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PROCESSING OF TELEMETRY DATA GENERATED BY SENSORS MOVING IN A VARYING FIELD

by

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PROCESSING OF TELEMETRY DATA GENERATED BY SENSORS MOVING IN A VARYING FIELD

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Abstract: We consider two problems that arise in the processing of telemetry data. We first consider a situation in which a sensor moves through a spatially varying nonrandom field and relays samples of its observations to a ground station at a fixed time rate. The problem is then to reconstruct the entire field from a knowledge of only the samples. The reasons why the sin x/x functions constitute a good choice for reconstruction are discussed and the sources of error in this reconstruction pointed out. Quantitative expressions are given for the errors contributed by the following sources: aliasing, noise in the sensor output, distortion caused by sensor response, the effect of using only a finite number of samples, and effects due to spatially nonuniform samples. Although the exact forms of these expressions are difficult to evaluate, the analysis indicates the considerations that are of importance in the design of the experiment. The second problem of interest is the observation of certain counting processes (as in gamma ray astronomy) to discover the presence of sources of interest. The probability distribution of the counts is discussed and used to derive expressions for the probability of a false alarm (declaring a source present when none is) and the probability of a miss (declaring no source present when one is).

1. Reconstruction of a spatial function from a finite number of sample values

1.1. DESCRIPTION OF THE PROBLEM

Consider a function $f(x, t)$ which may in general be a function of both time t and some spatial coordinate x. This function is observed over a spatial interval $[0, X]$ during a time interval $[0, T]$. During that time, the observer's position is specified by

$$x = x(t) \qquad 0 \le t \le T$$
$$x(0) = 0; \qquad x(T) = X. \tag{1.1}$$

The observer thus sees the time function

$$g(t) = f[x(t), t]. \tag{1.2}$$

We shall assume here that temporal changes in the function are negligible during the observation interval $[0, T]$; thus our interest is in $f(x, 0)$ and $f[x(t), 0]$ which we denote for simplicity by $f(x)$ and $g(t)$ respectively.

Our problem is as follows: the time function $g(t)$ is sampled once every $T/2WX$ seconds during the interval $[0, T]$ and the samples

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relayed to an observer on the ground. Those samples are subject to errors due to noise and instrument dynamics. The question of interest is then, how should the observer use these samples to construct an approximation, $\hat{f}(x)$, of $f(x)$, and how do different factors limit the approximation error? We will take here as a quantitative measurement of the approximation error the integral square error

$$\mathcal{E} = \int_0^X [f(x) - \hat{f}(x)]^2 dx \tag{1.3}$$

and give quantitative expressions indicating the dependence of this error on:

the percentage of the integral square value or "energy" of $f(x)$ lying within the frequency band $[-W, W]$;

errors introduced by aliasing, noise, and sensor dynamics;

nonuniformity of the sampling.

It must be emphasized at the outset that our expressions for the error will be complex and unwieldy to the extent of precluding their evaluation in some cases of practical interest. However, the analysis has general value in pointing out explicitly the sources of error, thus indicating the directions that should be taken in the design of the on-board data preprocessing and ground station reconstruction.

For convenience, we will sometimes regard the functions involved as elements of a vector space, with the inner (dot) product of two functions $f(x)$ and $g(x)$ defined as

$$(f, g) = \int_{-\infty}^{\infty} f(x)g(x)dx \tag{1.4}$$

and the norm or "length" of a function f defined by

$$\|f\| = (f, f)^{1/2} = \left[\int_{-\infty}^{\infty} f^2(x)dx\right]^{1/2}. \tag{1.5}$$

We will say that two functions f and g are orthogonal or perpendicular if $(f, g) = 0$. Note that if f and g are perpendicular then $\|f + g\|^2 = \|f\|^2 + \|g\|^2$.

1.2. RECONSTRUCTION BY SAMPLING FUNCTIONS - BOUNDS ON APPROXIMATION ERROR

Our observations of $f(x)$ are confined to the $2WX + 1$ sampled values $f(x_k)$, where

$$x_k = x\left(\frac{kT}{2WX}\right), \qquad k = 0, 1, \ldots, 2WX. \tag{1.6}$$

Given any set of functions $\psi_m(x)$, $m = 0, 1, \ldots, 2WX$ whose sets of values $\psi_m(x_0), \psi_m(x_1), \ldots, \psi_m(x_{2WX})$

$$m = 0, 1, \ldots, 2WX$$

form $2WX + 1$ linearly independent $2WX + 1$ dimensional vectors, we could assume an approximation for $f(x)$ of the form

$$\hat{f}(x) = \sum_{m=0}^{2WX} f_m \psi_m(x), \qquad 0 \leq x \leq X \tag{1.7}$$

and the unknown f_k could be found by solving the set of $2WX + 1$ linear equations

$$f(x_k) = \sum_{m=0}^{2WX} f_m \psi_m(x_k), \qquad k = 0, 1, \ldots, 2WX. \tag{1.8}$$

The first question we need to consider is what set of functions $\psi(x)$ should be used to provide a good approximation to $f(x)$. There are two primary considerations here:

(1) the ease of finding the coefficients f_k in eq. (1.7), or, equivalently, having a tractable expression for $f(x)$ directly in terms of the $f(x_k)$;

(2) the degree of approximation as measured, say, by $\| f(x) - \hat{f}(x) \|$.

Our attention will be focused on the sin x/x functions by virtue of the two criteria above; that is, we will consider the $\psi_m(x)$ to be

$$\psi_m(x) = \frac{(-1)^m \sin \Omega x}{(\Omega x - m\pi)} = \frac{\sin [\Omega x - m\pi]}{[\Omega x - m\pi]} . \tag{1.9}$$

A plot of sin x/x is given in fig. 1. The Fourier transform of $\psi_m(x)$ is

$$\Psi(\omega) = \int_{-\infty}^{\infty} \psi_m(x)\, e^{-j\omega x} dx = \begin{cases} \dfrac{e^{-j\omega m\pi}}{2W} & -\Omega \le \omega \le +\Omega \\ 0 & \text{elsewhere} \end{cases} \tag{1.10}$$

in which $\Omega = 2\pi W$.

With regard to the first point above, it is desirable not to have to solve the set of eqs. (1.7) in each case by inverting the matrix (whose elements are $c_{km} = \psi_m(x_k)$) but to have an explicit expression for $f(x)$ in terms of the $f(x_k)$. This is possible for <u>any</u> choice of the sample points x_k if one uses the sin x/x functions; to the best of the author's knowledge, this is not true for any other (nontrivial) choice of $\psi_m(x)$.

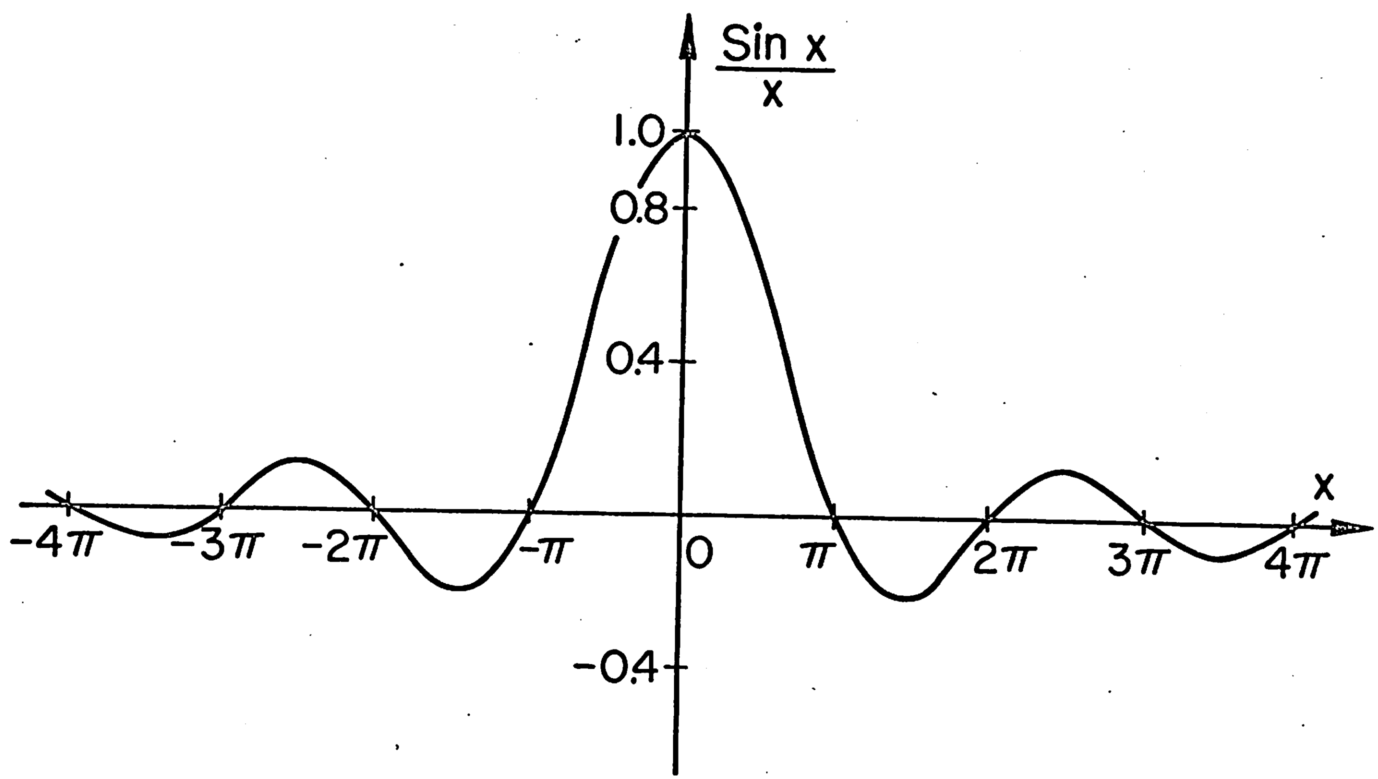

Fig. 1. Plot of the sin x/x function.

Insofar as the second point is concerned, the selection of the $\psi_m(x)$ to minimize $\|f(x) - \hat{f}(x)\|$ depends upon the a-priori knowledge available concerning the nature of $f(x)$. In the applications pertinent to space experiments, any specialized knowledge is usually lacking, and we seek a set of functions that will be generally useful with only vague assumptions regarding the nature of $f(x)$. In this regard, the sin x/x functions are again suitable, for if we let

$$F(\omega) = \int_0^X f(x)\, e^{-j\omega x} dx \tag{1.11}$$

and

$$f_L(x) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} F(\omega)\, e^{j\omega x} d\omega \tag{1.12}$$

in which

$$\Omega = 2\pi W \tag{1.13}$$

and

$$\epsilon_W{}^2 = \frac{\|f - f_L\|^2}{\|f\|^2} = \frac{\int_{|\omega| > \Omega} |F(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega} \tag{1.14}$$

then the representation error using the sin x/x functions becomes small as ϵ_W becomes small. Thus our criterion for a good approximation using the sin x/x functions requires only that the function $f(x)$ have small spectral content outside the band $[-\Omega, \Omega]$. This type of criterion is suitably general (and intuitively seems appropriate) for most of the applications in mind.

For these two reasons, we will consider in detail approximation of f(x) by sin x/x functions and give bounds on the various sources of error. We will start by considering approximating f(x) from uniformly spaced samples of $f_L(x)$ and note the additional errors that enter in as we successively relate our situation more closely to reality.

The function $f_L(x)$ defined by eq. (1.12) is strictly bandlimited; thus it may be represented as [1]

$$f_L(x) = \sum_{m=-\infty}^{\infty} f_L(m/2W)\frac{(-1)^m \sin \Omega x}{(\Omega x - m\pi)} . \tag{1.15}$$

Thus if we were able to pass f(x) through an ideal lo-pass filter of radian bandwidth $[-\Omega, \Omega]$ and sample the output periodically at a rate 2W, we could regenerate $f_L(x)$ to approximate f(x), and the integral square error would be

$$\begin{aligned} \mathcal{E}_0 = \|f_L(x) - f(x)\|^2 &= \int_0^X [f(x) - f_L(x)]^2 dx \\ &\leq \int_{-\infty}^{\infty} [f(x) - f_L(x)]^2 dx \\ &= \frac{1}{2\pi} \int_{|\omega|>\Omega} |F(\omega)|^2 d\omega \\ &= \epsilon_W{}^2 \|f\|^2 \end{aligned} \tag{1.16}$$

in which we have used Parseval's Theorem [2] to express the integral in the frequency domain. Now let us take into account the fact that we may observe only the samples on the interval [0, X]; that is, we observe the samples $f_L(m/2W)$, $m = 0, 1, \ldots, 2WX$ and generate the approximating function

$$\hat{f}_1(x) = \sum_{m=0}^{2WX} f_L(m/2W) \frac{(-1)^m \sin \Omega x}{(\Omega x - m\pi)}. \tag{1.17}$$

We now wish to bound the increase in error caused by this truncation. Such a bound is given by Landau and Pollak [3]. We note that if ϵ_W is small compared to one, then $f_L(x)$ is approximately limited to the interval [0, X]. To make this precise, let us denote

$$Df_L(x) = \begin{cases} f_L(x) & 0 \leq x \leq X \\ 0 & \text{elsewhere} \end{cases} \tag{1.18}$$

and

$$(1 - D)f_L(x) = f_L(x) - Df_L(x). \tag{1.19}$$

Now

$$f_L(x) = f(x) + [f_L(x) - f(x)]$$

$$(1 - D)f_L = (1 - D)f + (1 - D)[f_L - f]$$

$$= (1 - D)[f_L - f].$$

Thus

$$\|(1 - D)f_L\|^2 = \|(1 - D)[f_L - f]\|^2$$

$$= \int_{\substack{x < 0 \\ x > X}} [f_L(x) - f(x)]^2 dx$$

$$\leq \int_{-\infty}^{\infty} [f_L(x) - f(x)]^2 dx = (\epsilon_W)^2 \|f\|^2 \tag{1.20}$$

and

$$\frac{\|(1 - D)f_L\|^2}{\|f_L\|^2} \leq (\epsilon_W)^2 \frac{\|f\|^2}{\|f_L\|^2} = \frac{(\epsilon_W)^2}{1 - (\epsilon_W)^2} . \tag{1.21}$$

Thus by Theorem 2 of Landau and Pollak [3]

$$\frac{\|\hat{f}_1 - f_L\|^2}{\|f_L\|^2} \leq \pi \frac{\epsilon_W}{\sqrt{1 - (\epsilon_W)^2}} + \frac{(\epsilon_W)^2}{1 - (\epsilon_W)^2}$$

or

$$\|\hat{f}_1 - f_L\|^2 \leq [\pi \epsilon_W \sqrt{1 - \epsilon_W} + (\epsilon_W)^2] \|f\|^2 . \tag{1.22}$$

We now wish to bound $\|f - \hat{f}_1\|^2$; we write

$$\|f - \hat{f}_1\|^2 = \|(f - f_L) + (f_L - \hat{f}_1)\|^2 .$$

The Fourier transforms of both f_L and $\hat{f}_1$ are zero outside $[-\Omega, \Omega]$ while $f - f_L$ has a transform which is nonzero only outside this interval.

Thus, by Parseval's Theorem [2], it follows directly that

$$(f - f_L, f_L - \hat{f}_1) = 0$$

and hence

$$\mathcal{E}_1 = \|f - \hat{f}_1\|^2 = \|f - f_L\|^2 + \|f_L - \hat{f}_1\|^2$$

$$\leq \|f\|^2[\pi \epsilon_W \sqrt{1 - \epsilon_W{}^2} + 2\epsilon_W{}^2]. \tag{1.23}$$

In passing let us note that

$$\hat{f}_1(x) - f_L(x) = \sum_{\substack{m<0 \\ m>2WX}} f_L(m/2W)\frac{(-1)^m \sin \Omega x}{(\Omega x - m\pi)} \tag{1.24}$$

and since

$$\left(\frac{(-1)^m \sin \Omega x}{\Omega x - m\pi}, \frac{(-1)^n \sin \Omega x}{\Omega x - n\pi}\right)$$

$$= \int_{-\infty}^{\infty} \frac{(-1)^{n+m} \sin^2 \Omega x}{(\Omega x - n\pi)(\Omega x - m\pi)} dx$$

$$= \frac{1}{2W}\delta_{mn} \tag{1.25}$$

$$\sum_{\substack{m<0 \\ m>2WX}} f_L{}^2(m/2W) = 2W\|\hat{f}_1 - f_L\|^2$$

$$\leq 2W\|f\|^2[\pi \epsilon_W \sqrt{1 - \epsilon_W} + \epsilon_W{}^2]. \tag{1.26}$$

This inequality will be useful later.

Let us next consider a further source of error. In practice we cannot observe $f_L(x)$ for several reasons:

(1) an ideal lo-pass filter has associated with it an infinite delay--if we passed $f(x)$ through a realizable filter (one with only finite delay) the output would be only an approximation of $f_L(x)$. If $f(x)$ is sampled directly, the resulting error is referred to as aliasing.

(2) the signal $f(x)$ is distorted by the response of the sensor, and additive noise (e.g., thermal) is also present in the output of the sensor.

We will return to these sources of error in Section 1.3 and give a quantitative description of them and what can be done to combat them. For the time being, we will denote the actual filtered (or preprocessed) output of the sensor by $f'(x)$ and let

$$\epsilon(x) = f'(x) - f_L(x). \qquad (1.27)$$

The approximating function that is pertinent is thus

$$\hat{f}_2(x) = \sum_{m=0}^{2WX} f'(m/2W) \frac{(-1)^m \sin \Omega x}{\Omega x - m\pi} . \qquad (1.28)$$

If we calculate the approximation error, the term $\epsilon(x)$ will add in an additional term

$$\mathcal{E}_2 = \|f - \hat{f}_2\|^2 = \|(f - f_L) + (f_L - \hat{f}_1) + (\hat{f}_1 - \hat{f}_2)\|^2$$

$$= \|f - f_L\|^2 + \|f_L - \hat{f}_1\|^2 + \|\hat{f}_1 - \hat{f}_2\|^2$$

$$= \mathcal{E}_1 + \|\hat{f}_1 - \hat{f}_2\|^2. \tag{1.30}$$

The three terms add as the sum of the squares because the three functions $(f - f_L)$, $(f_L - f_1)$, and $(\hat{f}_1 - \hat{f}_2)$ are all pairwise orthogonal. We have already pointed out that $(f - f_L)$ is orthogonal to any linear combination of $\sin \Omega x/\Omega x$ functions. Moreover $(f_L - f_1)$ is a linear combination of $\sin \Omega x/\Omega x$ functions with indices m, $m < 0$ and $m > 2WX$, while $\hat{f}_1 - \hat{f}_2$ is also a linear combination of such functions with indices m, $0 \leq m \leq 2WX$: thus the orthogonality of $(f_L - \hat{f}_1)$ and $(\hat{f}_1 - \hat{f}_2)$ follows from eq. (1.25).

From eq. (1.25), the second term on the right hand side of eq. (1.30) is evaluated as

$$\|\hat{f}_1 - \hat{f}_2\|^2 = \frac{1}{2W} \sum_{m=0}^{2WX} \epsilon^2(m/2W). \tag{1.31}$$

Thus

$$\|f - \hat{f}_2\|^2 = \mathcal{E}_1 + \frac{1}{2W} \sum_{m=0}^{2WX} \epsilon^2(m/2W). \tag{1.32}$$

We lastly give consideration to the effects caused by nonuniform sampling. In most space experiments the data transmission is uniform in time and the velocity of the sensor will be nonuniform in time. Thus the positions at which we take observations

$$x_k = x\left(\frac{kT}{2WX}\right), \qquad k = 0, 1, \ldots, 2WX$$

will depart from the uniformly spaced sample positions $k/2W$ considered above. Our treatment in analyzing the errors introduced by the nonuniform sampling is based on the nonuniform sampling theorem of Yen [4].

The function $f_L(x)$ is bandlimited and hence can be written [1]

$$f_L(x_k) = \sum_{n=0}^{2WX} f_L(n/2W)\frac{(-1)^n \sin \Omega x_k}{(\Omega x_k - n\pi)} + r(x_k) \qquad (1.33)$$

in which

$$r(x_k) = \sum_{\substack{n<0 \\ n>2WX}} f_L(n/2W)\frac{(-1)^n \sin \Omega x_k}{\Omega x_k - n\pi}. \qquad (1.34)$$

The quantities $r(x_k)$ will be small if ϵ_W is small by virtue of eq. (1.26). For, using the Schwartz inequality,

$$r^2(x_k) = \left[\sum_{\substack{n<0 \\ n>2WX}} f_L(n/2W)\frac{(-1)^n \sin \Omega x_k}{\Omega x_k - n\pi}\right]^2$$

$$\leq \left[\sum_{\substack{n<0 \\ n>2WX}} f_L^2(n/2W)\right]\left[\sum_{\substack{n<0 \\ n>2WX}} \frac{\sin^2 \Omega x_k}{[\Omega x_k - n\pi]^2}\right]$$

$$\leq \sin^2 \Omega x_k \; 2W\|f\|^2[\pi \epsilon_W\sqrt{1 - \epsilon_W^2} + \epsilon_W^2] \cdot \sum_{\substack{n<0 \\ n>2WX}} \frac{1}{[\Omega x_k - n\pi]^2}.$$

(1.35)

The sum in this equation can be bounded by appropriate integrals, yielding

$$r^2(x_k) \leq \left[\frac{\sin \Omega x_k}{\pi}\right]^2 \|f\|^2 [\pi \epsilon_W \sqrt{1 - \epsilon_W^2} + \epsilon_W^2] \, 2W$$

$$\cdot \left[\frac{1}{2Wx_k + 1} + \frac{1}{(2Wx_k + 1)^2} + \frac{1}{[(2W)(X - x_k) + 1]^2} + \frac{1}{(2W)(X - x_k) + 1}\right]. \quad (1.35)$$

Now let C be the matrix whose k-n th entry is

$$c_{kn} = \frac{(-1)^n \sin \Omega x_k}{\Omega x_k - n\pi}, \qquad k, n = 0, 1, 2, \ldots, 2WX \quad (1.36)$$

$$D = [d_{pq}] = C^{-1} \quad (1.37)$$

and $\underset{\sim}{f}_L(x_k)$ the vector whose k-th component is $f_L(x_k)$. Then eq. (1.33) can be expressed

$$\underset{\sim}{f}_L(x_k) + \underset{\sim}{r}(x_k) = C \, \underset{\sim}{f}_L(n/2W).$$

Solving for $\underset{\sim}{f}_L(n/2W)$ yields

$$\underset{\sim}{f}_L(n/2W) = D \, \underset{\sim}{f}_L(x_k) + D \, \underset{\sim}{r}(x_k) \quad (1.38)$$

or, in terms of the function $f'(x)$ that we are actually able to observe

$$\underset{\sim}{f}_L(n/2W) = D \, \underset{\sim}{f}'(x_k) - D \, [\underset{\sim}{r}(x_k) + \underset{\sim}{\epsilon}(x_k)]. \quad (1.39)$$

Let us denote

$$\hat{\underset{\sim}{f}}_3(n/2W) = D\,\underset{\sim}{f}'(x_k) \tag{1.40}$$

and

$$\underset{\sim}{\epsilon}_3(n/2W) = D[\underset{\sim}{r}(x_k) + \underset{\sim}{\epsilon}(x_k)]. \tag{1.41}$$

The quantities $\hat{f}_3(n/2W)$, $n = 0, 1, \ldots, 2WX$, are the estimates we can obtain of $f_L(n/2W)$ from the observed samples $f'(x_k)$, $k = 0, 1, \ldots, 2WX$. The corresponding estimate of $f(x)$ is

$$\hat{f}_3(x) = \sum_{m=0}^{2WX} \hat{f}_3(m/2W) \frac{(-1)^m \sin \Omega x}{(\Omega x - m\pi)} . \tag{1.42}$$

Using eq. (1.25) it follows directly that

$$\begin{aligned} \|\hat{f}_1 - \hat{f}_3\|^2 &= \sum_{k=0}^{2WX} [\epsilon_3(k/2W)]^2 \\ &= \|\underset{\sim}{\epsilon}_3(n/2W)\|^2 \\ &= \|D[\underset{\sim}{r}(x_k) + \underset{\sim}{\epsilon}(x_k)]\|^2. \end{aligned} \tag{1.43}$$

The remarks subsequent to eq. (1.30) are again pertinent, and the approximation error involved in representing $f(x)$ by $\hat{f}_3(x)$ is thus

$$\begin{aligned} \|f - \hat{f}_3\|^2 &= \|f - \hat{f}_1\|^2 + \|\hat{f}_1 - \hat{f}_3\|^2 \\ &= \mathcal{E}_1 + \sum_{k=0}^{2WX} [\epsilon_3(k/2W)]^2. \end{aligned} \tag{1.44}$$

Observe from eq. (1.43) that the effect of the nonuniform observations is the addition of the term $r(x_k)$ (which is zero if $x_k = \pi$ times any integer) and transformation by the matrix D.

The matrix D is crucial in using the approximating function $\hat{f}_3(x)$ both because it must be used in computing $\hat{f}_3(x)$ (through eq. (1.40)) and in evaluating the effect of the errors. Fortunately, it is possible to invert the matrix C analytically through the use of the expression [4], [5]:

$$\mathrm{DET}\left[\frac{1}{a_q + b_p}\right] = \frac{\prod_{0 \le p < q \le N} (a_q - a_p)(b_q - b_p)}{\prod_{p,q=0}^{N} (a_q + b_p)} .$$

The result is

$$d_{pq} = \frac{(-1)^p(\Omega x_q - p\pi)}{\sin \Omega x_q} \prod_{\substack{n=0 \\ n \neq p}}^{2WX} \left[\frac{\Omega x_q - n\pi}{p\pi - n\pi}\right] \prod_{\substack{k=0 \\ k \neq q}}^{2WX} \left[\frac{\Omega x_k - p\pi}{\Omega x_k - \Omega x_q}\right] . \qquad (1.45)$$

This expression can be used to obtain an explicit representation of $\hat{f}_3(x)$; combining eqs. (1.40) and (1.42)

$$\hat{f}_3(x) = \sum_{m=0}^{2WX} \frac{(-1)^m \sin \Omega x}{(\Omega x - m\pi)} \sum_{n=0}^{2WX} d_{mn} f'(x_n)$$

$$= \sum_{n=0}^{2WX} f'(x_n) \Psi_n(x) \qquad (1.46)$$

in which

$$\Psi_n(x) = \sum_{m=0}^{2WX} d_{mn} \frac{(-1)^m \sin \Omega x}{(\Omega x - m\pi)} . \qquad (1.47)$$

If the expression for d_{mn} is substituted into this sum, the result can be recognized as a partial fraction expansion, yielding

$$\Psi_q(x) = \frac{\sin \Omega x}{\sin \Omega x_q} \prod_{\substack{k=0 \\ k \neq q}}^{2WX} \left[\frac{\Omega x - \Omega x_k}{\Omega x_q - \Omega x_k} \right] \prod_{n=0}^{2WX} \left[\frac{\Omega x_q - n\pi}{\Omega x - n\pi} \right] . \tag{1.48}$$

1.3. SOURCES OF ERROR IN THE OBSERVATIONS

In this section we give quantitative expressions for the errors caused by

1) the dynamics of the sensor
2) noise in the sensor
3) imperfect lo-pass filtering of the sensor output.

These expressions involve time-varying operations on the data and are not readily evaluated. Their value lies in pointing out what functions the on-board data processing should fulfill.

We assume that the sensor is being operated in its linear region. The response of a (possibly time-varying) linear system is defined by its impulse response $h(t, \tau)$, which represents the output of the system at time t caused by an impulse input at time $t-\tau$. If $g(t)$ is the input to such a system, the output is

$$y(t) = \int_0^\infty h(t, \tau) g(t - \tau) d\tau = \int_0^t h(t, t - \zeta) g(\zeta) d\zeta . \tag{1.49}$$

If the system is time-invariant, then $h(t, \tau)$ does not depend on t, but only on τ, the "age" of the input, and $h(t, \tau) = h(\tau)$.

We will denote by $g(t) = f[x(t)]$ the time function observed and assume that the sensor is time-invariant with impulse response $h_s(\tau)$. The output of the sensor is thus

$$y(t) = \int_0^t h_s(\tau)g(t - \tau)d\tau + n(t) \tag{1.50}$$

in which $n(t)$ is the additive noise in the sensor output (due, for example, to thermal noise in the sensor and first stage of electronic amplification following the sensor). The spatial function that we wish to match is

$$f_L(x) = \int_0^X \frac{\sin \Omega(x - \nu)}{\pi(x - \nu)} f(\nu)d\nu \tag{1.51}$$

or letting $\nu = x(\zeta)$ in the above we can obtain this as a function of time

$$g_L(t) = f_L[x(t)] = \int_0^T \frac{\sin \Omega[x(t) - x(\zeta)]}{\pi[x(t) - x(\zeta)]} x'(\zeta)g(\zeta)d\zeta \tag{1.52}$$

in which $x'(\zeta)$ denotes $\frac{dx(\tau)}{d\tau}\Big| \tau = \zeta$. Note that, if the velocity is non-uniform and $x'(\zeta)$ is not a constant, eq. (1.50) represents a time-varying linear operation on $g(t)$.

Filtering or preprocessing of $y(t)$ will be useful to obtain a better estimate of $g_L(t)$ than is afforded by $y(t)$ itself. This filtering is required to perform three somewhat conflicting objectives:

1) compensate for the dynamics of the sensor; the sensor will in general have a frequency response which drops off at high frequencies requiring the filter to accentuate high frequencies

2) reduce the effects of the noise; most noise sources are broad band, thus to perform objective 2) a given filter must sacrifice somewhat in fulfilling objective 1) and vice versa.

3) filter out all frequency components of $f(x)$ above Ω; since $g(t) = f[x(t)]$, a nonuniform velocity will require a time-varying filter.

A functional diagram of the signals and processing involved is given in fig. (2).

Lo-pass filtering cannot be achieved without some delay (a perfect lo-pass filter requires infinite delay); we denote the allowable delay by Δ, the impulse response of the filter by $h_f(t, \tau)$, and the output of the filter by

$$g'(t) = f'[x(t)] = \int_{-\Delta}^{t} h_f(t, \tau) y(t - \tau) d\tau . \tag{1.53}$$

We are interested in the error at the sample times

$$t_k = \frac{k}{2WX} T, \qquad k = 0, 1, \ldots, 2WX.$$

Let the cascade response of the filter and sensor at time t_k be denoted by $h(t_k, \tau)$

$$h(t_k, \tau) = \int_0^{\infty} h_f(t_k, \tau - \zeta) h_s(\zeta) d\zeta . \tag{1.54}$$

The error at time t_k is then

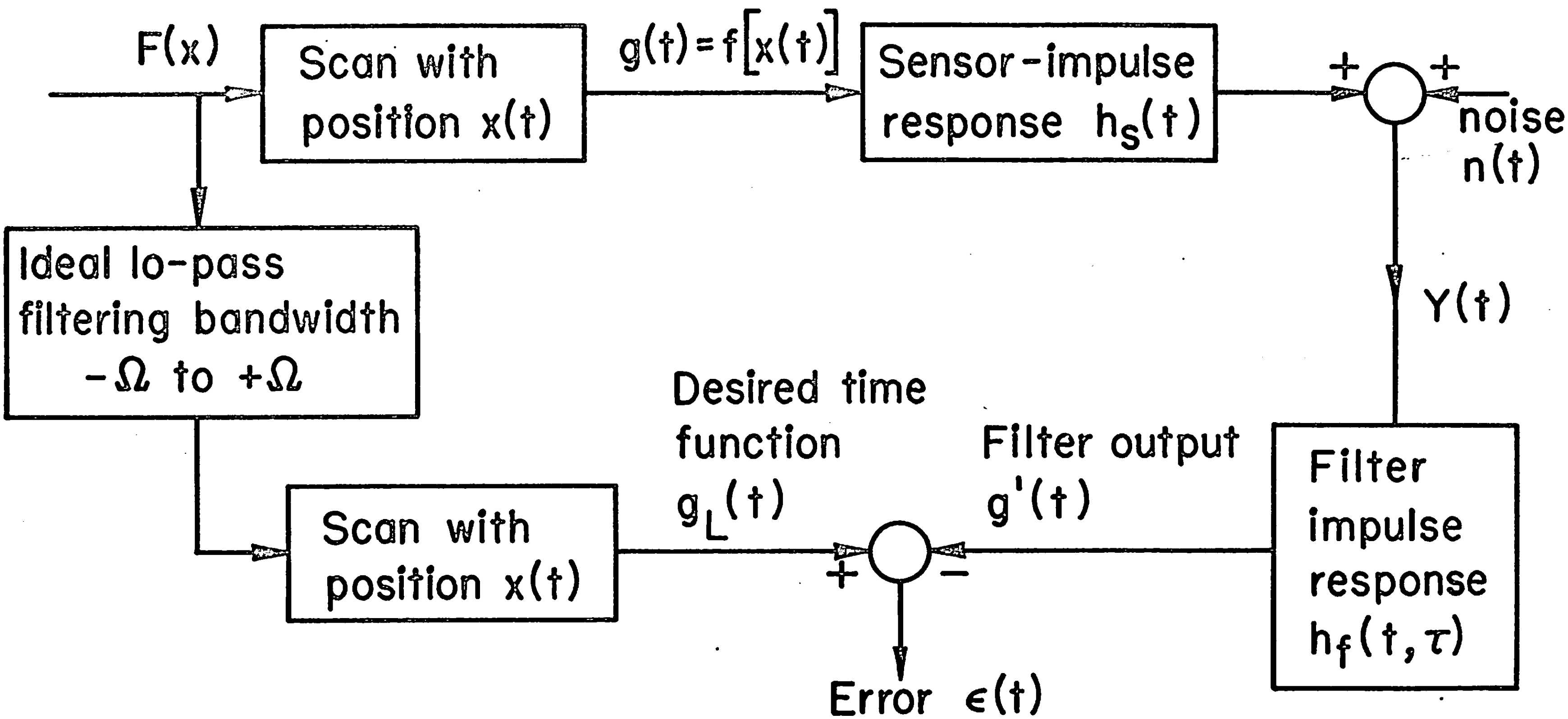

Fig. 2. Functional diagram of the operations involved in preprocessing the data.

$$g'(t_k) - g_L(t_k) = \int_0^T \left[h(t_k, t_k - \zeta) - \frac{\sin \Omega[x(t) - x(\zeta)]}{\pi[x(t) - x(\zeta)]} x'(\zeta) \right] g(\zeta) d\zeta$$

$$+ \int_0^{t_k + \Delta} h_f(t_k, t_k - \zeta) n(\zeta) d\zeta. \tag{1.55}$$

The first term in eq. (1.55) represents the errors due to imperfect lo-pass filtering of the signal $g(t)$ and the response of the sensor; the second term is due to the additive noise in the sensor output. The mean square value of this noise term is

$$\mathcal{E}(t_k) = \int_0^{t_k+\Delta} \int_0^{t_k+\Delta} h_f(t_k, t_k - \zeta_1) h_f(t_k, t_k - \zeta_2) \varphi_n(\zeta_1 - \zeta_2) d\zeta_1 d\zeta_2 \tag{1.56}$$

in which

$$\varphi_n(\zeta_1 - \zeta_2) = E\{n(\zeta_1) n(\zeta_2)\} \tag{1.57}$$

is the correlation function of the noise. If the noise is broadband with respect to the signal $g(t)$ and has a flat spectral density of N_o watts/cycle per sec, then

$$\varphi_n(\zeta_1 - \zeta_2) = N_o \, \delta(t_1 - t_2)$$

and

$$\mathcal{E}(t_k) = N_o \int_0^{t_k+\Delta} h_f^2(t_k, t_k - \zeta) d\zeta. \tag{1.58}$$

If the signal generated by the sensor, g(t) is regarded as a random process, existing theory [6] specifies the impulse response of the filter that minimizes the total mean square error of eq. (1.58) as the solution to an integral equation. It seems of dubious value to pursue this point for two reasons:

1) This integral equation involves the correlation function of the signal g(t); in many cases a good estimate of this correlation function will not be available prior to an experiment

2) Exact solution of the integral equation for a nonconstant velocity would be extremely difficult; a nonoptimum filter which could be found by simpler means would be of more practical importance.

2. Detection of a source from observations of a counting process

2.1. INTRODUCTION

Consider a detector, such as a gamma-ray detector, scanning some arc of the sky. At random event times the detector will count the arrival of an event, so that at the end of the scan we have recorded the angles $\theta_1, \theta_2, \ldots, \theta_N$, at which the arrivals occurred; N, the number of counts, being a random variable. We consider here the problem of using this information to detect the presence of a possible source. Let I_s denote the angular interval suspected, by virtue of other means of observation, of containing a source and let I_b denote the remainder of the angular observation interval. Let Θ_s and Θ_b denote the respective angular magnitudes of these two intervals and N_s and N_b the number of counts observed in each of the two respective intervals. Although N_s and N_b are random variables, their mean or expected values

$$m_s = E\{N_s\} \qquad m_b = E\{N_b\}$$

yield a direct indication of the presence or absence of a source in the interval I_s. We thus wish to use the observations of N_s and N_b to test the hypothesis that the mean counting rate is larger in the interval I_s than in the interval I_b.

In those situations in which the numbers N_s and N_b are both large, the percentage variation in these numbers from their respective means will be quite small and one can reliably ascertain whether a source is present or not. However, in some applications such as gamma-ray astronomy [7], the number of counts, particularly N_s, will be quite small. We wish here to develop expressions which give a measure of the credence one can place in the conclusion as to the presence or absence of a source. In particular, we discuss the detection strategy and give quantitative expressions for the probability of a false alarm (falsely concluding a source present) and the probability of a miss (falsely concluding no source is present). First we consider the probability distribution of the random variables N_s and N_b.

2.2. THE PROBABILITY DISTRIBUTION OF THE NUMBER OF COUNTS

Let θ be an arbitrary angle; the assumptions we make regarding the distribution of the number of counts are as follows:

(1) Let dt denote an infinitesimal interval of time. We assume the probability of two or more counts occurring simultaneously at any arbitrary angle θ in any time interval of length dt to be zero, and the probability of a single count occurring to be $a(\theta)dt = [a_s(\theta) + a_b]dt$; $a_s(\theta)$ denoting the average counting rate due to the source and a_b the average rate due to isotropic background radiation.

(2) Let I_1 and I_2 be any two disjoint time intervals and N_1 and N_2 the number of counts occurring in the two respective intervals. We assume that N_1 and N_2 are statistically independent.

Let the scan angle as a function of time be given by the relation $\theta = \theta(t)$. By modifying slightly the results of a similar problem done in a different physical context [8], it can be shown that $P(k, T_s)$, the probability of observing exactly k counts in the angular interval I_s during the time interval $[0, T_s]$, is given by

$$P(k, T_s) = \exp\left\{-\int_0^{T_s} a[\theta(t)]dt\right\}\left\{\int_0^{T_s} a[\theta(t)]dt\right\}^k \Big/ k! \qquad (2.1)$$

$$k = 0, 1, 2, \ldots$$

Thus N_s is Poisson distributed with a mean value

$$m_s = E\{N_s\} = \sum_{k=0}^{\infty} k\,P(k, T_s) = \int_0^{T_s} a[\theta(t)]\,dt. \qquad (2.2)$$

If the angular rate of sweep across I_s is constant, so that

$$\frac{d\theta}{dt} = v_s = \frac{\Theta_s}{T_s} \qquad (2.3)$$

then, denoting

$$a_s = \frac{1}{\Theta_s}\int_{I_s} a_s(\theta)d\theta \qquad (2.4)$$

the expression for the mean value of N_s becomes

$$m_s = [T_s/\Theta_s]\int_{I_s} [a_b + a_s(\theta)]\,d\theta = T_s(a_b + a_s). \qquad (2.5)$$

Next, consider making a number of successive scans of the interval I_s, and let the number of counts observed in these successive scans be denoted by $N_{s_1}, N_{s_2}, \ldots, N_{s_m}$. The number of counts observed in the different scans are [by virtue of assumption (2)] statistically independent random variables. Thus [9] N_s, the total number of counts observed in all m scans, is again Poisson distributed with a mean that is equal to the sum of the means of each of the m counts. If the m scans are made with the respective (uniform) velocities $\Theta_s/T_{s_1}, \Theta_s/T_{s_2}, \ldots, \Theta_s/T_{s_m}$, then m_s, the mean value of N_s, the total number of counts observed, will be

$$m_s = [a_b + a_s]\left[\sum_{k=1}^{m} T_{s_k}\right] = [a_b + a_s]T_s \tag{2.6}$$

in which T_s is the total observation time for all m scans of the interval I_s.

Over the interval I_b, the radiation is presumed isotropic with an expected number of counts per second of a_b, independent of θ. The above remarks apply here to the effect that after q scans of the interval I_b with respective (uniform) velocities $\Theta_s/T_{b_1}, \Theta_s/T_{b_2}, \ldots, \Theta_s/T_{b_q}$, the distribution of N_b, the total number of counts is Poisson with mean

$$m_b = a_b \sum_{k=1}^{q} T_{b_k} = a_b T_b \tag{2.7}$$

in which T_b is the total time of observation for the interval Ib.

2.3. SOURCE DETECTION: FALSE ALARM AND MISS PROBABILITIES

We now return to the problem of ascertaining the presence or absence of a source in the interval I_s. There will be difficulty in doing this only if the time intervals T_b and T_s and counting rates a_b and a_s are such that either one or both of the mean counts m_b and m_s is small. For clarity of presentation, we will assume that m_b is sufficiently large that it can be considered known (the standard deviation of N_b is m_b, so that for N_b large, N_b gives an accurate estimate of m_b). The following discussion can be altered in a reasonably straightforward manner to apply to the case in which both m_s and m_b are small.

Let us point out in passing that for the purposes of gaining any information about m_b and m_s, the only attributes of the observations that are required are the two total counts N_b and N_s (i.e., the location of the occurrence of the counts within the two intervals I_b and I_s is irrelevant). This is easily shown by making use of the concept of a sufficient statistic [10]. It can be shown that the two counts N_b and N_s form a sufficient statistic for forming any inference about m_b and m_s [11]. It can also be shown [12] that, regardless of the criterion of quality, it is not possible to improve upon the quality of any statistical inference about m_b and m_s by making use of properties of the observations besides the sufficient statistic (N_b and N_s).

Our problem is now as follows: having made m scans of I_s with total scan time T_s, we wish to decide if N_s was caused just by the isotropic background radiation or is due to the background radiation plus an additional source. Statistically, we wish to decide upon the validity of the two alternate hypotheses:

H_o : N_s is Poisson distributed with mean

$$m_b = T_s a_b$$

or

H_1 : N_s is Poisson distributed with mean

$$m_s = T_s(a_b + a_s)$$

in which a_b is known (through m_b) and a_s is an unknown parameter. If a_s were known, this would be referred to as testing a simple hypothesis [13]; i.e., a decision between two known alternatives. Our situation, in which a_s is unknown, is referred to as a composite hypothesis. The following terminology is used to describe the two possible errors that can be committed:

declaring H_1 true, when H_o is in fact true, is termed an error of the first kind or a false alarm

declaring H_o true, when in fact H_1 is true, is termed an error of the second kind or a miss.

For a given decision rule or hypothesis test the probability of a false alarm is referred to as the size of the rule and one minus the probability of a miss is referred to as the power of the rule. In the case of testing a simple hypothesis, the rule having the greatest power among all tests of fixed size α, $0 \leq \alpha \leq 1$, is referred to as the most powerful at level α. In our composite hypothesis situation, it may be that a rule that would be good for one value of a_s could be quite poor (relative to some other rule) for another value of a_s. In the happy case that it is possible to find a rule of level α that is most powerful independently of the value of a_s, we say that this test is uniformly most powerful at level α.

Consider the following decision rule for our case: if $N_s > k$ for some fixed number k, we declare H_1 true (a source present); if $N_s \leq k$, we declare H_o to be true (only background radiation present). From the expression for the Poisson distribution, we have for the probabilities of the two types of errors:

$$P_{fa} = P\text{ (false alarm)} = \exp[-m_b] \sum_{n>k}^{\infty} (m_b)^n/n! \tag{2.8}$$

$$m_b = T_s a_b$$

$$P_m = P\text{ (miss)} = \exp[-m_s] \sum_{n=0}^{k} (m_s)^n/n! \tag{2.9}$$

$$m_s = T_s (a_s + a_b).$$

It can be verified that this decision rule is uniformly most powerful. That is, if k is picked to yield a fixed value of P_{fa} as given by eq. (2.8), then no other decision rule with the same false alarm probability will yield a smaller value of P_m than that given by eq. (2.9), regardless of what the value of a_s. This follows from the Neyman-Pearson Lemma [13] and the fact that the ratio of the probability that $N_s = n$ under hypothesis H_1 to the probability that $N_s = n$ under hypothesis H_o

$$\frac{\exp[-m_s][m_s]^n/n!}{\exp[-m_b][m_b]^n/n!} = \exp[-a_b T_s][(a_s + a_b)/a_b]^n \qquad (2.10)$$

increases monotonically with n independently of a_s.

The values of P_{fa} and P_m given by eqs. (2.8) and (2.9) thus indicate the ultimate in reliability that can be achieved in deciding upon the presence or absence of a source in the interval I_s. These expressions can easily be evaluated with the use of existing tables [14] for a wide variety of values of m_b, m_s, and k. Table 1 gives the values of P_{fa} and P_m for different threshold values k for the cases $a_s = a_b/2$, $a_s = a_b$, and $a_s = 2a_b$. The table shows that for $a_s \geq 2a_b$, a reliable decision can be achieved for a small number of total counts, while for values of a_s smaller than this a substantial number of total counts is required to achieve reasonable confidence in the decision.

Given that one has confirmed the presence of a source in the interval I_s, the remaining problem is to accurately estimate the value of m_s [or equivalently the value of $a_s = (m_s/T_s) - a_b$]. Intuitively it seems that one should take N_s as the estimate of m_s. This estimate is unbiased, i.e.,

$$E\{N_s\} = m_s$$

and has variance

$$E\{(N_s - m_s)^2\} = m_s.$$

The Cramer'-Rao inequality [10] bounds the variance of any unbiased estimate, $\widehat{m}_s$, of m_s by

$$E\{(\widehat{m}_s - m_s)^2\} \geq \left[E\left\{\left[\frac{\partial}{\partial m_s} \log P(n, T_s; m_s)\right]^2\right\}\right]^{-1} \qquad (2.11)$$

$$= -\left[E\left\{\frac{\partial^2}{\partial m_s^2} \log P(n, T_s; m_s)\right\}\right]^{-1}.$$

Using

$$P(n, T_s; m_s) = \exp[-m_s][m_s]^n/n!$$

this bound can be evaluated to yield

$$E\{(\widehat{m}_s - m_s)^2\} \geq m_s.$$

Since $\widehat{m}_s = N_s$ actually achieves this bound, no other unbiased estimate can yield a smaller mean square error, thus using N_s as an estimate of m_s is optimum in this sense as well as intuitively correct.

TABLE 1

Values of Miss and False Alarm Probabilities for Different Source to Background Ratios and Counting Intervals

$m_b = T_s a_b \qquad m_s = (a_s + a_b) T_s$

Decision Rule: Accept H_o (No Source Present) if $N_s \leq k$

Accept H_1 (Source Present) if $N_s > k$

Case I: $a_s = a_b/2 : m_s = 3/2\ m_b$

$m_b = 10$; $m_s = 15$			$m_b = 20$ $m_s = 30$			$m_b = 40$ $m_s = 60$		
k	P_{fa}	P_m	k	P_{fa}	P_m	k	P_{fa}	P_m
11	.3032	.1847	22	.2794	.0805	45	.1903	.0265
12	.2085	.2676	23	.2125	.1146	46	.1521	.0365
13	.1253	.3632	24	.1567	.1572	47	.1195	.0491
14	.0835	.4656	25	.1121	.2083	48	.0924	.0650
			26	.0778	.2673	49	.0703	.0844
			27	.0524	.3328	50	.0526	.1076
						51	.0387	.1350
						52	.0280	.1666
						53	.0199	.2024

Case II: $a_s = a_b$; $m_s = 2m_b$

$m_b = 5$; $m_s = 10$			$m_b = 10$ $m_s = 20$			$m_b = 15$ $m_s = 30$		
k	P_{fa}	P_m	k	P_{fa}	P_m	k	P_{fa}	P_m
6	.2378	.1301	11	.3032	.0213	18	.1805	.0052
7	.1333	.2202	12	.2084	.0390	19	.1247	.0093
8	.0680	.3328	13	.1355	.0661	20	.0829	.0159
9	.0318	.4579	14	.0834	.1048	21	.0531	.0259
			15	.0487	.1565	22	.0327	.0406
			16	.0270	.2210	23	.0194	.0609
			17	.0142	.2970	24	.0111	.0881
						25	.0061	.1228
						26	.0033	.1655

Case III: $a_s = 2a_b$; $m_s = 3m_b$

$m_b = 3$ $m_s = 9$			$m_b = 5$ $m_s = 15$		
k	P_{fa}	P_m	k	P_{fa}	P_m
4	.1847	.0402	6	.2378	.0076
5	.0839	.0885	7	.1333	.0180
6	.0335	.1649	8	.0681	.0374
7	.0119	.2686	9	.0318	.0698
8	.0038	.3918	10	.0137	.1184
			11	.0054	.1847

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