
&= P(x^1 \leq \tilde{F}_i(a_1) / \lambda^1 = i) \ldots P(x^n \leq \tilde{F}_\ell(a_n) / \lambda^n = \ell) \\
&= a_1, a_2, \ldots, a_n
\end{aligned}$$

we can conclude that the U^i's are uniformly distributed and that the sequences $\{U^n\}$ and $\{\lambda^n\}$ are independent.

Therefore we get the desired result

$$x^n = \tilde{F}_{\lambda^n}(U^n) = \emptyset(\lambda^n, U^n) \qquad \text{Q.E.D.} \tag{48}$$

B. CONVERGENCE OF ESTIMATES

Lemma 2.4: The process $\{(\lambda^n, U^n)\}$ is a Markov process satisfying Doeblin's hypothesis.*

The proof of this lemma can be found in Ref. 2.

Lemma 2.5: If $\lambda^1, \lambda^2, \ldots$ is a finite state ergodic Markov chain with arbitrary initial distribution then

$$P\left[\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} h_i(x^k) = \pi_i \ (i = i, \ldots, r)\right] = 1 \tag{49}$$

and an exponential bound can be obtained for

$$P\left[\left|\frac{1}{n} \sum_{k=1}^{n} h_i(x^k) - \pi_i\right| \geq \epsilon \quad \text{for some} \quad n \geq m\right] . \tag{50}$$

Proof: If it is assumed that the initial probability distribution on the state space of the Markov chain is the unique stationary one, then the Markov chain is a stationary process which is ergodic (metrically transitive). It is well known that under this condition $\{(\lambda^n, U^n)\}$ is also a stationary and ergodic process, and since a function of an ergodic process is ergodic, we can conclude that $\{x^n\}$ is a stationary and ergodic process. Using Birkhoff's ergodic theorem** and noting that $E[h_i(x^k)] = \pi_i \ \forall k$ we get (49). Or equivalently, since (λ^n, U^n) is a Markov process which satisfies Doeblin's hypothesis (by Lemma 2.4) we get (49) by a theorem in Doob.***

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* Ref. 3, page 192.

** Ref. 3, page 465.

*** Ibid., 3, Th. 6.1, page 219.

If it is a Markov chain with an arbitrary initial distribution then (49) still holds by another theorem in Doob. *

Now we want to find an exponential bound for (50). We shall use a result of M. Katz, Jr., and A. J. Thomasian.[4] They have proved the following theorem. Let $\{Y^k : k = 1, 2, \ldots\}$ be a discrete parameter Markov process satisfying Doeblin's condition, f a bounded, real-valued, measurable function. Denote $S_n = \sum_{k=1}^{n} f(Y^k)$, and $\mu = \int_{R^1} f(x)\, \pi(dx)$ where π is the unique stationary measure. Then for every $\epsilon > 0$ there exists two constants, c and $\gamma < 1$, such that for all m and any initial distribution

$$P\left[\left|\frac{1}{n} S_n - \mu\right| \geq \epsilon \text{ for some } n \geq m\right] \leq c\, \gamma^m .$$

Using Lemma 2.3 and noting that $h \circ \emptyset$ is a bounded function, we can use the above theorem to obtain the desired exponential bound for (50).

Lemma 2.6: If $\lambda^1, \lambda^2, \ldots$ is a finite state ergodic Markov chain with arbitrary initial distribution, then

$$P\left[\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} h_i(x^k)\, h_j(x^{k+1}) = \pi_i\, p_{ij} (i, j = 1, \ldots, r)\right] = 1 \quad (51)$$

and an exponential bound can be obtained for

$$P\left[\left|\frac{1}{n} \sum_{k=1}^{n} h_i(x^k)\, h_j(x^{k+1}) - \pi_i\, p_{ij}\right| \geq \epsilon \text{ for some } n \geq m\right]. \quad (52)$$

Proof: The following is a well known theorem: The process $\{x^n\}$ is a stationary and ergodic process if and only if

$$\frac{f(x^1, \ldots x^m) + f(x^2, \ldots, x^{m+1}) + \ldots + f(x^n, \ldots, x^{n+m-1})}{n}$$

$$\xrightarrow[n \to \infty]{} E(f(x^1, \ldots, x^m)) \qquad \forall f, \quad \forall m.$$

*Ibid. 3 Th. 6.2, page 220.

From this theorem we conclude that if $\{x^n\}$ is a stationary and ergodic process, i.e., if the initial distribution of the Markov chain is the unique stationary one, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi'(x^k, x^{k+1}) = E[\phi'(x^k, x^{k+1})]. \tag{53}$$

Now, (51) follows from (53), (38), and (30).

Consider now the process $\{Y^n = (\lambda^n, U^n)\}$. Let Y be a space of points $\xi = (\lambda, U)$. Replace Y by the space $\widetilde{Y}$ of points $\widetilde{\xi} : (\xi^1, \xi^2), \xi^i \in Y$, replace $\mathcal{F}_Y$ by the product σ-field $\widetilde{\mathcal{F}}_Y = \mathcal{F}_Y \times \mathcal{F}_Y$, and replace the space of points $(\xi^1, \xi^2, \ldots)$, $\xi^j \in Y$ by the space of points $\widetilde{\omega}: (\widetilde{\xi}^1, \widetilde{\xi}^2, \ldots)$, $\widetilde{\xi}^j \in \widetilde{Y}$. Let $\widetilde{Y}^j$ be the new j-th coordinate function, so that $\widetilde{Y}^j(\widetilde{\omega}) = \widetilde{\xi}^j$. Define $\widetilde{Y}^1, \widetilde{Y}^2, \ldots, \widetilde{\omega}$ probabilities to be the same as $\hat{Y}^1, \hat{Y}^2, \ldots, \omega$ probabilities where $\hat{Y}^j$ is the 2-tuple (Y^j, Y^{j+1}). Then $\{\widetilde{Y}^j\}$ process is a Markov process satisfying Doeblin's hypothesis if the $\{Y^n\}$ process is such a process and we know by Lemma 2.4 that it is. The function f of (ξ^1, ξ^2) defines a function $\widetilde{f}$ of $\widetilde{\xi}^1$, and the ω random variables $\{f(Y^m, Y^{m+1}), m \geq 1\}$ have the same joint distributions as the $\widetilde{\omega}$ random variables $\{\widetilde{f}(\widetilde{Y}^m), m \geq 1\}$.*

Thus we have reduced the problem of convergence of the sequence

$$\frac{1}{n} \sum_{k=1}^{n} h_i(x^k)\, h_j(x^{k+1}) = \frac{1}{n} \sum_{k=1}^{n} \phi(x^k, x^{k+1})$$

$$\frac{1}{n} \sum_{k=1}^{n} \phi'(\phi(\lambda^k, U^k), \phi(\lambda^{k+1}, U^{k+1})) = \frac{1}{n} \sum_{k=1}^{n} f(Y^k, Y^{k+1})$$

to the corresponding problem of convergence of $1/n \sum_{k=1}^{n} \widetilde{f}(\widetilde{Y}^k)$. We get (51) for the non stationary case and the bound for (52) exactly by the same arguments given in the proof of Lemma 2.5.

III. ADAPTIVE DECISION MAKING

A. ADAPTIVE DECISION PROCEDURES

Using the estimates derived in Sec. II, we shall construct an "adaptive" decision procedure t which is ϵ-asymptotically optimal relative to $\underline{P}$ (Definition 2, Sec. I).

Define

$$\phi_{\underline{\hat{P}}}^{m}(a, \underline{x}^n) = \sum_{i=1}^{r} [L(a,i)\hat{P}(\lambda^n = \underbrace{i/x^n \ldots x^{n-m}}_{m+1})] \tag{54}$$

where

$$\hat{P}(\lambda^n = 1/x^n \ldots x^{n-m})$$

$$= \frac{\sum_{k=1}^{r} \cdots \sum_{j=1}^{r} \sum_{i=1}^{r} \hat{g}_i^n \hat{g}_{ij}^n \ldots \hat{g}_{k,1}^n \, f_i(x^{n-m}) \ldots f_1(x^n)}{\underbrace{\sum_{\ell=1}^{r} \sum_{k=1}^{r} \cdots \sum_{j=1}^{r} \sum_{i=1}^{r}}_{(m+1) \text{ sums}} \hat{g}_i^n \hat{g}_{ij}^n \ldots \hat{g}_{k,\ell}^n \, f_i(x^{n-m}) \ldots f_\ell(x^n)} \tag{55}$$

By (27) and (29) it is clear that for a fixed m

$$P[\lim_{n \to \infty} | \hat{P}(\lambda^n = i/x^n \ldots x^{n-m}) - P(\lambda^n = i/x^n \ldots x^{n-m}) | = 0 (i = 1, \ldots, r) \forall m \text{ finite}] = 1. \tag{56}$$

Let $A = \{a_0, \ldots, a_s\}$ be a finite set, choose $t_{\underline{\hat{P}}}(\underline{x}^{n-m})$ such that for almost every $(\mu)\underline{x}^n$

$$\phi_{\underline{\hat{P}}}^{m}(t_{\underline{\hat{P}}}(\underline{x}^{n-m}), x^n) = \min_{a_i \in A} \phi_{\underline{\hat{P}}}(a_i, \underline{x}^n) \text{ for } n \leq m \tag{57}$$

$$\phi^{m}_{\underline{\hat{P}}}\left(t_{\underline{\hat{P}}}(\underline{x}^{n-m}), \underline{x}^{n}\right) = \min_{a_i \in A} \phi^{m}_{\underline{\hat{P}}}(a_i, \underline{x}^{n}) \quad \text{for } n > m$$

thus for $n \leq m$

$t_{\underline{\hat{P}}}(\underline{x}^{n-m}) = a_k$ where k is any integer $0 \leq k \leq s$ such that

$$\phi_{\underline{\hat{P}}}(a_k, \underline{x}^{n}) = \min \left[\phi_{\underline{\hat{P}}}(a_0, \underline{x}^{n}), \ldots, \phi_{\underline{\hat{P}}}(a_s, \underline{x}^{n}) \right]$$

for $n > m$ $t_{\underline{\hat{P}}}(\underline{x}^{n-m}) = a_k$ where k is any integer $0 \leq k \leq s$ such that

$$\phi^{m}_{\underline{\hat{P}}}(a_k, \underline{x}^{n}) = \min \left[\phi^{m}_{\underline{\hat{P}}}(a_0, \underline{x}^{n}), \ldots, \phi^{m}_{\underline{\hat{P}}}(a_s, \underline{x}^{n}) \right] . \tag{58}$$

In what follows it is assumed that the Markov chain is started with the unique stationary probability distribution. The $\{x^n\}$ process then, is a stationary and ergodic process.

Let $Y = \{(y^{-1}, y^{-2}, \ldots\ldots) = \underline{y}\}$. We define the following probability for every finite subfamily of y's.

$$\overline{P}(y^{-1} \in A_1, \ldots, y^{-n} \in A_n) = P(x^n \in A_1, \ldots, x^n \in A_n).$$

This probability is consistent* since

$$\overline{P}(y^{-1} \in A_1, \ldots, y^{-n} \in \Omega) = P(x^n \in A_1, \ldots, x^1 \in \Omega)$$

$$= P(x^n \in A_1, \ldots, x^2 \in A_{n-1}) = P(x^{n-1} \in A_1, \ldots, x^1 \in A_{n-1})$$

$$= \overline{P}(y^{-1} \in A_1, \ldots, y^{-n+1} \in A_{n-1}) .$$

Then by applying the consistency** theorem, we obtain a law defined for $(y^{-1}, \ldots)$.

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* Ref. 5 page 92.

** Ibid., page 93.

Define the following transformation $T_n(y^{-1},\ldots) = (y^{-n},\ldots,y^{-1})$. Let $\mathcal{F}^n = \mathcal{F}(x^1,\ldots,x^n)$ the smallest σ-field for which $(x^1,\ldots x^n)$ are measurable.

Then

$$\overline{P}(T_n^{-1}(A)) = P(A) \qquad \forall A \in \mathcal{F}^n$$

and define

$$f_n(i,\ (y^{-1},\ldots)) = P(\lambda^n = i/T_n\underline{y}) . \tag{59}$$

Lemma 3.1: $f_n(i, (y^{-1}y, \overset{-2}{\ldots}))$ is a martingale sequence.*

Proof: Let B be an n-dim Borel set and Ω 1-dim space. Then

$$\int_{T_{n+1}^{-1}(\Omega x B)} f_{n+1}(i,\underline{y})d\overline{P} = \int_{T_{n+1}^{-1}(\Omega x B)} P(\lambda^{n+1}=i/T_{n+1}\underline{y})d\overline{P}$$

$$= \int_{\Omega x B} P(\lambda^{n+1}=i/x^1\ldots x^{n+1})dP = P(\lambda^{n+1}=i,\ (x^2\ldots x^{n+1})\in B)$$

$$= P(\lambda^n = i,\ (x^1\ldots x^n)\in B) = \int_B P(\lambda^n = i/x^1\ldots x^n)dP$$

$$= \int_{T_n^{-1}B} P(\lambda^n/T_n\underline{y})d\overline{P} = \int_{T_n^{-1}B} f_n(i,\underline{y})\ d\overline{P} .$$

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* For $T = (1, 2, \ldots)$ a stochastic process $\{x^n, n\in T\}$ is called a martingale if $E\{|x^n|\} < \infty$ for all n and if

$$E(x^{n+1}/\mathcal{F}^n) = x^n \text{ a.s.},$$

i.e., if $A\in\mathcal{F}^n$ $\displaystyle\int_A x^{n+1}\,dP = \int_A x^n\,dP.$

Theorem 3.2: There exists an m_0 such that $t_{\hat{\underline{P}}}^{m_0}(\underline{x}^n)$ is ϵ-asymptotically optimal relative to $\underline{P}$.

Proof: We want to show that

$$\lim_{n\to\infty} R_{(n-m_0)}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{P}) = R_{n-m_0}(\underline{P}) \leq R(\underline{P}) + \epsilon \tag{60}$$

$$0 \leq R_{n-m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{P}) - R_{n-m_0}(\underline{P}) \text{ by (22)}$$

$$0 \leq R_{n-m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{P}) - R_{n-m_0}(\underline{P}) \tag{61}$$

$$= E[\phi_{\underline{P}}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{x}^n) - \phi_{\underline{P}}(t_{\underline{P}}(\underline{x}^{n-m_0}), \underline{x}^n)]$$

$$= E[\phi_{\underline{P}}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{x}^n) - \phi_{\hat{\underline{P}}}^{m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{x}^n)$$

$$+ \phi_{\hat{\underline{P}}}^{m_0}(t_{\hat{\underline{P}}}(x^{n-m_0}), \underline{x}^n) - \phi_{\hat{\underline{P}}}^{m_0}(t_{\underline{P}}(x^{n-m_0}), \underline{x}^n)$$

$$+ \phi_{\hat{\underline{P}}}^{m_0}(t_{\underline{P}}(x^{n-m_0}), \underline{x}^n - \phi_{\underline{P}}(t_{\underline{P}}(x^{n-m_0}), \underline{x}^n)]$$

$$= E[\phi_{\underline{P}}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{x}^n) - \phi_{\underline{P}}^{m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{x}^n)$$

$$+ \phi_{\underline{P}}^{m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{x}^n) - \phi_{\hat{\underline{P}}}^{m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{x}^n)$$

$$+ \phi_{\hat{\underline{P}}}^{m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{x}^n) - \phi_{\hat{\underline{P}}}^{m_0}(t_{\underline{P}}(\underline{x}^{n-m_0}), \underline{x}^n)$$

$$+ \phi_{\hat{\underline{P}}}^{m_0}(t_{\underline{P}}(\underline{x}^{n-m_0}), \underline{x}^n) - \phi_{\underline{P}}^{m_0}(t_{\underline{P}}(\underline{x}^{n-m_0}), \underline{x}^n)$$

$$+ \phi_{\underline{P}}^{m_0}(t_{\underline{P}}(\underline{x}^{n-m_0}),\ \underline{x}^n) - \phi_{\underline{P}}(t_{\underline{P}}(\underline{x}^{n-m_0}),\ \underline{x}^n)\,] \ .$$

Note that

$$\phi_{\hat{\underline{P}}}^{m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}),\ \underline{x}^n) - \phi_{\hat{\underline{P}}}^{m_0}(t_{\underline{P}}(x^{n-m_0}),\ \underline{x}^n) \le 0 \ .$$

Then we have

$$R_{n-m_0}(t_{\hat{\underline{P}}}(x^{n-m_0}),\ \underline{P}) - R_{n-m_0}(\underline{P}) \tag{62}$$

$$\le E\,[\,\phi_{\underline{P}}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}),\ \underline{x}^n) - \phi_{\underline{P}}^{m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}),\ \underline{x}^n)\,]$$

$$+ E\,[\,\phi_{\underline{P}}^{m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}),\ \underline{x}^n) - \phi_{\hat{\underline{P}}}^{m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}),\ \underline{x}^n)\,]$$

$$+ E\,[\,\phi_{\hat{\underline{P}}}^{m_0}(t_{\underline{P}}(\underline{x}^{n-m_0}),\ \underline{x}^n) - \phi_{\underline{P}}^{m_0}(t_{\underline{P}}(\underline{x}^{n-m_0}),\ \underline{x}^n)\,]$$

$$+ E\,[\,\phi_{\underline{P}}^{m_0}(t_{\underline{P}}(\underline{x}^{n-m_0}),\ \underline{x}^n) - \phi_{\underline{P}}(t_{\underline{P}}(\underline{x}^{n-m_0}),\ \underline{x}^n)\,] \ .$$

Let us look at each of the terms above

$$E\,[\,\phi_{\underline{P}}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}),\ \underline{x}^n) - \phi_{\underline{P}}^{m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}),\ \underline{x}^n)\,]$$

$$= E\left[\sum_{i=1}^{r} L(t_{\hat{\underline{P}}}(x^{n-m_0}), i)P(\lambda_n = i/x^1, \ldots, x^n)\right.$$

$$\left. - \sum_{i=1}^{r} L(t_{\hat{\underline{P}}}(x^{n-m_0}), i)P(\lambda^n = i/x^n, \ldots, x^{n-m_0})\right].$$

By the construction above, (59), and the fact that $P(\lambda^n = i/(x^n, \ldots, x^{n-m_0}) = x) = P(\lambda^{m_0+1} = i/(x^1, \ldots, x^{m_0+1}) = x)$, we have

$$= E\left[\sum_{i=1}^{r} L(t_{\hat{\underline{P}}}(T_{m_0+1}\underline{y}), i) f_n(i, \underline{y}) - \sum_{i=1}^{r} L(t_{\hat{\underline{P}}}(T_{m_0+1}\underline{y}), i) f_{m_0+1}(i, \underline{y})\right]$$

$$= E\left[\sum_{i=1}^{r} L(t_{\hat{\underline{P}}}(T_{m_0+1}\underline{y}), i)\,(f_n(i, \underline{y}) - f_{m_0+1}(i, \underline{y}))\right].$$

Since $f_n(i, y)$ is a martingale sequence we can choose m_0 so large that $\lim_{n\to\infty} |f_n(i, \underline{y}) - f_{m_0+1}(i, \underline{y})| \leq (\epsilon_1/2r\, L)$ by the martingale convergence theorem,* where $L = \max_{j,i} L(a_j, i) < \infty$.

Since

$$\left[\sum_{i=1}^{r} L(t_{\hat{\underline{P}}}(T_{m_0+1}\underline{y}), i)\,[f_n(i, \underline{y}) - f_{m_0+1}(i, \underline{y})]\right] < rL < \infty$$

using the Dominated Convergence Theorem,** we get

$$\lim_{n\to\infty} E\left[\phi_{\underline{P}}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{x}^n) - \phi_{\underline{P}}^{m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{x}^n)\right]$$

$$\leq \lim_{n\to\infty} E\left[\sum_{i=1}^{r} L(t_{\underline{P}}(T_{m_0+1}\underline{y}), i)\,[f_n(i, \underline{y}) - f_{m_0+1}(i, \underline{y})]\right]$$

$$\leq \frac{\epsilon_1}{2}$$

By the same argument we get

* Ref. 3, page 319.

** Ref. 5, page 125.

$$\lim_{n\to\infty} E\left[\phi_{\underline{P}}^{m_0}(t_{\underline{P}}(\underline{x}^{n-m_0}), \underline{x}^n) - \phi_{\underline{P}}(t_{\underline{P}}(x^{n-m_0}), \underline{x}^n)\right] \le \frac{\epsilon_1}{2}$$

and

$$\lim_{n\to\infty} E\left[\phi_{\underline{P}}^{m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{x}^n) - \phi_{\hat{\underline{P}}}^{m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{x}^n)\right]$$

$$= \lim_{n\to\infty} E\left[\sum_{i=1}^{r} L(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{x}^n)\left[(P(\lambda^n = i/x^n \ldots x^{n-m_0}) - \hat{P}(\lambda^n = i/x^n \ldots x^{n-m_0})\right]\right] = 0$$

by (56), and using the dominated convergence theorem.

Similarly

$$\lim_{n\to\infty} E\left[\phi_{\hat{\underline{P}}}^{m_0}(t_{\underline{P}}(\underline{x}^{n-m_0}), \underline{x}^n) - \phi_{\underline{P}}^{m_0}(t_{\underline{P}}(\underline{x}^{n-m_0}), x^n)\right] = 0 .$$

Finally we have

$$\lim_{n\to\infty} R_{n-m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{P}) - R_{n-m_0}(\underline{P}) \le \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} = \epsilon_1 .$$

For this value of m_0 we have by (17) $R_{n-m_0}(\underline{P}) \le R(\underline{P}) + \epsilon_2$

therefore $\lim_{n\to\infty} R_{n-m_0}(t_{\hat{\underline{P}}}(\underline{x}^{n-m_0}), \underline{P}) \le R(P) + \epsilon$. Thus,

$t_{\hat{\underline{P}}}(\underline{x}^{n-m_0})$ is ϵ-asymptotically optimal relative to $\underline{P}$. This completes the proof of the theorem.

Consider the following decision rule t^*. Let ϵ_n be any sequence of constants tending to zero. Use $t_{\hat{\underline{P}}}(x^n)$ for n_1 steps until for $n > n_1$ we have

$$|\hat{P}(\lambda^n = i/x^n x^{n-1}) - P(\lambda^n = i/x^n x^{n-1})| < \epsilon_1 .$$

Then start using $t_{\hat{\underline{P}}}(x^n x^{n-1})$ for additional n_2 steps until for $n \ge n_2 + n_1$ we have

$$|\hat{P}(\lambda^n = i/x^n x^{n-1} x^{n-2}) - P(\lambda^n = i/x^n x^{n-1} x^{n-2})| < \epsilon_2$$

then start using $t_{\hat{\underline{P}}}(x^n, x^{n-1}, x^{n-2})$, and so on. In general use $t_{\hat{\underline{P}}}(\underline{x}^{n-m_0})$ for n_{m_0+1} steps until for $n \ge n_1 + n_2 + \ldots + n_{m_0+1}$

$$|\hat{P}(\lambda^n = i/\underline{x}^{n-m_0}) - P(\lambda^n = i/\underline{x}^{n-m_0})| < \epsilon_{m_0+1},$$

then start using $t_{\hat{\underline{P}}}(\underline{x}^{n-(m_0+1)})$.

We see that in this rule we keep increasing m_0 the number of steps that we look back for our decision. Referring to the proof of Theorem 3.2 we see that in the expression $R(\underline{P}) + \epsilon$ increasing m_0 will decrease ϵ and in the limit we shall converge to $R(\underline{P})$. So we have the following theorem.

Theorem 3.3: Rule t^* is asymptotically optimal relative to $\underline{P}$.

B. REMARKS

It would be interesting to settle the question of whether or not the following convergence holds

$$\left| \hat{P}(\lambda^n = i/x^1, \ldots, x^n) - P(\lambda^n = i/x^1, \ldots, x^n) \right| \xrightarrow{n \to \infty} 0 \quad (i=1, \ldots, r) \tag{63}$$

almost surely or in probability.

If the convergence holds, almost surely or in probability, the following decision rule t^* is asymptotically optimal. For, if

$$\phi_{\hat{\underline{P}}}(a, \underline{x}^n) = \sum_{i=1}^{r} [L(a, i)\hat{P}(\lambda^n = i/x^1 \ldots x^n)]$$

then choose $t^*_{\hat{\underline{P}}}(\underline{x}^n)$ such that for almost every $(\mu)\underline{x}^n$

$$\phi_{\hat{\underline{P}}}(t^*_{\hat{\underline{P}}}(\underline{x}^n), \underline{x}^n) = \min_{a_i \epsilon A} \phi_{\hat{\underline{P}}}(a_i, \underline{x}^n)$$

or

$t^*_{\hat{\underline{P}}}(\underline{x}^n) = a_k$ where k is any integer $0 \leq k \leq r$ such that

$$\phi_{\hat{\underline{P}}}(a_k, \underline{x}^n) = \min [\phi_{\hat{\underline{P}}}(a_0, \underline{x}^n), \ldots, \phi_{\hat{\underline{P}}}(a_r, \underline{x}^n)] .$$

The proof that t^* is asymptotically optimal if (63) holds proceeds as follows:

$$0 \leq [R_n(t^*_{\hat{\underline{P}}}(\underline{x}^n), \underline{P}) - R_n(\underline{P})] \tag{64}$$

$$= E[\phi_{\underline{P}}(t^*_{\hat{\underline{P}}}(\underline{x}^n), \underline{x}^n) - \phi_{\underline{P}}(t_{\underline{P}}(\underline{x}^n), x^n)]$$

now

$$0 \leq [\phi_{\underline{P}}(t^*_{\hat{\underline{P}}}(\underline{x}^n), \underline{x}^n) - \phi_{\underline{P}}(t_{\underline{P}}(\underline{x}^n), x^n)] \tag{65}$$

$$= [\phi_{\underline{P}}(t_{\underline{\hat{P}}}^{*}(\underline{x}^n), \underline{x}^n) - \phi_{\underline{\hat{P}}}(t_{\underline{\hat{P}}}^{*}(\underline{x}^n), \underline{x}^n)]$$

$$+ [\phi_{\underline{\hat{P}}}(t_{\underline{\hat{P}}}^{*}(\underline{x}^n), \underline{x}^n) - \phi_{\underline{\hat{P}}}(t_{\underline{P}}(\underline{x}^n), \underline{x}^n)]$$

$$+ [\phi_{\underline{\hat{P}}}(t_{\underline{P}}(\underline{x}^n), \underline{x}^n) - \phi_{\underline{P}}(t_{\underline{P}}(\underline{x}^n), \underline{x}^n)] \quad .$$

Since $0 \geq [\phi_{\underline{\hat{P}}}(t_{\underline{\hat{P}}}^{*}(\underline{x}^n), \underline{x}^n) - \phi_{\underline{\hat{P}}}(t_{\underline{P}}(\underline{x}^n), \underline{x}^n)]$ we have

$$[\phi_{\underline{P}}(t_{\underline{\hat{P}}}^{*}(\underline{x}^n, \underline{x}^n) - \phi_{\underline{P}}(t_{\underline{P}}(\underline{x}^n), \underline{x}^n)]$$

$$\leq [\phi_{\underline{P}}(t_{\underline{\hat{P}}}^{*}(\underline{x}^n), \underline{x}^n) - \phi_{\underline{\hat{P}}}(t_{\underline{P}}(\underline{x}^n), \underline{x}^n)]$$

$$+ [\phi_{\underline{\hat{P}}}(t_{\underline{P}}(\underline{x}^n), \underline{x}^n) - \phi_{\underline{P}}(t_{\underline{P}}(\underline{x}^n), \underline{x}^n)]$$

$$= \left[\sum_{i=1}^{r} L(t_{\underline{\hat{P}}}^{*}(\underline{x}^n), i)P(\lambda^n = i/x^1 \ldots x^n) - \sum_{i=1}^{r} L(t_{\underline{\hat{P}}}^{*}(\underline{x}^n), i) \cdot \hat{P}(\lambda^n = i/x^1 \ldots x^n)\right]$$

$$+ \left[\sum_{i=1}^{r} L(t_{\underline{P}}(\underline{x}^n), i)P(\lambda^n = i/x^1 \ldots x^n) - \sum_{i=1}^{r} L(t_{\underline{P}}(\underline{x}^n), i) \cdot P(\lambda^n = i/x^1 \ldots x^n)\right]$$

$$= \left[\sum_{i=1}^{r} L(t_{\underline{\hat{P}}}^{*}(\underline{x}^n), i)\,[P(\lambda^n = i/x^1 \ldots x^n) - \hat{P}(\lambda^n = i/x^1 \ldots x^n)]\right]$$

$$+ \left[\sum_{i=1}^{r} L(t_{\underline{P}}(\underline{x}^n), i)\,[\hat{P}(\lambda^n = i/x^1 \ldots x^n) - P(\lambda^n = i/x^1 \ldots x^n)]\right] .$$

If (63) holds we have

$$P\left[\lim_{n\to\infty}\left[\phi_{\underline{\hat{P}}}(t^*_{\underline{P}}(\underline{x}^n),\underline{x}^n) - \phi_{\underline{P}}(t_{\underline{P}}(\underline{x}^n),\ x^n)\right] = 0\right] = 1$$

(or convergence in probability if this is the convergence in (64)).

Using the dominated convergence theorem our proof is complete, since from $\lim_{n\to\infty} R_n(\underline{P}) = R(\underline{P})$ we deduce that $\lim_{n\to\infty} R_n(t^*_{\underline{\hat{P}}}(x^n),\underline{P}) = R(\underline{P})$ and hence that t^* is asymptotically optimal.

Note that the rule t^* is the same type as the rule given in Theorem 3.3, the difference being that $m_0+1 = n$ and we increase m_0 at every step. This amounts to always basing our decision at the n-th step on all the observations up to this point.

C. EXAMPLE:

Let us consider a discrete state discrete time system S whose state transitions constitute an ergodic Markov chain with two possible states μ_1 and μ_2. Denote the state space of S by $\Lambda = \{\mu_1, \mu_2\}$. It is known that when S is in state μ_1 the distribution of the observable random variable x is gaussian with mean μ_1 and variance σ^2, and when S is in state μ_2 the distribution is gaussian with mean μ_2 and the same variance σ^2.

Thus, the two possible densities of x are

$$f_1(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp - \left\{\frac{1}{2\sigma^2}(x-\mu_1)^2\right\}$$

and

$$f_2(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp - \left\{\frac{1}{2\sigma^2}(x-\mu_2)^2\right\}.$$

Let the action space consist of three possible actions, $A = \{a_1, a_2, a_3\}$. The action a_1 is "guess that S is in state μ_1." The action a_2 is "guess that S is in state μ_2," and the action a_3 is "guess that S is in state μ_3." Three possible actions were chosen to demonstrate the fact that the number of elements in the action space does not necessarily have to

Thus we have

$$\phi_{\underline{P}}(a_3, \underline{x}^n) = (\mu_1-\mu_3)^2 \; P(\lambda^n = \mu_1/\underline{x}^n) + (\mu_2-\mu_3)^2 \; P(\lambda^n = \mu_2/\underline{x}^n)$$

$$\phi_{\underline{P}}(a_2, \underline{x}^n) = (\mu_1-\mu_2)^2 \; P(\lambda^n = \mu_1/\underline{x}^n)$$

$$\phi_{\underline{P}}(a_1, \underline{x}^n) = (\mu_2-\mu_1)^2 \; P(\lambda^n = \mu_2/\underline{x}^n)$$

where

$$P(\lambda^n = \mu_i/\underline{x}^n)$$

$$= \frac{\sum_{k=1}^{2} \cdots \sum_{j=1}^{2} \sum_{i=1}^{2} \pi_i \, p_{ij} \cdots p_{k\mu_i} \; f_i(x^1) \cdots f_k(x^{n-1}) f_{\mu_i}(x^n)}{\sum_{\ell=1}^{2} \sum_{k=1}^{2} \cdots \sum_{j=1}^{2} \sum_{i=1}^{2} \pi_i \, p_{ij} \cdots p_{k\ell} \; f_i(x^1) \cdots f_k(x^{n-1}) f_\ell(x^n)}$$

and $i, j, \ldots, \ell$ can take on the values μ_1 and μ_2.

Suppose we are faced with the same problem but the transition matrix $\underline{P}$ is unknown. The first question which arises is, can we solve the problem in this case? By Theorem 2.1 and Lemma 2.2 we can find the sequences of estimates, which we shall use for our decisions, if and only if $f_1(x)$ and $f_2(x)$ are linearly independent. Notice that μ, as defined on page 14, is Lebesque measure in this case, and it is clear that the f_i's are square integrable. Using Gram's criterion for linear dependence of vectors it can be easily verified that if the μ_i's are distinct, f_1 and f_2 are linearly independent.

equal the number of states of S. From the derivation on page 42 it is clear that when using the loss function defined below, a_3 might be used in the optimal decision rule even though S cannot be in state μ_3.

The loss function L is the square of the difference between the state guessed and the true state of S. Thus

$$L(a_1, \mu_1) = 0 \qquad\qquad L(a_1, \mu_2) = (\mu_2-\mu_1)^2$$

$$L(a_2, \mu_1) = (\mu_1-\mu_2)^2 \qquad\qquad L(a_2, \mu_2) = 0$$

$$L(a_3, \mu_1) = (\mu_1-\mu_2)^2 \qquad\qquad L(a_3, \mu_2) = (\mu_2-\mu_3)^2 \quad .$$

Based on the observations $x^1, \ldots, x^n$, we are required to take one of the three possible actions a_1, a_2, a_3. Let us notice that the process $\{x^n\}$ can be represented as the following sum,

$$x^n = \lambda^n + N$$

where $\{\lambda^n\}$ is the Markov chain with state space Λ, and N is a gaussian random variable with zero mean and variance σ^2. This is illustrated in Fig. 2.

If the transition matrix $\underline{P}$ is known then by (4) the optimal decision rule t_p^n is as follows.

If we observe $\underline{x}^n$ then

$$t_{\underline{P}}(\underline{x}^n) = a_k \quad \text{where } k \text{ is any integer } 0 < k \leq 3 \text{ such that}$$

$$\phi_{\underline{P}}(a_k, \underline{x}^n) = \min\left[\, \phi_{\underline{P}}(a_1, \underline{x}^n),\ \phi_{\underline{P}}(a_2, \underline{x}^n),\ \phi_{\underline{P}}(a_3, \underline{x}^n)\,\right]$$

where, as defined in (2)

$$\phi_{\underline{P}}(a, \underline{x}^n) = E[L(a, \lambda^n)/\underline{x}^n] = \sum_{i=1}^{2} L(a, \mu_i)\, P(\lambda^n = \mu_i/\underline{x}^n).$$

We shall now derive h_1 and h_2. Let us remember that the h_i's are defined by (35) as $(h_i, f_j) = \delta_{ij}$. h_1, h_2 are then a reciprocal basis to the basis vectors f_1, f_2.

The following form for the h_i's is easily arrived at using the Schmidt orthagonalization process.

$$h_1(x) = \frac{\frac{1}{\sigma\sqrt{2\pi}} \exp - \left\{\frac{1}{2\sigma^2} (x-\mu_1)^2\right\}\left(\frac{1}{2\sigma\sqrt{\pi}}\right)}{\frac{1}{4\sigma^2\pi} - \frac{\exp - \left\{\frac{(\mu_1-\mu_2)^2}{2\sigma^2}\right\}}{4\sigma^2\pi}}$$

$$- \frac{\frac{1}{4\sigma^2\pi} \exp - \left\{\frac{(\mu_1-\mu_2)^2}{2\sigma^2}\right\} \frac{1}{\sigma\sqrt{2\pi}} \exp - \left\{\frac{1}{2\sigma^2} (x-\mu_2)^2\right\}}{\frac{1}{4\sigma^2\pi} - \frac{\exp - \left\{\frac{(\mu_1-\mu_2)^2}{2\sigma^2}\right\}}{4\sigma^2\pi}}$$

$$h_1(x) = \frac{\sqrt{2} \exp - \left\{\frac{1}{2\sigma^2} (x-\mu_1)^2\right\} - \frac{1}{\sigma\sqrt{2\pi}} \exp - \left\{\frac{(\mu_1-\mu_2)^2}{2\sigma^2}\right\} \exp - \left\{\frac{1}{2\sigma^2} (x-\mu_2)^2\right\}}{1 - \exp - \left\{\frac{(\mu_1-\mu_2)^2}{2\sigma^2}\right\}}$$

$$h_2(x) = \frac{\sqrt{2}\ \exp - \left\{\frac{1}{2\sigma^2}(x-\mu_2)^2\right\} - \frac{1}{\sigma\sqrt{2\pi}}\ \exp - \left\{\frac{(\mu_1-\mu_2)^2}{2\sigma^2}\right\} \exp - \left\{\frac{1}{2\sigma^2}(x-\mu_1)^2\right\}}{1 - \exp - \left\{\frac{(\mu_1-\mu_2)^2}{2\sigma^2}\right\}}$$

By (40) we have

$$\overline{\hat{g}}_i^n = \frac{1}{n} \sum_{k=1}^{n} h_i(x^k) \ ;$$

$$\overline{\hat{g}}_1^n = \frac{1}{n} \sum_{k=1}^{n} h_1(x^k) = \frac{1}{n\left(1-\exp - \left\{\frac{(\mu_1-\mu_2)^2}{2\sigma^2}\right\}\right)} \sum_{k=1}^{n} \left[\sqrt{2}\exp - \left\{\frac{1}{2\sigma^2}(x^k-\mu_1)^2\right\}\right.$$

$$\left. - \frac{1}{\sigma\sqrt{2\pi}}\ \exp - \left\{\frac{(\mu_1-\mu_2)^2}{2\sigma^2}\right\} \exp - \left\{\frac{1}{2\sigma^2}(x^k-\mu_2)^2\right\}\right].$$

Similarly we get an expression for g_2^n. Using (41) we get

$$\hat{g}_i^n = \frac{[\hat{g}_i^n]^+}{[\hat{g}_1^n]^+ + [\hat{g}_2^n]^+} \qquad i = 1, 2 \ .$$

By (43) we have

$$\overline{\hat{g}}_{12}^n = \frac{1}{n} \sum_{k=1}^{n} [h_1(x^k)\ h_2(x^{k+1})]$$

$$\bar{\hat{g}}^n_{12} = \frac{1}{n\left(1-\exp -\left\{\frac{(\mu_1-\mu_2)^2}{2\sigma^2}\right\}\right)} \sum_{k=1}^{n}\Bigg[\Bigg(\sqrt{2}\exp - \left\{\frac{1}{2\sigma^2}(x^k-\mu_1)^2\right\} - \frac{1}{\sigma\sqrt{2\pi}}$$

$$\exp -\left\{\frac{(\mu_1-\mu_2)^2}{2\sigma^2}\right\}\exp - \left\{\frac{1}{2\sigma^2}(x^k-\mu_2)^2\right\}\Bigg)\Bigg(2\ \exp -\Bigg\{\frac{1}{2\sigma^2}$$

$$(x^{k+1}-\mu_1)^2\Bigg\} - \frac{1}{\sigma\sqrt{2\pi}}\ \exp -\left\{\frac{(\mu_1-\mu_2)^2}{2\sigma^2}\right\}\exp - \left\{\frac{1}{2\sigma^2}(x^{k+1}-\mu_2)^2\right\}\Bigg)\Bigg].$$

Similarly we find the expressions for $\bar{\hat{g}}^n_{11}$, $\bar{\hat{g}}^n_{21}$, $\bar{\hat{g}}^n_{22}$. Using (44) we get

$$\hat{g}^n_{ij} = \frac{[\bar{\hat{g}}^n_{ij}]^+}{[\bar{\hat{g}}^n_{i1}]^+ + [\bar{\hat{g}}^n_{i2}]^+}.$$

By (57) and Theorem 3.2 we shall adopt the following decision rule:
for $n \leq m_0$

$t_{\underline{\hat{P}}}(\underline{x}^n) = a_k$ where k is any integer $0 < k \leq 3$ such that

$$\phi_{\underline{\hat{P}}}(a_k, \underline{x}^n) = \min[\phi_{\underline{\hat{P}}}(a_1, \underline{x}^n), \phi_{\underline{\hat{P}}}(a_2, \underline{x}^n), \phi_{\underline{\hat{P}}}(a_3, \underline{x}^n)].$$

for $n > m_0$

$t_{\underline{\hat{P}}}(\underline{x}^{n-m_0}) = a_k$ where k is any integer $0 < k \leq 3$ such that

$$\phi^{m_0}_{\underline{\hat{P}}}(a_k, \underline{x}^n) = \min[\phi^{m_0}_{\underline{\hat{P}}}(a_1, \underline{x}^n), \phi^{m_0}_{\underline{\hat{P}}}(a_2, \underline{x}^n), \phi^{m_0}_{\underline{\hat{P}}}(a_3, \underline{x}^n)].$$

as defined in (55)

$$\phi^{m_0}_{\underline{\hat{P}}}(a, \underline{x}^n) = [L(a, \mu_1)\hat{P}(\lambda^n = \mu_1/\underline{x}^{n-m_0}) + L(a, \mu_2)\hat{P}(\lambda^n = \mu_2/\underline{x}^{n-m_0})]$$

and

$$\phi_{\underline{\hat{P}}}(a, \underline{x}^n) = [\, L(a, \mu_1)\, \hat{P}(\lambda^n = \mu_1/\underline{x}^n) + L(a, \mu_2)\, \hat{P}(\lambda^n = \mu_2/\underline{x}^n)\,]$$

where

$$\hat{P}(\lambda^n = \mu_i/\underline{x}^{n-m_0}) = \frac{\sum_{k=1}^{2} \cdots \sum_{j=1}^{2} \sum_{i=1}^{2} [\hat{g}_i^n \hat{g}_{ij}^n \cdots \hat{g}_{k\mu_i}^n f_i(x^{n-m_0} \cdots f_k(x^{n-1}) f_{\mu_i}(x^n)]}{\sum_{\ell=1}^{2} \sum_{k=1}^{2} \cdots \sum_{j=1}^{2} \sum_{i=1}^{2} [\hat{g}_i^n \hat{g}_{ij}^n \cdots \hat{g}_{k\ell}^n\, f_i(x^{n-m_0}) \cdots f_k(x^{n-1}) f_\ell(x^n)]}$$

The above procedure is illustrated in Fig. 3, where $\hat{\lambda}^n$ denotes the guess about λ^n, and can take on the values μ_1, μ_2, μ_3.

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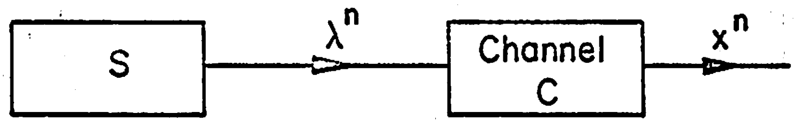
S
λ^n
Channel
C
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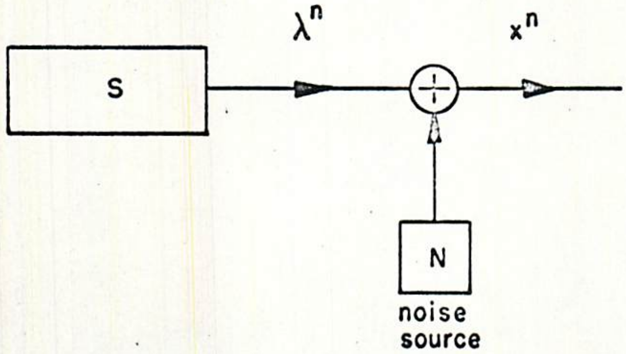

S
λ^n
x^n
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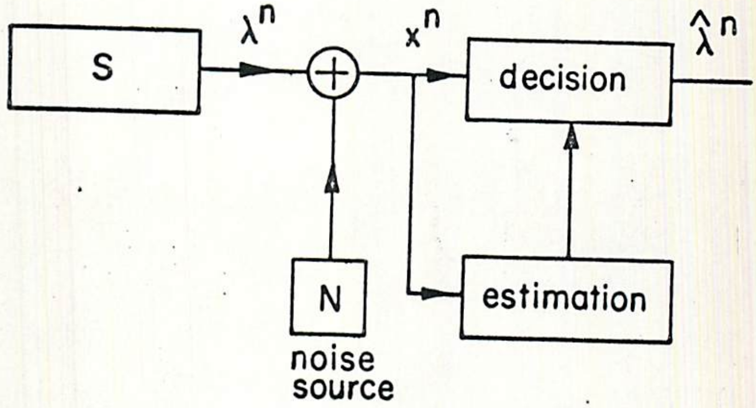
source



S
λ^n
x^n
decision
$\hat{\lambda}^n$
N
noise source
estimation