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SOME RESULTS IN STABILITY THEORY

by

C. A. Desoer

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Charles A. Desoer
Department of Electrical Engineering
Electronics Research Laboratory
University of California, Berkeley, California*

The first result we wish to establish is a generalization of the Nyquist criterion.

ASSUMPTIONS

Following Nyquist[1] we consider the linear time-invariant single-loop feedback system shown on Figure 1. It will be referred to as the closed-loop system. The block labeled k is a constant gain factor (i. e., independent of time and frequency); if its input is $\eta(t)$ its output is $k\ \eta(t)$ where k is a fixed number. The block labeled G satisfies the following conditions.**

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**The discussion of stability for the case where the transfer functions are not rational is far from trivial. Any reader who doubts this should consider the function defined for $t > 0$ by $e^t \sin(e^t)$ and note that its Laplace transform is analytic for all finite s. This example shows that the discussion of stability cannot be settled by "looking at the singularity that is the furthest to the right," which is a legitimate procedure with rational transfer functions.

(G.1) Its input-output relation relating the output y, the zero-input response z and the input ζ is

$$y(t) = z(t) + \int_0^t g(t - \tau)\zeta(\tau)d\tau \qquad \text{for all } t \geq 0. \tag{1}$$

(G.2) For all initial states, the zero-input response z is bounded on $[0, \infty)$ and tends to zero as $t \to \infty$. Let $z_M = \sup_{t \geq 0} |z(t)|$.

(G.3) The unit impulse response g is given by

$$g(t) = r + g_1(t) \qquad \text{for } t \geq 0$$

where the constant r is ≥ 0 and g_1 is bounded on $[0, \infty)$, tends to zero as $t \to \infty$, and is an element of $L^1(0, \infty)$. We shall write $G(s) = (r/s) + G_1(s)$.

The conclusion is stated in the form of

Theorem I: Suppose the linear time-invariant single-loop-feedback system shown on Figure 1 satisfies assumptions (G.1), (G.2) and (G.3) and that $k > 0$. If the modified Nyquist diagram* of $G(s)$ does not encircle nor go through the critical point $(-1/k, 0)$, then

(a) the impulse response of the closed-loop system is bounded, tends to zero as $t \to \infty$, and is an element of $L^1(0, \infty)$;

(b) for any initial state, the zero-input response of the closed-loop system is bounded and goes to zero as $t \to \infty$;

(c) for any initial state and for any bounded input, the response of the closed-loop system is bounded.

*The modified Nyquist diagram is the map under G of the imaginary axis from which the interval $[-j\epsilon, j\epsilon]$ has been removed and replaced by the semi-circle $\epsilon e^{j\theta} \mid -(\pi/2) \leq \theta \leq (\pi/2)$. Here ϵ taken arbitrarily small.

If the Nyquist diagram of $G(s)$ encircles the critical point $(-1/k, 0)$ a finite number of times then the impulse response of the closed-loop system grows exponentially as $t \to \infty$.

It should be stressed that the only assumption that is made concerning the box G is that it fulfills the conditions (G.1), (G.2), and (G.3). Such conditions are often fulfilled by the impulse response of systems described by ordinary differential equations, difference-differential equations, and those whose input-output relation is obtained through the solution of partial differential equations. The latter is the case for distributed circuits and for many control systems.

ANALYSIS

Let u be the bounded input applied to the system and let $u_M \triangleq \sup_{t \geq 0} |u(t)|$. The response of the closed-loop system starting from an arbitrary initial state is given by

$$y(t) = z(t) + k \int_0^t g(t-\tau)[u(\tau) - y(\tau)]\, d\tau \quad \text{for all } t \geq 0. \tag{2}$$

The outline of the proof is as follows: (a) Using the Gronwall-Bellman[2, 3] inequality we establish that the solution of (2) is of exponential order, hence that Laplace transform techniques are applicable. (b) Simple considerations[4, 5] establish the uniqueness of the solution. (c) h, the unit impulse response of the closed-loop system, is shown to be bounded, to tend to zero and $\in L^1(0, \infty)$. The fact that $h \in L^1(0, \infty)$ is obtained by the use of the Principle of the Argument[7] and from a theorem of Paley and Wiener.[8] Reference to the integral equation defining h shows easily that h is bounded and goes to zero. The latter requires that it be shown that h is uniformly continuous. (d) z_c, the zero-input response of the closed-loop system, is bounded and goes to zero. (e) To obtain the converse, (i.e., if the Nyquist diagram encircles the critical point then the closed-loop impulse response grows exponentially) we must use a theorem of Doetsch.[10]

CONCLUSION

If we collect that which is known about the Nyquist criterion we may state that its virtues are the following: (a) It is a test which utilizes only experimentally available data (results of sinusoidal steady state measurements). (b) It is a necessary and sufficient condition, i.e., it does not lead to conditions that are too restrictive, as is often the case with Lyapunov techniques. (c) It covers an extremely wide class of linear time-invariant systems. (d) It leads to very specific conclusions regarding the impulse response, zero-input response and the total response of the closed loop system.

The second result, closely related to the first, is an extension of the Popov criterion, in the principal case.[11] Aizerman and Grantmacher have written a detailed monograph on this subject with an extensive bibliography.[12] In the U.S. it has also received attention.[13, 14] The approach used in these presentations is to make very detailed and specific assumptions concerning the internal dynamics of the linear part of the system. On the other hand one of the most important and appealing features of the Popov criterion is that it shares all the desirable features of the Nyquist criterion; the essence of the Nyquist criterion is that in its general formulation, as given above, it does not make any assumptions concerning the internal dynamics but rather makes assumptions only on its input-output relation. We propose to do the same for the Popov criterion.

DESCRIPTION OF THE SYSTEM

Figure 2 shows the single-input single-output feedback system under consideration; N is a time-varying memoryless nonlinearity, G is a linear time-invariant subsystem. N is assumed to be characterized as follows:

(N.1) If, at time t, its input is $\eta(t)$, then its output is $\varphi(\eta(t), t)$, where, for each $t \geq 0$, φ is a continuous function of its first argument and there are two positive numbers ϵ and k such that

$$0 < \epsilon \leq \frac{\varphi(\xi, t)}{\xi} \leq k - \epsilon \qquad \text{for all } \xi \neq 0, \text{ for all } t \geq 0. \tag{1'}$$

G is assumed to be characterized as follows: if α is its input and z is its zero-input response, its output y is given by

$$y(t) = z(t) + \int_0^t g(t - \tau)\, \alpha(\tau)\, d\tau \qquad \text{for all } t \geq 0. \tag{2'}$$

In addition the following conditions are satisfied:

(G.1) for any initial state, the zero-input response z of G is (i) bounded on $[0, \infty)$, (ii) uniformly continuous on $[0, \infty)$, and (iii) both z and $\dot{z}$ are elements of $L^2(0, \infty)$. Let $z_M \triangleq \sup_{t \geq 0} |z(t)|$.

(G.2) g, the unit impulse response of G, is an element of $L^1(0, \infty)$.

It is very important to note that since g is a special instance of a zero-input response, (G.1) implies that (i) g is bounded, (ii) g is uniformly continuous and (iii) g and $\dot{g}$ are elements of $L^2(0, \infty)$.

By Lemma 1 below, (G. 2) and (ii) imply that $g(t) \to 0$ as $t \to \infty$. Similarly $z(t) \to 0$ as $t \to \infty$. For future reference, let

$$g_M \triangleq \sup_{t \geq 0} |g(t)| .$$

The main result may be stated in the form of

Theorem II: Consider the single-input single-output nonlinear time-varying single-loop-feedback system shown on Figure 2 where N is a time-varying memoryless nonlinearity which satisfies condition (N. 1) and G be a linear time-invariant subsystem satisfying conditions (G. 1) and (G. 2). If there exists a real number $q \neq 0$ and a positive δ such that

$$\mathrm{Re}\left[(1 + qj\omega)G(j\omega) + \frac{1}{k}\right] \geq \delta > 0 \qquad \text{for all } \omega \geq 0 \qquad (P_\delta)$$

then, for any initial state, the zero-input response y is bounded on $[0, \infty)$ and tends to zero as $t \to \infty$.

The proof of the theorem is based on three lemmas which are easy to prove and the proofs are in the appendix.

Lemma 1: Let f map $[0, \infty)$ into the real line R and be uniformly continuous. Let g map R into R, be continuous; let $g(0) = 0$ and let $x \neq 0$ imply $g(x) > 0$. Under these conditions, if

$$\int_0^\infty g[f(t)] \, dt < \infty \qquad (3)$$

then

$$\lim_{t \to \infty} f(t) = 0. \qquad (4)$$

Lemma 2: Let f_1, f_2, f_3 map $[0, \infty)$ into R and belong $L^2[0, \infty)$. Call F_1, F_2, and F_3 their Fourier transforms. Let H be the Fourier transform of a real-valued function h.

If

$$F_1(j\omega) = -H(j\omega)F_3(j\omega) + F_2(j\omega) \qquad \text{for all } \omega \tag{5}$$

and if there is a number σ_0 such that

$$\text{Re } H(j\omega) \geq \sigma_0 > 0 \qquad \text{for all } \omega \geq 0 \tag{6}$$

then

$$\int_0^{\infty} f_1(t)f_3(t)\, dt \leq \frac{1}{8\pi} \int_{-\infty}^{+\infty} \frac{|F_2(j\omega)|^2}{\text{Re } H(j\omega)} d\omega \leq C \tag{7}$$

where

$$C \triangleq (8\pi\, \sigma_0)^{-1} \int_{-\infty}^{+\infty} |F_2(j\omega)|^2 d\omega = (4\sigma_0)^{-1} \int_{-\infty}^{+\infty} |f_2(t)|^2 dt. \tag{8}$$

Comment: Let us interpret $H(j\omega)$ as the impedance of a linear time-invariant network which is passive in view of (6). Let f_3 be the current and f_2 be the internal The venin equivalent voltage source. (See Figure 3.) The product $f_1(t)f_3(t)$ is the power delivered at time t by the network to the "external world." Note that the "external world" is any circuit (not necessarily linear nor time-invariant), which to the current f_3 creates the voltage drop f_1. Thus (7) asserts that, whatever may be the external circuit to which the network is connected, the energy delivered by the network in the interval $[0, \infty)$ is bounded as indicated by the inequality (8). Note also that the equation is satisfied with the equality sign if the network is terminated by a linear time-invariant circuit defined by its impedance $Z(j\omega) = H^*(j\omega) = \text{Re } H(j\omega) - j\text{Im } H(j\omega)$: indeed the current is then $F_3(j\omega) = F_2(j\omega)/2\text{Re } H(j\omega)$ and the power delivered at the frequency ω is $|F_2(j\omega)|^2 / [4\text{Re } H(j\omega)]$.

Lemma 3: In the proof of the theorem, q may be assumed to be positive.

Proof of the theorem: Let M be the linear time-invariant system whose transfer function is for each real ω given by $(1 + qj\omega)\, G(j\omega) + 1/k$. According to Lemma 3 we may assume $q > 0$. If $M\alpha$ represents the zero-state response of M to α, we have

$$\eta + q\dot{\eta} - \frac{\alpha}{k} = -M\alpha - z - q\dot{z} .$$

Let η_T be the function defined by

$$\eta_T(t) = \begin{cases} \eta(t) & 0 \le t \le T \\ 0 & \text{elsewhere.} \end{cases}$$

Let α_T, z_T, $\dot{z}_T$ be similarly defined. Since

$$\dot{\eta}(t) = -\dot{z}(t) - g(0+)\alpha(t) - \int_0^t \dot{g}(t - \tau)\alpha(\tau)\, d\tau .$$

(G.1) and (G.2) imply that $z + q\dot{z}$ belongs to $L^2(0, \infty)$, and that α_T, $\eta_T + q\dot{\eta}_T - \alpha_T/k$ are in $L^2(0, T)$ for all positive T. These conditions together with P_δ imply, by Lemma 2, that

$$\int_0^T (\eta_T(t) + q\dot{\eta}_T(t) - \alpha_T(t)/k)\, \alpha_T(t)\, dt \le C \qquad \text{for all } T > 0$$

where the constant C is independent of T because $z + q\dot{z} \in L^2(0, \infty)$. Expressing α in terms of η we get

$$k^{-1}\int_0^T [\, k\eta(t) - \varphi(\eta(t), t)]\; \varphi(\eta(t), t)\, dt + q\int_0^T \varphi(\eta(t), t)\, \dot{\eta}(t)\, dt \le C. \tag{9}$$

Call J_1 the first term of the left-hand side. Define the function $\tilde{\eta}$ on $[-1, 0)$ to be $\eta(0)(1 + t)$ for $-1 \le t \le 0$. From (1) we obtain

$$\frac{\epsilon}{2}\eta(0)^2 \leq \int_{-1}^{0} \varphi(\tilde{\eta}(t), t)\, \tilde{\eta}(t)\, dt \leq \frac{k}{2}\, \eta(0)^2 .$$

Let us add this integral to both sides of (9), hence

$$J_1 + q\, \frac{\epsilon}{2}\, \eta(T)^2 \leq C + \frac{k}{2}\, \eta(0)^2 .$$

(1') implies that the integrand of J_1 is non-negative for all t.
Since $q > 0$ we get

$$\eta(T) \leq \sqrt{\frac{2C}{\epsilon\, q} + \frac{k}{\epsilon\, q}\, \eta(0)^2} \qquad \text{for all } T > 0 .$$

Since the right-hand side is independent of T, η is bounded on $[0, \infty)$. Let $\eta_M \triangleq \sup_{t \geq 0} |\eta(t)|$. We now observe that η is uniformly continuous on $[0, \infty)$: from the defining relation of η we get

$$|\eta(t + \xi) - \eta(t)| \leq |z(t + \xi) - z(t)| + \int_0^t |g(t + \xi - \tau) - g(t - \tau)|\, |\alpha(\tau)|\, d\tau$$
$$+ \int_t^{t \pm \xi} |g(t + \xi - \tau)\, \alpha(\tau)|\, d\tau .$$

Using (N.1) to bound α, we get

$$\leq |z(t + \xi) - z(t)| + k\eta_M \int_0^{\infty} |g(t' + \xi) - g(t')|\, dt' + k\eta_M g_M \xi .$$

(G.1) implies that the first term goes to zero as $\xi \rightarrow 0$ uniformly in t; $g \epsilon L^1(0, \infty)$ implies that the integral $\rightarrow 0$ as $\xi \rightarrow 0$, and the same obviously holds for the last term. Therefore η is uniformly continuous on $[0, \infty)$. Thus all assumptions of Lemma 1 apply to J_1 and hence $\eta \rightarrow 0$ as $t \rightarrow \infty$. This concludes the proof of the theorem.

CONCLUSION

The conclusions that can be drawn in this case concern only the zero-input response. On the other hand the class of nonlinear systems covered by Theorem II is very broad.

APPENDIX

Proof of Lemma 1: Suppose f does not go to zero as $t \to \infty$, then given any $\epsilon > 0$, there is an infinite sequence $t_1, t_2, \ldots, t_k, \ldots$ of points going to infinity such that

$$|f(t_k)| > 2\epsilon \qquad \text{for } k = 1, 2, \ldots$$

Since f is uniformly continuous, given any $\epsilon > 0$, there is a δ such that $|\xi| \leq \delta$ implies that $|f(t+\xi) - f(t)| < \epsilon$. Therefore in each one of the intervals $[t_k - \delta, t_k + \delta]$, $|f(t)| > \epsilon$. Since there is an infinite number of them, the integral in (3) of Lemma 1 must be infinite, which contradicts the hypothesis. Hence $f(t) \to 0$ as $t \to \infty$.

Proof of Lemma 2: The proof is a straightforward application of the completing the square technique.[12]

Proof of Lemma 3: Suppose the condition (P_δ) is satisfied by the given G for some q, k and δ. The equation of the system may be symbolically represented by

$$G\,\varphi(\eta, t) = -\eta$$

where G denotes the effect of the linear time-invariant system G upon $\varphi(\eta, t)$, which is the output of N. Let $\hat{\varphi}(\eta, t) \triangleq k\eta - \varphi(\eta, t)$. After a few algebraic manipulations we obtain

$$H\,\varphi(\eta, t) = -\eta$$

which is of the same form as the preceding relation and where H is the linear system whose transfer function is $-G(s)[1 + kG(s)]^{-1}$. First from the definition of $\hat{\varphi}$ and (1) it follows that

$$0 < \epsilon \leq \frac{\hat{\varphi}(\xi, t)}{\xi} \leq (k - \epsilon) \quad \text{for all } \xi \neq 0 \text{ and all } t \geq 0.$$

In other words, $\hat{\varphi}$ satisfies the same conditions as φ. Second, we must verify that H satisfies the conditions (G.1) and (G.2). The geometric interpretation of the condition (P_δ)[12] which is satisfied by G implies that the Nyquist diagram of G does not encircle the critical point $(-1/k, 0)$. It follows from the theory of the Nyquist criterion given above that h, the impulse response of $H(s) = G(s)[1 + k\,G(s)]^{-1}$, is bounded on $[0, \infty)$, belongs to $L^1(0, \infty)$ and $\to 0$ as $t \to \infty$. Since $h(t) = g(t) - k(h * g)(t)$ and $\dot{h}(t) = \dot{g}(t) - k\,g(0+)h(t) - k\,(h * \dot{g})(t)$, for any finite $T > 0$, $\dot{h}$ is in $L^2(0, T)$ because each term of the right-hand side of the last equation is in $L^2(0, T)$ for every $T > 0$. To test the conditions (G.1) note that z_c, the closed-loop zero-input response satisfies the equation $z_c(t) = z(t) - k(h * z)(t)$. Thus, immediately z_c is bounded and is uniformly continuous on $[0, \infty)$. Also $\dot{z}_c(t) = \dot{z}(t) - k\,h(0+)\,z(t) - k(h * \dot{z})(t)$. From these two equations, it follows that both z_c and $\dot{z}_c$ are in $L^2(0, \infty)$ since each term of the right-hand side is in $L^2(0, \infty)$. Thus by a process of relabelling we have transformed the given system characterized by G and φ into a system characterized by H and φ, such that H and φ satisfy the conditions (N.1), (G.1), and (G.2). Finally, a direct calculation establishes that, for all ω,

$$\operatorname{Re}\ (1 - qj\omega)\, H(j\omega) + \frac{1}{k} = \frac{1}{|1 + kG(j\omega)|^2}\ \operatorname{Re}\ (1 + qj\omega)G(j\omega) + \frac{1}{k} \quad .$$

Therefore if the condition (P_δ) is satisfied by G for some $q < 0$ and some k, the same condition P_δ (with a different but still positive δ) is satisfied by H, for $-q$ and the same k.

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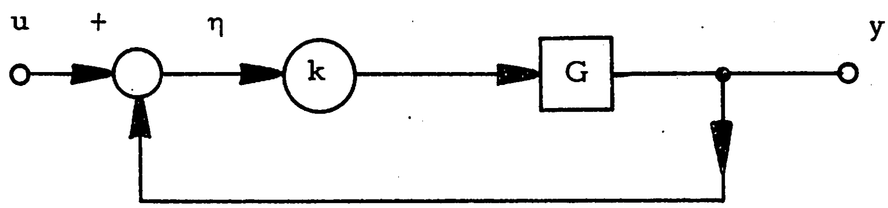

Figure 1

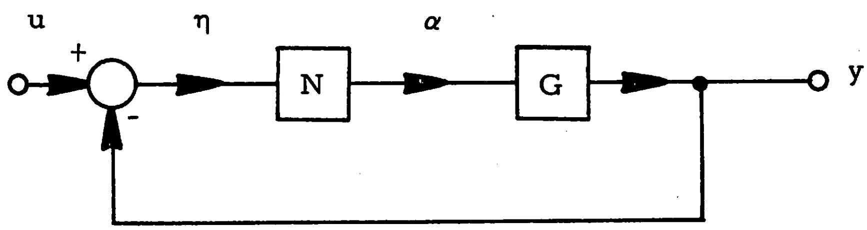

Figure 2

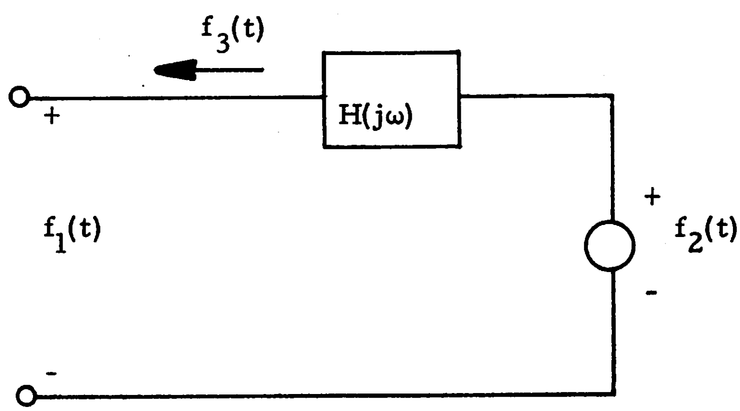

Figure 3