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NONLINEAR SAMPLED-DATA SYSTEMS AND MULTIDIMENSIONAL Z-TRANSFORM

by

A. Rault
E. I. Jury

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ABSTRACT

Discrete Volterra series is shown to approximate arbitrarily closely the response of nonlinear sampled-data systems. A multi-dimensional Z-transform calculus is developed which makes it possible to generalize the property of convolution to nonlinear discrete systems; operator notation can then be used to analyze the problem. A typical example is taken and a representation of the system is obtained from which the transient response is easily computed.

I. INTRODUCTION

The analysis of continuous nonlinear systems using Volterra series has been developed by Brilliant[1], George[2], Van Trees[3], and Flake[4]. In connection with the application of Volterra series, a multidimensional Laplace transform calculus was defined and used in solving for the kernels. Brilliant proved that for a bounded input function, a nonlinear time-invariant system of the form:

$$L[y(t)] + F[y(t), y'(t), \dots, y^{(n)}(t)] = x(t) \tag{1}$$

where L represents a linear differential operator and F is a nonlinear analytic function, can be approximated arbitrarily closely by the following polynomial form:

$$y(t) = h_0 + \int_{-\infty}^{+\infty} h_1(\tau)x(t-\tau)d\tau + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_2(\tau_1, \tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2$$

$$+ \dots + \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h_N(\tau_1, \dots, \tau_N)\prod_{i=1}^{N} x(t-\tau_i)d\tau_i \tag{2}$$

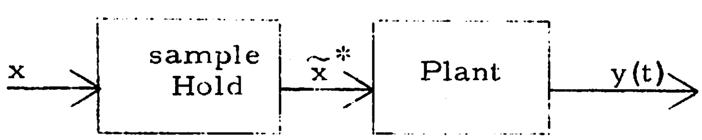
The h_i's are the kernels respectively of i-th order.

A similar procedure is taken to solve nonlinear discrete systems and the multidimensional Z-transform calculus concurrently introduced makes it possible to solve easily for the kernels.

Analysis of nonlinear discrete systems has not yet been solved although Alper[5] gave some related results.

II. REPRESENTATION OF NONLINEAR SYSTEMS IN DISCRETE VOLTERRA SERIES

Consider the following system:



The plant is time-invariant and "continuous" in the sense defined by Brilliant.

If x(t) is the input to the sample, the output $x^*(t)$ of the sample is

$$x^*(t) = \sum_{k=0}^{\infty} x(t)\,\delta(t-kT). \tag{3}$$

The output of the hold having impulse response function Q(t) is

$$\tilde{x}^*(t) = \int_{-\infty}^{+\infty} Q(t-\tau)x^*(\tau)d\tau. \tag{4}$$

Approximating the output of the plant by the n first terms of the Volterra series we have

$$y(t) = \sum_{i=1}^{n} \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h_i(\tau_1,\ldots,\tau_i)\prod_{j=1}^{i}\tilde{x}^*(t-\tau_j)d\tau_j \tag{5}$$

$$\tilde{x}^*(t-\tau_j) = \int_{-\infty}^{+\infty} Q(t-\tau_j-\tau)\sum_{k=0}^{\infty} x(\tau)\,\delta(\tau-kT)d\tau \tag{6}$$

Without loss of generality we can consider the second degree term; the other ones will take the same general form.

$$Y_2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_2(\tau_1,\tau_2)x^*(t-\tau_1)x^*(t-\tau_2)d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{+\infty} d\tau_1 \int_{-\infty}^{+\infty} d\tau_2 h_2(\tau_1,\tau_2)\int_{-\infty}^{+\infty} Q(t-\tau_1-\tau)\sum_{k_1=0}^{\infty} x(\tau)\,\delta(\tau-k_1T)$$

$$\times \int_{-\infty}^{+\infty} d\tau' Q(t-\tau_2-\tau')\sum_{k_2=0}^{\infty} x(\tau')\delta(\tau'-k_2T). \tag{7}$$

Changing the order of integration,

$$Y_2 = \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' x^*(\tau) x^*(\tau') \left[\int_{-\infty}^{+\infty} d\tau_1 \int_{-\infty}^{+\infty} d\tau_2 h_2(\tau_1, \tau_2) Q(t-\tau_1-\tau) \times Q(t-\tau_2-\tau) \right]. \quad (8)$$

The term between brackets represents the hold in cascade with the second order kernel; let it be designated by $g_2(t-\tau, t-\tau')$. So then we have

$$Y_2 = \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' g_2(t-\tau, t-\tau') \sum_{k_1=0}^{\infty} x(\tau)\,\delta(\tau-k_1T) \sum_{k_2=0}^{\infty} x(\tau')\,\delta(\tau'-k_2T) \quad (9)$$

Interchanging the order of integration and summation, one obtains

$$Y_2 = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g_2(t-\tau, t-\tau') x(\tau) x(\tau')\, \delta(\tau-k_1T)\, \delta(\tau'-k_2T)\, d\tau\, d\tau'. \quad (10)$$

So the second order term of the overall system has the discrete representation shown above; $g_2(t-k_1T, t-k_2T)$ is the second order discrete kernel. In general the output of the system can therefore be represented by the following discrete series:

$$y(t) = \sum_{i=1}^{\infty} \left\{ \sum_{k_1=0}^{\infty} \cdots \sum_{k_i=0}^{\infty} g_i(t-k_1T, t-k_2T, \ldots, t-k_iT) \prod_{j=1}^{i} x(k_jT) \right\} \quad (11)$$

If the output is sampled, the following series, which we shall call Volterra discrete series, is obtained:

$$y(m) = \sum_{i=1}^{n} \left\{ \sum_{k_1=0}^{n} \cdots \sum_{k_i=0}^{n} g_i(m-k_1, \ldots, m-k_i) \prod_{j=1}^{i} x(k_j) \right\} \tag{12}$$

where the sampling period T has been deleted from the notation for simplification. From now on, $y(m)$ means $y(mT)$.

The discrete Volterra series has been shown to approximate, within an arbitrarily small error, the output of a "discrete" continuous nonlinear system. The next step would be to extend the validity of this representation to digital systems. It involves the notion of compactness in the space of discrete variables and an extension of the Stone-Weierstrass theorem.

Definitions: a system that can be exactly represented by

$$y(m) = \sum_{i=1}^{n} \left\{ \sum_{k_1=0}^{m} \cdots \sum_{k_i=0}^{m} h_i(k_1, \ldots, k_i) \prod_{j=1}^{i} x_i(m-k_j) \right\} \tag{13}$$

will be called an analytic system if it is stable in the bounded input, bounded output sense. The norm of the n-th order kernel is defined as

$$||h_n|| = \sum_{k_1=0}^{m} \cdots \sum_{k_n=0}^{m} |h_n(k_1, k_2, \ldots, k_n)| \tag{14}$$

If all the norms $||h_i||$ are finite and the power series

$B_H(x) = \sum_{n=0}^{\infty} ||h_n|| x^n$ has a nonzero radius of convergence R_H, then for an input bounded by a constant $R < R_H$, each functional converges absolutely and the series will converge absolutely. The output will be bounded by $B_H(R)$; the function B_H is then called the bound function of the system.

Any kernel can be altered by permuting its arguments $k_1, \ldots k_i$ except if it is symmetric. So analytic systems are represented by a unique set of functionals if the kernels are symmetric. In case the kernels are unsymmetric, symmetric ones will be obtained by taking the arithmetic mean of all the kernels obtained after permuting the arguments in all possible ways.

Properties of the kernels

The difference equation of an analytic system can take the general form

$$L[y(m)] + F[y(m), y(m+1), \ldots, y(m+n)] = x(m).$$

L is a linear time-invariant difference operator and F is a multinomial in y(m) and its n following samples. h(k) is the sampled impulse response of the linear part of the system. $h_1(k)$ is the first order kernel in the Volterra series representation.

If the Volterra series representation of y(m) is substituted in the difference equation, it is immediately seen that the difference equation corresponding to $h_1(k)$ is the same as that for h(k). Therefore the first order kernel corresponds to the sampled impulse response of the linear part alone.

In the particular case when the nonlinearity is composed only of an n-th order term $[y(m)]^n$, a relationship between the order of the existing kernels is derived. First of all, being given a Volterra series representation of the output y(m), let us investigate the Volterra series representation of $[y(m)]^2$,

$$y(m) = \sum_{i=1}^{p} \left\{ \sum_{k_1=0}^{m} \cdots \sum_{k_i=0}^{m} h_i(k_1 k_2, \ldots, k_i) \prod_{j=1}^{i} (x(m-k_j)) \right\} \qquad (15)$$

which can be written as

$$y(m) = Y_1 + Y_2 + \ldots + Y_n + \ldots$$

$[y(m)]^2$ will be obtained as the result of the discrete convolution of two signals $y(m)$; therefore the n-th functional V_n of $[y(m)]^2$ is related to $y(m)$ as follows:

$$V_n = \sum_{j=1}^{n-1} \left\{ \sum_{k_1=0}^{m} \ldots \sum_{k_n=0}^{m} h_j(k_1, \ldots, k_j) h_{n-j}(k_{j+1}, \ldots, k_n) \prod_{i=1}^{n} x(m-k_i) \right\} \tag{16}$$

Similarly $[y(m)]^3$ is obtained as the discrete convolution of $[y(m)]^2$ and $y(m)$. Therefore if $[y(m)]^3$ is written as the sum of the contributing functionals

$$[y(m)]^3 = W_1 + W_2 + \ldots + W_n + \ldots \tag{17}$$

$$W_n = \sum_{\ell=1}^{n-1} V_\ell Y_{n-\ell} \tag{18}$$

$$V_\ell = \sum_{j=1}^{\ell-1} \left\{ \sum_{k_1=0}^{m} \ldots \sum_{k_\ell=0}^{m} h_j(k_1, \ldots, k_j) h_{\ell-j}(k_{j+1}, \ldots, k_\ell) \prod_{i=1}^{\ell} x(m-k_i) \right\} \tag{19}$$

$$Y_{n-\ell} = \sum_{k_\ell=0}^{m} \ldots \sum_{k_n=0}^{m} h_{n-\ell}(k_{\ell+1}, \ldots, k_n) \prod_{i=\ell}^{n} x(m-k_i). \tag{20}$$

Hence

$$W_n = \sum_{\ell=1}^{n-1} \sum_{j=1}^{\ell-1} \left\{ \sum_{k_1=0}^{m} \cdots \sum_{k_n=0}^{m} h_j(k_1, \ldots, k_j) h_{\ell-j}(k_{j+1}, \ldots, k_\ell) \right.$$

$$\left. h_{n-\ell}(k_{\ell+1}, \ldots, k_n) \prod_{i=1}^{n} x(m-k_i) \right\}. \tag{21}$$

Note that at the maximum $\ell-1 = n-2$, thus the n-th order kernel of $[y(m)]^3$ takes the following form

$$h_{n3} = \sum_{\ell=1}^{n-1} \sum_{j=1}^{n-2} h_j h_{\ell-j} h_{n-\ell}. \tag{22}$$

This result could be easily generalized by an induction proof to yield the n-th order kernel of $[y(m)]^p$ as

$$h_{np} = \sum_{\ell=1}^{n-1} \sum_{j=1}^{n-2} \cdots \sum_{r=1}^{n-p+1} h_r h_{g-r} \cdots h_{\ell-j} h_{n-j} \tag{23}$$

which is a combination of the kernels of y(m) of order less than n taken p at a time. Now consider a difference equation of the type $L[y(m)] + F[y(m)] = x(m)$ where L is a linear difference operator and F is an n-th order power operator. The two first nonzero kernels are h_1 and h_n. The next one is obtained as a combination of n terms in which the only kernels appearing are h_1 and h_n; therefore the next least order kernel is obtained as

$$h_n \underbrace{h_1 \cdots h_1}_{n-1 \text{ terms}} \quad \text{which is of order } n+(n-1)$$

The following one is then

$$\underbrace{h_n h_n h_1 \cdots h_1}_{n-2 \text{ terms}} \quad \text{which is of order } n+(n-1)\,2$$

From the two initial kernels h_1 and h_n, only kernels of order $n+(n-1)p$ appear where p is an integer. In general, higher order kernels obtained are formed as combinations of kernels of order 1, n, $n+(n-1)j_i$ (j_i are integers):

$$\prod_{i=1}^{p} h_{n+(n-1)j_i} \prod_{j=1}^{m} h_n \prod_{r=1}^{n-(m+p)} h_1$$

The order of such kernels is

$$pn + (n-1)\sum_{i=1}^{p} j_i + mn + n - (m+p) = (n-1)\left[\sum_{i=1}^{p} j_i + p + n\right] + n$$

Therefore the order of all possibly existing kernels is

$$n + (n-1)p \quad \text{where } p = 1, 2, \ldots, q$$

and 1 and n.

Note that if we allow $p = -1, 0, 1, 2, \ldots, q$, all the existing kernels are obtained.

In order to extend the property of ordinary Z-transform which transforms the convolution in the discrete domain into multiplication, a multidimensional Z-transform is introduced.

III. MULTIDIMENSIONAL Z-TRANSFORMS

(a) Definition. Similarly to the ordinary Z-transform, one defines

$$F(z_1, z_2, \ldots, z_n) = Z\{f(m_1, m_2, \ldots, m_n)\} \tag{24}$$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} f(m_1, m_2, \ldots, m_n) z^{-m_1} z^{-m_2}, \ldots, z^{-m_n} \tag{25}$$

Let us consider the n-th functional in the Volterra series representation of y(m)

$$Y_n(m) = \sum_{k_1=0}^{m} \cdots \sum_{k_n=0}^{m} f(k_1, k_2, \ldots, k_n) \prod_{i=1}^{n} x(m-k_i) \tag{26}$$

Associate with this form

$$Y_n(m_1, m_2, \ldots, m_n) \text{ such that } Y_n(m) = Y_n(m_1, m_2, \ldots, m_n) \tag{27}$$

when $m_1 = m_2 = \ldots = m_n = m.$

$$Y_n(m_1, m_2, \ldots, m_n) = \sum_{k_1=0}^{m_1} \cdots \sum_{k_n=0}^{m_n} f(k_1, \ldots, k_n) \prod_{i=1}^{n} x(m_i-k_i) \tag{28}$$

If we consider the multidimensional Z-transform of this expression the following is obtained

$$Z\{Y_n(m_1, m_2, \ldots, m_n)\} = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \sum_{k_1=0}^{m_1} \cdots \sum_{k_n=0}^{m_n} f(k_1, \ldots, k_n)$$

$$\times \prod_{i=1}^{n} x(m_i-k_i) z_i^{-m_i} \tag{29}$$

$$= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} f(k_1, \ldots, k_n) \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \prod_{i=1}^{n} x(m_i - k_i) z_i^{-m_i} \tag{30}$$

Let $m_i - k_i = p_i$, then

$$Y_n(z_1, \ldots z_n) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} f(k_1, \ldots, k_n) \prod_{i=1}^{n} z_i^{-k_i} \sum_{p_i = -k_i}^{\infty} \cdots \sum^{\infty} \prod_{i=1}^{n} x(p_i) z_i^{-p_i} \tag{31}$$

but p_i and k_i are independent indices and range from 0 to ∞, therefore we have

$$Y_n(z_1, \ldots, z_n) = F(z_1, \ldots, z_n) x(z_1, \ldots, z_n) \tag{32}$$

So it is seen that the relation between convolution and multiplication which makes the Z-transform so useful in the case of linear systems has been extended to the nonlinear systems by the introduction of the multidimensional Z-transform.

But dummy variables $m_1, m_2, \ldots, m_n$ had to be introduced and an extra problem of associating these variables appears when the inverse transform is sought.

(b) Association of Variables

The problem is, knowing $H_n(z_1, \ldots, z_n)$, how can we find $h_n(m)$? The procedure of taking a number of variables $m_1, \ldots, m_n$ to be equal is called "associating the variables." Theoretically it would be possible to invert $H_n(z_1, \ldots, z_n)$ into $h_n(m_1, \ldots, m_n)$ and then, by associating the variables, find $h_n(m)$. A better procedure is to associate the discrete time variables in the transform domain, that is, given $H_n(z_1, \ldots, z_n)$ as the transform of $h_n(m_1, \ldots, m_n)$, $H_n(z)$ is the transform of $h_n(m)$ which will then be found directly.

The inverse multidimensional Z-transform is found as

$$h_n(m_1, m_2, \ldots, m_n) = \left(\frac{1}{2\pi j}\right)^n \int_{\Gamma_1} \cdots \int_{\Gamma_n} H_n(z_1, z_2, \ldots, z_n) z_1^{m_1-1} z_2^{m_2-1} \cdots z_n^{m_n-1} dz_1 \ldots dz_n \quad (33)$$

where Γ_i is the contour containing the singularities of $H_n(z_i)$.

Associating the two first variables $m_1 = m_2$

$$h_n(m_1, m_2, m_3, \ldots, m_n) = \left(\frac{1}{2\pi j}\right)^n \int_{\Gamma_2} \cdots \int_{\Gamma_n} \left\{ \int_{\Gamma_1} H_n(z_1, z_2, \ldots, z_n)(z_1 z_2)^{m_1-1} \times dz_1 \right\} z_3^{m_3-1} \ldots z_n^{m_n-1} dz_2 \ldots dz_n. \quad (34)$$

Let us call $z_1 z_2 = u_1$. In the integral between braces, z_1 is the variable and the other z_i are held constant so $dz_1 = \left(\frac{du_1}{z_2}\right) h_n(m_1, \ldots, m_n)$

$$= \left(\frac{1}{2\pi j}\right) \int_{\Gamma_2} \cdots \int_{\Gamma_n} \left\{ \int_{\Gamma} H_n\left(\frac{u_1}{z_2}, z_2, \ldots, z_n\right) \frac{du_1}{z_2} \right\} z_3^{m_3-1} \ldots z_n^{m_n-1} dz_2 \ldots dz_n \quad (35)$$

where Γ is the contour containing the singularities of the function of u_1. Now assuming the function H_n is absolutely convergent and continuous in all variables, we can interchange the order of integration

$$= \left(\frac{1}{2\pi j}\right)^{n-1} \int_{\Gamma} \int_{\Gamma_3} \cdots \int_{\Gamma_n} \left\{ \frac{1}{2\pi j} \int_{\Gamma_2} H_n\left(\frac{u_1}{z_2}, z_2, \ldots, z_n\right) \frac{dz_2}{z_2} \right\} z_3^{m_3-1} \ldots z_n^{m_n-1} \times du_1 dz_3 \ldots dz_n \quad (36)$$

Let us consider the integral between braces and make the following change of variables

$$u_1 = z_1$$
$$z_2 = u$$

We then have

$$G_{n-1}(z_1, z_3, \ldots, z_n) = \frac{1}{2\pi j}\int_{\Gamma_2} H_n\left(\frac{z_1}{u}, u, z_3, \ldots, z_n\right)\frac{du}{u} \tag{37}$$

Γ_2 is the domain containing the singularities of the function in u; note that at the maximum this domain is the unit circle in the u plane, so if some poles are outside they will not contribute to the function.

To obtain $H_n(z)$ one should proceed associating variables two by two using the derived formula.

<u>Example:</u> Consider

$$H_2(z_1, z_2) = \frac{z_1}{z_1 - e^{-\alpha}}\;\frac{z_2}{z_2 - e^{-\alpha}}\;\frac{z_1 z_2}{z_1 z_2 - e^{-\beta}} \qquad \alpha \text{ and } \beta > 0$$

$$H_2(z) = \frac{1}{2\pi j}\oint \frac{\frac{z}{u}}{\frac{z}{u} - e^{-\alpha}}\;\frac{u}{u - e^{-\alpha}}\;\frac{z}{z - e^{-\beta}}\;\frac{du}{u}$$

$$H_2(z) = \frac{z}{z - e^{-\beta}}\;\frac{1}{2\pi j}\oint \frac{z}{(z - ue^{-\alpha})(u - e^{-\alpha})}du$$

Only the pole $u = e^{-\alpha}$ is inside the unit circle, so

$$H_2(z) = \frac{z}{z - e^{-\beta}}\;\frac{z}{z - e^{-2\alpha}}\,.$$

Now inverting H(z), it yields

$$h_2(m) = \frac{1}{1-e^{\beta-2\alpha}} \; e^{-\beta m} + \frac{1}{1-e^{2\alpha-\beta}} \; e^{-2\alpha m}$$

The technique and the formula previously given are general and whatever the form of the function to invert is, simple poles, multiple poles, delay term, the inverse multidimensional Z-transform can be obtained.

In the particular case of simple poles, characteristic forms always appear and an inspection technique can be developed. Considering the previous example, the first factor $\frac{z_1}{z_1-e^{-\alpha}}$ has for transform $e^{-\alpha m_1}$; similarly, to $\frac{z_2}{z_2-e^{-\alpha}}$ corresponds $e^{-\alpha m_2}$. Associating the variables $m_1 = m_2 = m$, it yields $e^{-2\alpha m}$ which has for Z-transform $\frac{z}{z-e^{-2\alpha}}$. The third factor of $H_2(z_1, z_2)$, $\frac{z_1 z_2}{z_1 z_2 - e^{-\beta}}$ has been seen to yield after association of variables $\frac{z}{z-e^{-\beta}}$. Therefore $H_2(z)$ is easily found as

$$H_2(z) = \frac{z}{z-e^{-2\alpha}} \; \frac{z}{z-e^{-\beta}}$$

It is repeated that this technique is limited to functions with simple poles, only case when the transform has this simple configuration. Further properties of the multidimensional Z-transform are investigated.

(c) Properties of the Multidimensional Z-transform

(α) Initial value theorem

$$y(0) = \lim_{\substack{z_1 \to \infty \\ \vdots \\ z_n \to \infty}} Y(z_1, z_2, \ldots, z_n) \tag{38}$$

The proof is obvious from the definition of the multidimensional transform.

(β) Foward shifting theorems

The problem is to find $Z[y(m_1+1,\ldots,m_n+1)]$. By definition

$$Z[y(m_1+1,\ldots,m_n+1)] = \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty} y(m_1+1,\ldots,m_n+1)\prod_{i=1}^{n} z_i^{-(m_i+1)} \tag{39}$$

Make the following change of variables $m_i+1 = p_i$

$$Z[y(p_1,\ldots,p_n)] = \sum_{p_1=1}^{\infty}\cdots\sum_{p_n=1}^{\infty} y(p_1,\ldots,p_n)\prod_{i=1}^{n} z_i^{-p_i}$$

$$Z[y(p_1,\ldots,p_n)] = \sum_{p_1=0}^{\infty}\cdots\sum_{p_n=0}^{\infty} y(p_1,\ldots,p_n)\prod_{i=1}^{n} z_i^{-p_i} \tag{40}$$

$$-\sum_{i=1}^{n}\tilde{y}(z_1,\ldots,z_{i-1},0,z_{i+1},\ldots,z_n) + (n-1)y(0,0,\ldots,0) \tag{41}$$

where

$$\tilde{y}(z_1,\ldots,z_{i-1},0,z_{i+1},\ldots,z_n) = \sum_{p_1=0}^{\infty}\cdots\sum_{p_{i-1}=0}^{\infty}\sum_{p_{i+1}=0}^{\infty}\cdots\sum_{p_n=0}^{\infty}$$

$$y(p_1,\ldots,p_{i-1},0,p_{i+1},p_n)z_1^{-p_1}\ldots z_{i-1}^{-p_{i-1}},z_{i+1}^{-(p_{i+1})},z_n^{-p_n} \tag{42}$$

Thus

$$Z[y(m_1+1,m_2+1,\ldots,m_n+1)] = Y(z_1,\ldots,z_n)-\sum_{i=1}^{n}y(z_1,\ldots,z_{i-1},0,$$

$$\times z_{i+1},\ldots,z_n) + (n-1)y(0). \tag{43}$$

(γ) Final value theorem

A proof of the final value theorem is given only for the two dimensional Z-transform; in the general case, a similar approach would be taken but would yield more complicated developments.

Consider the following expression

$$Z[y(m_1+1, m_2+1) - y(m_1+1, m_2) - y(m_1, m_2+1) + y(m_1, m_2)] \tag{44}$$

It is easily shown that

$$Z[y(m_1+1, m_2)] = z_1 Y(z_1, z_2) - z_1 y(0, z_2). \tag{45}$$

Therefore expression (44) is written

$$z_1 z_2 Y(z_1, z_2) - z_1 z_2 \tilde{y}(z_1, 0) - z_1 z_2 \tilde{y}(0, z_2) + z_1 z_2 y(0, 0) - z_1 Y(z_1, z_2)$$

$$+ z_1 \hat{y}(0, z_2) - z_2 Y(z_1, z_2) + z_2 \hat{y}(z_1, 0) + Y(z_1, z_2) \tag{46}$$

$$= (z_1-1)(z_2-1) Y(z_1, z_2) - z_1(z_2-1)\tilde{y}(0, z_2)$$

$$-z_2(z_1-1)\tilde{y}(0, z_1) + z_1 z_2 y(0,0) \tag{47}$$

Relation (44) could have been written

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \{y(m_1+1, m_2+1) - y(m_1+1, m_2) - y(m_1, m_2+1) + y(m_1, m_2)\} z_1^{-m_1} z_2^{-m_2}. \tag{48}$$

If we let $z_1 \to 1$, $z_2 \to 1$, and $m_1 \to \infty$, $m_2 \to \infty$, then after expanding the summation one obtains

$$\lim_{\substack{m_1 \to \infty \\ m_2 \to \infty}} \{y(m_1, m_2) - y(m_1, 0) - y(0, m_2) + y(0, 0)\}$$

If the limit exists, equality with the Z-transformed terms holds and

$$\lim_{\substack{z_1 \to 1 \\ z_2 \to 1}} (z_1-1)(z_2-1)Y(z_1, z_2) + y(0,0) - z_1(z_2-1)\tilde{y}(0, z_2) - z_2(z_1-1)\tilde{y}(z_1, 0)$$

$$= y(\infty, \infty) - y(\infty, 0) - y(0, \infty) + y(0, 0) \tag{49}$$

which yields

$$\lim_{\substack{z_1 \to 1 \\ z_2 \to 1}} (z_1-1)(z_2-1)Y(z_1, z_2) = y(\infty) \tag{50}$$

if the limit exists.

(δ) Relationship between multidimensional Laplace and Z-transforms

Let us consider the multidimensional time function $f(t_1, \ldots, t_n)$. Assuming the sampling rate is the same in each dimension, its sampled value is then

$$f^*(t_1, \ldots, t_n) = f(t_1, \ldots, t_n) \sum_{m_1=0}^{\infty} \delta(t_1 - m_1 T_1) \sum_{m_2=0}^{\infty} \delta(t_2 - m_2 T) \ldots \sum_{m_n=0}^{\infty} \delta(t_n - m_n T) \tag{51}$$

$$= \sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} f(m_1 T, \ldots, m_n T) \prod_{i=1}^{n} \delta(t_i - m_i T) \tag{52}$$

Note that

$$\mathcal{L}[\delta(t_i - m_i T)] = \int_0^{\infty} \delta(t_i - m_i T) e^{-s_i t_i} dt_i = e^{-m_i T s_i} \tag{53}$$

Thus we obtain

$$F^*(s_1, \ldots, s_n) = \mathcal{L}^n[f^*(t_1, \ldots, t_n)] = \sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} f(m_1, \ldots, m_n) \prod_{i=1}^{n} e^{-m_i T s_i} \tag{54}$$

where $\mathcal{L}^n[\cdot]$ denotes the n-dimensional Laplace transform. If we replace

e^{Ts_i} by z_i,

$$\mathscr{F}(z_1, \ldots, z_n) = F^*(s_1, s_2, \ldots, s_n)\Big|_{s_i = T^{-1}\ln z_i} \tag{55}$$

and the relationship between multidimensional Laplace and Z-transforms is expressed as follows

$$\mathscr{F}(z_1, \ldots, z_n) = F^*(s_1, \ldots, s_n) \stackrel{\Delta}{=} \mathscr{L}^n[f^*(t_1, \ldots, t_n)] \tag{56}$$

$$= \mathscr{L}^n\left[f(t_1, \ldots, t_n)\prod_{i=1}^{n}\sum_{m_i=0}^{\infty}\delta(t_i - m_i T)\right] \tag{57}$$

Given the n-dimensional $\mathscr{L}^n$ transform of a function, the next goal is to obtain directly the n-dimensional Z-transform. Using the fact that convolution holds for the $\mathscr{L}^n$ transform and that

$$\mathscr{L}\left[\sum_{m_i=0}^{\infty}\delta(t_i - m_i T)\right] = \frac{1}{1-e^{-s_i T}} \tag{58}$$

$$\mathscr{L}^n[g(t)f(t)] = \left(\frac{1}{2\pi j}\right)^n \int_{\Gamma_1}\cdots\int_{\Gamma_n} F(\lambda_1, \ldots, \lambda_n)\prod_{i=1}^{n} G(s_i - \lambda_i)\, d\lambda_i \tag{59}$$

$$G(s_i - \lambda_i) = \frac{1}{1 - e^{-T(s_i - \lambda_i)}} \tag{60}$$

Therefore

$$\mathscr{F}(z_1, \ldots, z_n) = \left(\frac{1}{2\pi j}\right)^n \int_{\Gamma_1}\cdots\int_{\Gamma_n} F(\lambda_1, \ldots, \lambda_n)\prod_{i=1}^{n}\frac{z_i}{z_i - e^{+T\lambda_i}}\, d\lambda_i \tag{61}$$

(d) Multidimensional Modified Z-Transforms

The approximate solution of nonlinear continuous systems with pure delay will be solved assuming the system sampled at a rate equal to the delay. In the case of sampled-data systems with delay, as well as when the value of the response is desired between samples, a multidimensional modified Z-transform must be used. Similarly to the modified Z-transform, it is defined as:

$$\mathscr{F}(z_1, z_2, \ldots, z_k, m) \triangleq \sum_{n_1=0}^{\infty} \sum_{n_k=0}^{\infty} f(n_1-1+m, n_2-1+m, \ldots, n_k-1+m) z_1^{-n_1} \ldots z_k^{-n_k} \tag{62}$$

IV. BLOCK DIAGRAM REPRESENTATION

Given a nonlinear system N, its output can be written $y = N(x)$ or as exposed in part II

$$y = N_o + N_1(x) + N_2(x) + \ldots + N_n(x) + \ldots \tag{63}$$

which could be considered as an operator power series expansion of the nonlinear operator N. If the system is stationary and continuous

$$N_n = \int_0^t \ldots \int_0^t h_n(t-\tau_1, \ldots, t-\tau_n) \prod_{i=1}^{n} x(\tau_i) d\tau_i \tag{64}$$

If the system is sampled

$$N_n = \sum_{k_1=0}^{m} \ldots \sum_{k_n=0}^{m} g_n(m-k_1, \ldots, m-k_n) \prod_{i=1}^{n} x(k_i). \tag{65}$$

The block diagram representation of a nonlinear system is then as shown in Fig. 1.

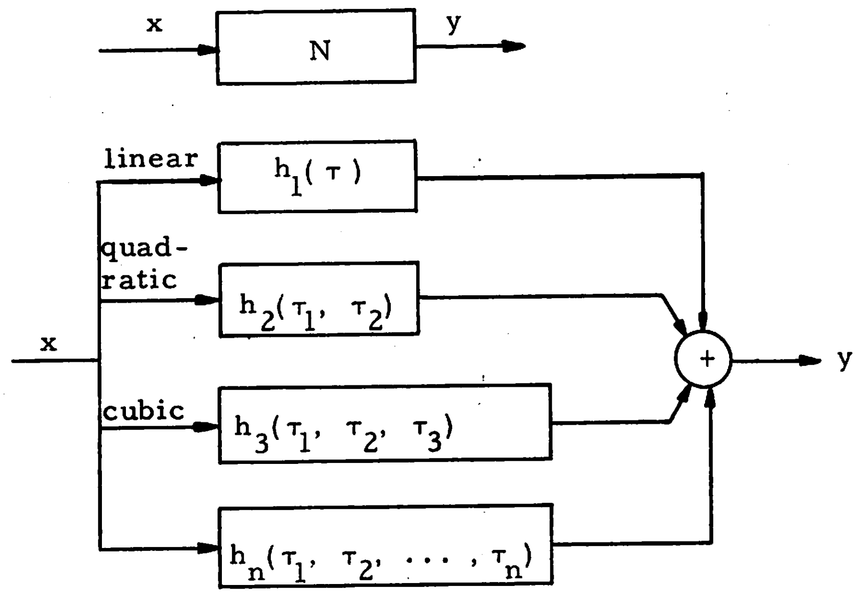


Fig. 1

Similar representation is obtained for discrete system. To determine the order of a subsystem, the fact that this operator representation is homogeneous in x is used and instead of considering input x, consider ϵx, where ϵ is a constant.

If Y_n represents the n-th functional $Y_n(\epsilon x) = N_n[\epsilon x] = \epsilon^n N_n[x]$ the order of the n-th component is seen to be n. George[2] gives an extensive treatment of the block diagram manipulation and some of his results will be exposed shortly and used in order to simplify the solution of nonlinear discrete systems. The basic operations on nonlinear systems are addition, multiplication and cascading and they have respective notation $+, \cdot, *$. The properties of these operations are given in detail by George[2] but most of them are fairly obvious and only some of the cascading operations are noted.

$$(J + K) * L = J * L + K * L \tag{66}$$

but

$$L * (J + K) \neq L * J + L * K \tag{67}$$

because

$$J * K \neq K * J \tag{68}$$

This is easily verified from the block diagram representation.

We have shown that a nonlinear operator was expandable as follows

$$H = H_1 + H_2 + \ldots + H_n. \tag{69}$$

Now consider the system

$$L = A_2 * (B_n + C_m). \tag{70}$$

We are looking for an expansion of that expression

$$\begin{aligned} L[x] &= (A_2 * (B_n + C_m))[x] \\ &= A_2[B_n[x] + C_m[x]]. \end{aligned} \tag{71}$$

Let

$$y = B_n[x] \quad \text{and } z = C_m[x]$$

$$L[x] = A_2[y+z]$$

which from the form of the operator A_2 can be written as

$$\begin{aligned} L[x] &= A_2((y+z)^2) \\ &= A_2(y^2) + 2A_2(yz) + A_2(z^2) \end{aligned} \tag{72}$$

Substituting the respective expression of y and z gives

$$L[x] = A_2((B_n[x])^2) + 2A_2(B_n[x]\cdot C_m[x]) + A_2((C_m[x])^2) \tag{73}$$

If we use the operator "o" defined as follows by George[2]

$$(A_2 \circ (B_n \cdot C_m))[x] = A_2(B_n[x]\cdot C_m[x]) \tag{74}$$

the system L becomes

$$L = A_2 \circ (B_n^2) + 2A_2 \circ (B_n \cdot C_m) + A_2 \circ (C_m^2) \tag{75}$$

Let us apply these properties to the solution of nonlinear discrete systems.

V. SOLUTION OF NONLINEAR SAMPLED DATA SYSTEMS

Simple block diagram properties and operator manipulation are sufficient to obtain the functional equations which are then solved using the properties of multidimensional Z-transforms. Let us consider the sampled-data system given in Fig. 2a where the feedback element is nonlinear. H_1 is a linear operator which has for transfer function $\frac{1 - e^{-Ts}}{s(s+1)}$.

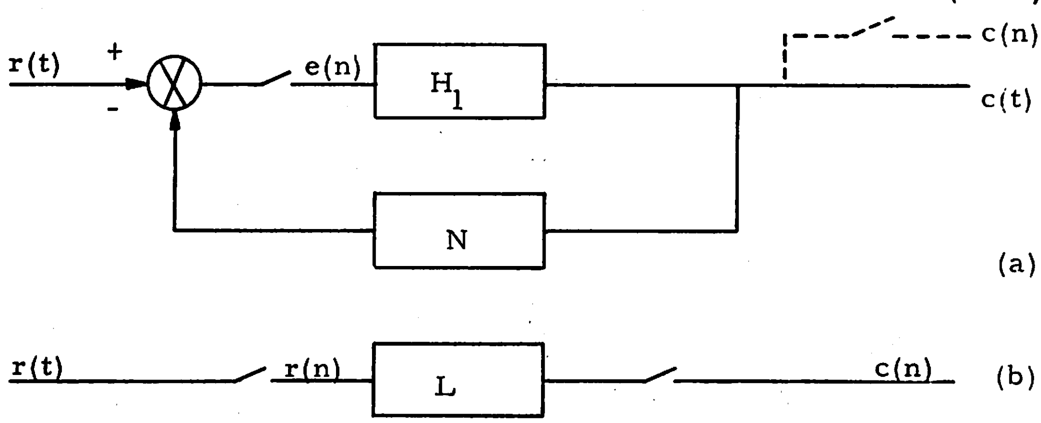


Fig. 2

N is a nonlinear operator such that

$$N[c(n)] = c(n) + 0.1(c(n))^3 \tag{76}$$

which can be written as $N = I + N_3$. We know that the output of the overall system can be represented as a discrete Volterra series which suggests the representation of Fig. 2b; therefore $L = L_1 + L_2 + \ldots + L_n + \ldots$

$$c(n) = H_1[e(n)] \tag{77}$$

$$e(n) = r(n) - N[c(n)] \tag{78}$$

$$c(n) = L[r(n)] \tag{79}$$

So

$$L[r(n)] = H_1[r(n) - N[L[r(n)]]] \tag{80}$$

which yields in operator notation

$$L = H_1*I - H_1*N*L \quad \text{where I is the identity operator.}$$

$$L = H_1*I - H_1*L - H_1*N_3*L \tag{81}$$

But $N_3*L = n_3L^3$ where n_3 is a constant gain term. Thus

$$L = H_1*(I - L) - n_3H_1*L^3 \tag{82}$$

Since the nonlinearity is of third order, the only operators present in the expansion of L are of order

$$3 + (3 - 1)p \quad \text{with } p = -1, 0, 1, 2, \ldots$$

So

$$L = L_1 + L_3 + L_5 + \ldots$$

Equation (82) is then written

$$L_1 + L_3 + L_5 + \ldots = H_1*(I-L_1-L_3-L_5\ldots) - n_3H_1*(L_1+L_3+L_5+\ldots)^3 \tag{83}$$

$$= H_1*(I-L_1)-H_1*L_3-H_1*L_5\ldots-n_3H_1*L_1^3-3n_3H_1*L_1^2L_3\ldots \tag{84}$$

Proceeding by identification of orders, this operator equation yields

$$L_1 = H_1*(I - L_1) \tag{85}$$

$$L_3 = -H_1*L_3 - n_3H_1*L_1^3 \tag{86}$$

$$L_5 = -H_1*L_5 - 3n_3H_1*L_1^2L_3 \tag{87}$$

Solving for the operators L_1, L_3, L_5

$$L_1 = (I + H_1)^{-1}*H_1 \tag{88}$$

$$L_3 = -n_3(I + H_1)^{-1}*H_1*L_1^3 \tag{89}$$

$$L_5 = -3n_3(I + H_1)^{-1}*H_1*L_1^2L_3 \tag{90}$$

Cascading corresponds to the convolution in discrete domain which becomes multiplication in the transform domain. For the linear operator the transform is the ordinary Z-transform while L_3 and L_5 correspond respectively to the 3 and 5 dimensional Z-transforms. $H_1(z)$ is found easily from $H_1(s)$:

$$H_1(z) = \frac{1 - e^{-T}}{z - e^{-T}} \quad \text{and if } T = 1$$

$$H_1(z) = \frac{0.632}{z - 0.368} \tag{91}$$

$$(1 + H_1(z))^{-1} = \frac{z - 0.368}{z + 0.264}$$

$$L_1(z) = \frac{0.632}{z + 0.264} \tag{92}$$

Before computing $L_3(z_1, z_2, z_3)$ and $L_5(z_1, \ldots, z_5)$ we should point out the relationship between the impulse response of cascaded systems and its transform.

Consider $L_n = A_1 * B_n$. The impulse response function of the system is

$$\ell_n(m_1, m_2, \ldots, m_n) = \sum_{k=0}^{\infty} a_1(k) b_n(m_1-k, m_2-k, \ldots, m_n-k) \tag{93}$$

The transform is

$$L_n(z_1, z_2, \ldots, z_n) = A_1(z_1, z_2, \ldots, z_n) B_n(z_1, z_2, \ldots, z_n) \tag{94}$$

Using this fact and

$$L_3 = -n_3 L_1 * L_1^3 \quad \text{with } n_3 = 0.1$$

$$L_3(z_1, z_2, z_3) = -\frac{0.0632}{z_1 z_2 z_3 + 0.264} \cdot \frac{0.632}{z_1 + 0.264} \cdot \frac{0.632}{z_2 + 0.264} \cdot \frac{0.632}{z_3 + 0.264} \tag{95}$$

Similarly one would get

$$L_5(z_1, \ldots, z_5) = L_1(z_1, \ldots, z_5).L_1(z_1).L_1(z_2).L_3(z_3, z_4, z_5) \tag{96}$$

Now, using the technique of association of variables, $L_3(z)$ and $L_5(z)$ can be obtained. If $\alpha = 0.632$, $\beta = 0.264$, $A = 0.0632$

$$L_3(z) = -\frac{A}{z+\beta}\,\frac{\alpha^3}{z+\beta^3} \tag{97}$$

Neglecting the fifth order term and higher terms, the response of the system to a unit step u(z) is

$$Y(z) = L_1(z)U(z) + L_3(z)U(z) \tag{98}$$

$$= \frac{\alpha}{z+\beta}\,\frac{z}{z-1} - \frac{A}{z+\beta}\,\frac{\alpha^3}{z+\beta^3}\,\frac{z}{z-1} \tag{99}$$

Taking the inverse Z-transform

$$y(m) = \frac{\alpha}{1+\beta}[1 - (-\beta)^m] - \frac{A\alpha^3}{1+\beta}\left[\frac{1}{1+\beta^3} + \frac{(-\beta)^m}{\beta - \beta^3} - \frac{(1+\beta)(-\beta)^{3m}}{(1+\beta^3)(\beta - \beta^3)}\right] \tag{100}$$

From which the transient response of the system is computed

$y(0) = 0$

$y(1) = 0.632$

$y(2) = 0.4488$

The final value theorem yields

$$y(\infty) = \lim_{z \to 1}\left(\frac{\alpha z}{z+\beta}\ \frac{A}{z+\beta}\ \frac{\alpha^3 z}{z+\beta^3}\right)$$

$y(\infty) = 0.498$

The final value theorem could be applied directly to the multidimensional transforms as shown in section III. It is a means of determining the number of terms necessary in the operator expansion, given a desired accuracy. For example if the transient response is desired with an accuracy of 10^{-3} units and the three first decimal points of $y(\infty)$ are the same considering either the 3 first terms of the Volterra series or 4 terms, the solution considering the 3 first terms is accurate within 10^{-3}. Since the Volterra series expansion is an approximation arbitrarily close to the solution, the solution with the 3 first terms is close to the exact value but nothing can be said regarding the accuracy with respect to the exact solution.

This example has been solved by Jury-Pai[6] using z convolution, and similar results were obtained. The solution of nonlinear difference equations is also obtained using Volterra series approach. If a block diagram simulation of the equation can be established, the problem is the same as a nonlinear sampled-data system. Otherwise, Volterra series expansion is used and the problem is one of simple identification, but the complexity of calculations is much greater.

Now we shall consider the case when the initial conditions are nonzero. The Volterra series solution considered previously for an n-th order sampled-data system or an n-th difference equation of the form $L[y(m)] + F[y(m)] = X(m)$ is the particular solution which is initially zero and whose (n-1) first samples are zero. The technique is then to introduce an auxiliary variable which is the difference between the desired solution and its initial value. This is fairly obvious when only the first sample is nonzero and the following n-1 are zero, then the new forcing function is defined by a simple translation of the variable. In the general case it is more intricate. An example of a second order nonlinear difference equation is given. Consider

$$y(m+2) + \alpha y(m) + \beta y^3(m) = x(m) \tag{101}$$

with

$$y(0) = a$$
$$y(1) = b$$

The first step is to remove the initial sample by translating the variables

$$z(m) = y(m) - a \tag{102}$$

$$z(m+2) + \alpha z(m) + \beta z^3(m) + 3\beta a z^2(m) = x(m) - a(\alpha+1) - \beta a^3 + 3\beta a^2 z(m)$$

$$z(m+2) + (\alpha + 3\beta a^2) z(m) + 3\beta a z^3(m) + \beta z^3(m) = p(m) \tag{103}$$

where

$$p(m) = x(m) - a(\alpha + 1) - \beta a^3 \tag{104}$$

To remove the second sample a new pseudo-forcing function is introduced.

$$f(m+1) + f(m) = p(m) \tag{105}$$

$$f(0) = b - a \tag{106}$$

The Volterra series solution using f(m) as the input function is

$$y(m) = \sum_{k=0}^{m} h_1(k) f(m-k) + \sum_{k_1=0}^{m} \sum_{k_2=0}^{m} h_2(k_1, k_2) f(m-k_2) + \ldots \tag{107}$$

Then proceed to obtain the solution.

Note on the stability problem

It should be noted that, as it is presented in section V, the solution of a nonlinear sampled-data system is obtained in general as an expanded operator form

$$L = L_1 + L_2 + L_3 + \ldots$$

which can take the block diagram representation of Fig. 1. Each of these operators has a multidimensional Z-transform; from the definition the multidimensional Z-transform converges outside the hypersphere of radius unity. Therefore the system will be stable if each transform has its poles inside the respective unit circle.

However, stability of the system will be analyzed before trying to obtain the Volterra series representation. Stability analysis can be performed on the system using Popov's method and its extensions as presented by Jury and Lee.[10] Then if the system is found stable, Volterra series representation can be sought.

Conclusion

A method of analysis of nonlinear sampled-data systems has been developed extending the concept of Volterra series and defining a multi-dimensional Z-transform. Approximate solution of analytic systems is obtained by solving for the kernels; transient response is then easily computed for general kinds of bounded inputs.

It is hoped that this representation of nonlinear sampled-data systems will be useful for future work in the areas of identification and optimal control, synthesis of nonlinear discrete systems and statistical design.

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