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A STABILITY INEQUALITY FOR A CLASS OF NONLINEAR FEEDBACK SYSTEMS

by

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Summary--For some systems the Popov stability criterion fails to verify Aizerman's conjecture, that is, when the Popov sector is not equal to the linear (Hurwitz) sector. In these cases the question of stability for a nonlinearity which exceeds the Popov sector but which is included in the Hurwitz sector is unanswered. This paper provides a partial answer to this question by taking into account the slope of the nonlinear function. By constraining this slope to the interval $[0, k]$ and thus the nonlinearity itself to the sector $[0, k]$, the following stability inequality is obtained

$$\mathrm{Re}\left[1 + \frac{j\omega q}{1+\mu\omega^2}\right] G(j\omega) + \frac{1}{k} > 0$$

where μ is a non-negative parameter. For $\mu = 0$ this inequality reduces to the Popov criterion.

Two examples are given, in the first of which the sector is extended up to the linear limit. The Popov theorem concerned only the zero-input response of the nonlinear feedback system, whereas here a

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restricted class of inputs to the system is allowed.

Introduction

In recent years the problem of absolute stability of nonlinear feedback systems has received much attention. Aizerman and Gantmacher [1] have presented a historical survey of the problem and a detailed exposition of the V. M. Popov theorem. For practical purposes, the Popov theorem was a significant new result, however, it actually only provided a new method of solution for a problem that had already been solved by Lur'e [2] since it has been shown by Yakubovitch [3] and by Kalman [4] that the Popov criterion is a necessary and sufficient condition for the existence of a Lyapunov function of the Lur'e type. The Popov theorem has been extended to systems with pure delay, systems with multiple nonlinearities and to sampled-data systems.

In the practical analysis of nonlinear control systems, a theorem concerning absolute stability is often far too general, as more information is known about the nonlinear element than the fact that it is contained in some sector. However, the main body of researchers in this field has concentrated on the problem of absolute stability and few have deviated from this problem to take account of additional information about the nonlinear element. Jury and Lee [5] have made a successful approach to this problem for sampled-data systems by considering the case where the slope of the nonlinear function is bounded. In this paper an equivalent approach is made to nonlinear continuous systems using the method of the Popov theorem. Yakubovitch [3] has obtained a stability inequality for monotonic nonlinearities and this result is included as a special case of theorem 1.

Several authors have also concerned themselves with the stability of the responses of nonlinear systems to various inputs. Naumov and Tsypkin [6] have recently presented a result for bounded inputs in which nonlinear functions similar to those in this paper are considered, however, the stability criterion they obtain is more restrictive than the Popov criterion. In this paper it is shown that the response of the system to a restricted class of inputs from arbitrary initial conditions is stable.

It must be noted that this paper provides additional information only when the Popov theorem fails to verify Aizerman's conjecture. In those cases where the Popov sector is equal to the linear (Hurwitz) sector, no additional information can be gained by further restricting the nonlinear element.

Description of System

The class of nonlinear feedback systems considered will have the configuration of Fig. 1. The block labelled N is a time-invariant memoryless nonlinear gain element whose output $\xi(t)$ is given by

$$\xi(t) = \phi[\sigma(t)] \tag{1}$$

where $\phi(\sigma)$ is a continuous function of σ,

$$\phi(0) = 0 \text{ and } 0 \leq \phi(\sigma)/\sigma \leq k < \infty, \ \forall \sigma \neq 0 \tag{2}$$

These inequalities restrict the nonlinear function to a sector in the σ, ϕ plane and we will refer to this as a nonlinearity in the sector $[0, k]$.

The following further restriction is made on N:

$$0 \leq \frac{d\phi}{d\sigma} \leq k' \tag{3}$$

Eq. (3) restricts the 'a.c.' gain of the nonlinear element and will clearly be a meaningful restriction only if

$$k' \geq k \tag{4}$$

This will be referred to as a nonlinearity with slope restriction $[0, k']$.

The block labelled G is a linear time invariant subsystem described by the equations

$$\begin{aligned} \dot{\underline{x}}(t) &= \underline{A}\,\underline{x}(t) + \underline{b}\,\xi(t), \qquad t \geq 0 \\ y(t) &= \underline{c}^T\,\underline{x}(t) \end{aligned} \tag{5}$$

where $\underline{A}$ is an $n \times n$ constant matrix, $\underline{x}(t)$ is an n-vector and $\underline{b}, \underline{c}$ are constant n-vectors.

The transfer function of G is given by

$$G(s) = \underline{c}^T (s\underline{I} - \underline{A})^{-1}\,\underline{b} \tag{6}$$

and putting $s = j\omega$ in (6) gives $G(j\omega)$, the frequency response of the subsystem G.

The following assumption is made: in the principal case, all eigenvalues of $\underline{A}$ have negative real parts or, equivalently, all poles of $G(s)$ have negative real parts. This restriction may be relaxed to allow simple poles on the imaginary axis of the s-plane in the particular cases.

An alternative formulation for the subsystem G is as follows[1]:

$$y(t) = z(t) + \int_0^t g(t-\tau)\,\xi(\tau)\,d\tau, \qquad t \geq 0 \tag{7}$$

Here $g(t)$ is the impulse response of G and $G(s) = \mathcal{L}[g(t)]$ where $\mathcal{L}$ denotes the Laplace transformation.

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[1] The formulation of G as a stable linear differential system is convenient because of the many properties of such a system which will be used in the Theorem. However, in terms of the more general formulation of Eq. (7), the following conditions may be shown to be sufficient [7]:

a) for all initial states, $z(t)$ and $\dot{z}(t)$ are bounded on $(0, \infty)$,
b) for all initial states, $z(t)$, $\dot{z}(t)$ and $\ddot{z}(t)$ are elements of $L_2(0, \infty)$,
c) $g(t)$ is an element of $L_1(0, \infty)$.

$z(t)$ is the zero-input response of G which depends on the initial state of G.

The input to the system $u(t)$ satisfies the following conditions:

a) $u(t)$ and $\dot{u}(t)$ are bounded on $[0, \infty)$

b) $u(t)$, $\dot{u}(t)$ and $\ddot{u}(t)$ are elements of $L_2(0, \infty)$

The above conditions imply that $u(t) \to 0$ as $t \to \infty$.

Stability Inequalities

Theorem 1: For the principal case, if there exists a finite number $q \geq 0$ and a finite number $\mu \geq 0$ such that for all $\omega \geq 0$

$$\mathrm{Re}\,(1 + j\omega q)\, G(j\omega) + \frac{1}{k} + \mu\omega^2 \left\{ \mathrm{Re}\, G(j\omega) + \frac{1}{k'} \right\} > 0 \tag{8}$$

then the system is asymptotically stable in the large for all nonlinearities with slope restriction $[0, k']$ in the sector $[0, k]$.

For a monotonic nonlinearity $\left(\frac{1}{k'} = 0\right)$ the inequality becomes[(2)]

$$\mathrm{Re}\,(1 + j\omega q + \mu\omega^2)\, G(j\omega) + \frac{1}{k} > 0 \tag{9}$$

Theorem 2: If there exists a finite number q and a finite number $\mu \geq 0$ such that for all $\omega \geq 0$

$$\mathrm{Re}\left[1 + \frac{j\omega q}{1 + \mu\omega^2}\right] G(j\omega) + \frac{1}{k} > 0 \tag{10}$$

then for the principal case, the system is asymptotically stable in the large for all nonlinearities with slope restriction $[0, k]$. In the particular cases, if the conditions for stability in the limit are satisfied[(3)] then the theorem remains true for nonlinearities with slope restriction $[0, k]$ in the sector $[\epsilon, k]$ where $\epsilon > 0$ is arbitrarily small.

Note that for $\mu = 0$ all three inequalities reduce to the

[(2)] Yakubovitch [3] has obtained this inequality using Lyapunov function techniques.

[(3)] These conditions require that the system be stable for a linear gain $\phi(\sigma) = \epsilon\sigma$ where $\epsilon > 0$ is small. This is a linear problem and conditions on $G(j\omega)$ for stability in the limit are given by a Theorem in [1].

V. M. Popov criterion.

The inequalities may be tested analytically or graphically as is done with the Popov inequality. However, the graphical procedure here is more complicated. An initial value must be taken for the parameter μ and then an iterative technique used to find the optimum value.

Inequality (8) is useful mainly for analysis problems where at least the value of k' is known. Inequalities (9) and (10) are more suited to design procedures where it is required to find the maximum value of k for stability.

The graphical technique is similar to that used with the Popov criterion. For inequality (9) we plot

$$Y = \omega \operatorname{Im} G(j\omega) \qquad \omega \geq 0$$

against $X = (1 + \mu\omega^2) \operatorname{Re} G(j\omega)$

for a fixed value of the parameter μ. Then(9) becomes

$$X - qY + \frac{1}{k} > 0$$

The tangent line of slope $1/q$ may then be drawn to find the value of the intercept on the X axis, $-1/k$. The optimum value of μ must be found by some iterative process, however, in the two examples given, this problem was not too difficult.

For inequality (10) we plot

$$Y = \frac{\omega}{1 + \mu\omega^2} \operatorname{Im} G(j\omega) \qquad \omega \geq 0$$

against $\qquad X = \operatorname{Re} G(j\omega)$

and proceed as above.

The following remarks unify and simplify the proof of Theorems 1 and 2.

<u>Remark 1</u> For the principal case, Theorem 2 follows from Theorem 1 by putting $k' = k$.

Remark 2 The inequalities guarantee the satisfaction of the Nyquist criterion for linear gains in the sector $[0, k]$. Let ω_0 be the frequency at which the X, Y plot crosses the negative X axis furthest from the origin. This will also be the frequency for the same intersection of the Nyquist plot and for inequality (8)

$$\text{Re } G(j\omega_0) + \frac{1}{k} + \mu\omega_0^2 \left[\text{Re } G(j\omega_0) + \frac{1}{k'}\right] > 0$$

Now $\frac{1}{k'} \leq \frac{1}{k}$ so this implies that

$$(1 + \mu\omega_0^2) \; [\text{ Re } G(j\omega_0) + \frac{1}{k} \;] > 0$$

and hence $\qquad \text{Re } G(j\omega_0) > \frac{-1}{k}$

which is the Nyquist criterion.

Remark 3 In all three inequalities we can, without loss of generality change the inequality

$$H(\omega, q, \mu) > 0$$

to the inequality

$$H(\omega, q, \mu) \geq \delta > 0$$

where δ is a positive number. This remark is explained fully in [1] for the Popov theorem and the reasoning is identical in this case.

Remark 4 In the proof of Theorem 2 we may restrict ourselves to the principal case since the particular cases may be reduced to the principal case by the transformation

$$\xi = k\sigma - \tilde{\xi} \tag{11}$$

This transformation changes the characteristics of the nonlinear function $\phi(\sigma)$ to $\tilde{\phi}(\sigma) = k\sigma - \phi(\sigma)$ and the frequency response of the subsystem $G(j\omega)$ to $\tilde{G}(j\omega)$ where

$$\tilde{G}(j\omega) = \frac{G(j\omega)}{1 + k\,G(j\omega)} \tag{12}$$

It must be noted that the transformed system differs from the original system only in notation. Now

$$\frac{d\tilde{\phi}}{d\sigma} = k - \frac{d\phi}{d\sigma}$$

and we have assumed for this particular case that the conditions for stability in the limit are satisfied. That is, that the system is stable for a linear gain $\phi(\sigma) = \epsilon\,\sigma$ (where $\epsilon > 0$ is arbitrarily small). Because of the fact that the Nyquist criterion is satisfied, this implies that the system is stable for $\phi(\sigma) = k\sigma$. Hence the transformed system is stable for $\tilde{\phi}(\sigma) = 0$, that is, the transformed system is a principal case.

From Eq. (12)

$$\mathrm{Re}\left[1 - \frac{j\omega q}{1+\mu\omega^2}\right]\tilde{G} + \frac{1}{k} = \mathrm{Re}\left[-1 + \frac{j\omega q}{1+\mu\omega^2}\right]\frac{G}{1+kG} + \frac{1}{k}$$

$$= \mathrm{Re}\left[\left(\frac{j\omega q}{1+\mu\omega^2}\,G + \frac{1}{k}\right)\Big/(1+kG)\right]$$

$$= \frac{1}{|1+kG|^2}\,\mathrm{Re}\left[\left(\frac{j\omega q}{1+\mu\omega^2}\,G + \frac{1}{k}\right)\left(1+k\bar{G}\right)\right]$$

$$= \frac{1}{|1+kG|^2}\,\mathrm{Re}\left[\bar{G} + \frac{j\omega q}{1+\mu\omega^2}\,G + \frac{1}{k}\right]$$

$$= \frac{1}{|1+kG|^2}\,\mathrm{Re}\left[\left(1 + \frac{j\omega q}{1+\mu\omega^2}\right)G + \frac{1}{k}\right] \qquad (13)$$

It is clear from Eq. (13) that for the particular cases, the inequality

$$\mathrm{Re}\left[1 - \frac{j\omega q}{1+\mu\omega^2}\right]\tilde{G} + \frac{1}{k} \geq 0 \qquad \text{for all } \omega \geq 0 \qquad (14)$$

follows from inequality (10). In (14), equality takes place for each $\omega = \omega_0$ where $j\omega_0$ is a pole of $G(s)$, but clearly it follows from (14) that for all $\omega \geq 0$

$$\mathrm{Re}\left[1 - \frac{j\omega q}{1+\mu\omega^2}\right]\tilde{G} + \frac{1}{k-\epsilon} > 0$$

Once Theorem 2 is proved for the principal case, we can apply this theorem to the transformed system and thus establish stability for $\tilde{\phi}(\sigma)$ with slope restriction $[0, k]$[4] in the sector $[0, k-\epsilon]$. This then implies stability of the original system for $\phi(\sigma)$ with slope restriction $[0, k]$ in the sector $[\epsilon, k]$ which was to be proved.

Remark 5 In proving Theorem 2 for the principal case, we may limit ourselves to the case $q > 0$.

Assume that Theorem 2 has been proved for the principal case for $q > 0$. Now let $q < 0$ and again apply the transformation (11). Since from remark 2, the original system is stable for $\phi(\sigma) = k\sigma$, the transformed system will be stable for $\tilde{\phi}(\sigma) = 0$, that is, the transformed system is also a principal case. Since, for the principal case, $\tilde{G}(j\omega)$ is finite for all ω, the inequality

$$\mathrm{Re}\left[1 - \frac{j\omega q}{1+\mu\omega^2}\right]\tilde{G}(j\omega) + \frac{1}{k} > 0 \qquad (15)$$

follows from inequality (10) and Eq. (13). This inequality differs from (10) only by the sign of q and the change of $G(j\omega)$ to $\tilde{G}(j\omega)$. So once Theorem 2 has been proved for the principal case for $q > 0$, it follows from inequality (15) that the transformed system is stable. But this implies stability of the original system since these two systems are, in fact, the same, differing only in notation. Hence the original system is stable for $q < 0$.

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(4) See Appendix 2.

As is shown in [1], the case $q = 0$ follows immediately from the case $q > 0$.

On the strength of the above remarks, only the proof of Theorem 1 will be given.

Preliminaries

From Fig. 1

$$\sigma(t) = u(t) - y(t) \tag{16}$$

using Eq. 7

$$\sigma(t) = u(t) - z(t) - \int_0^t g(t-\tau)\,\xi(\tau)\,d\tau\,, \quad t \geq 0 \tag{17}$$

Let $\sigma(t)$ be a solution of Eq. (17) with an arbitrary fixed function $\phi(\sigma)$ contained in the sector $[0, k]$.

Then $\xi(t) = \phi[\sigma(t)]$ is a fixed function of time. Let

$$\xi_T(t) = \begin{cases} \xi(t) & \text{for } 0 \leq t \leq T \\ \xi^T(t) & \text{for } t > T \end{cases} \tag{18}$$

where T is an arbitrary fixed positive number and $\xi^T(t)$ is the system trajectory obtained by replacing, for all $t > T$, the nonlinear function $\phi(\sigma)$ by the linear function $h\sigma$ where

$$h = \frac{\xi(T)}{\sigma(T)} \quad \text{and clearly } h \in [0, k] \tag{19}$$

Then $\xi_T(t)$ is continuous at $t = T$ and since, from the Nyquist criterion, the system is asymptotically stable for $\phi(\sigma) = h\,\sigma$,

$$\xi_T(t) \in L_2(0, \infty)$$

Let the Fourier transform of $\xi_T(t)$ be

$$X_T(j\omega) = \int_0^\infty \xi_T(t)\, e^{-j\omega t}\, dt \tag{20}$$

Let

$$\sigma_T(t) = u(t) - z(t) - \int_0^t g(t-\tau)\,\xi_T(t)\,d\tau, \quad t \geq 0 \tag{21}$$

$$= f(t) + \tilde{\sigma}_T(t) \tag{22}$$

$$\text{where } f(t) = u(t) - z(t) \tag{23}$$

Then it is clear that

$$\sigma_T(t) = \begin{cases} \sigma(t) & \text{for } 0 \leq t \leq T \\ \frac{1}{h}\,\xi^T(t) & \text{for } t > T \end{cases} \tag{24}$$

$\sigma_T(t)$ is continuous at $t = T$, and by the same reasoning as before

$\sigma_T(t) \in L_2(0, \infty)$

Referring to Eqs. (16) and (17), it is seen that $\tilde{\sigma}_T(t) \in L_2(0, \infty)$ and that its Fourier transform $\Sigma_T(j\omega)$ is given by

$$\Sigma_T(j\omega) = -G(j\omega)\,X_T(j\omega) \tag{25}$$

Let $\dot{\xi}_T(t) = \frac{d}{dt}\,\xi_T(t)$ and similarly differentiating Eq. (22) term by term

$$\dot{\sigma}_T(t) = \dot{f}(t) + \dot{\tilde{\sigma}}_T(t) \tag{26}$$

From the assumptions, it follows that $\dot{\xi}_T(t)$, $\dot{\sigma}_T(t)$, $\dot{f}(t)$ and $\dot{\tilde{\sigma}}_T(t)$ all belong to $L_2(0, \infty)$

It is seen from Eqs. (18) and (24) that

$$|\xi_T(t)| \leq k\,|\sigma_T(t)| \qquad \text{for all } t \geq 0$$

Our assumptions guarantee that $f(t)$ and $g(t)$ are bounded on $[0, \infty)$ so let

$$f_M = \sup_{t \geq 0} |f(t)| \quad \text{and} \quad g_M = \sup_{t \geq 0} |g(t)|$$

then, from Eq. (21)

$$|\sigma_T(t)| \leq f_M + k g_M \int_0^t |\sigma_T(t)|\,dt, \qquad t \geq 0 \tag{27}$$

From the Gronwall-Bellman inequality [8] it follows that

$$|\sigma_T(t)| \leq a\, e^{kg_m t}, \qquad t \geq 0 \tag{28}$$

where $a > 0$ is independent of T. This inequality will be required in the proof of the theorem.

The following further definitions are made in order to simplify the proof of the theorem:

Let
$$f_1(t) = f(t) + q\dot{f}(t), \qquad F_1(j\omega) = \mathcal{F}[f_1(t)]$$
$$f_2(t) = \dot{f}(t), \qquad F_2(j\omega) = \mathcal{F}[f_2(t)]$$

and

$$\theta_T(t) = \xi_T(t) - \xi(0)\, e^{-\alpha t}, \qquad \alpha > 0,\ t \geq 0. \tag{29}$$

Then $\theta_T(0) = 0$,

$$\boldsymbol{\Theta}_T(j\omega) = X_T(j\omega) - \frac{\xi(0)}{j\omega + \alpha}, \tag{30}$$

and

$$\dot{\theta}_T(t) = \frac{d}{dt}\theta_T(t) = \dot{\xi}_T(t) + \alpha\xi(0)\, e^{-\alpha t} \tag{31}$$

<u>Proof of Theorem 1</u>

Let
$$\begin{aligned} \lambda_T(t) &= \sigma_T(t) + q\dot{\sigma}_T(t) - \frac{1}{k}\theta_T(t) \\ &= \tilde{\sigma}_T(t) + q\dot{\tilde{\sigma}}_T(t) - \frac{1}{k}\theta_T(t) + f_1(t) \end{aligned} \tag{32}$$

and

$$\begin{aligned} \psi_T(t) &= \mu[\dot{\sigma}_T(t) - \frac{1}{k'}\dot{\theta}_T(t)] \\ &= \mu[\dot{\tilde{\sigma}}_T(t) - \frac{1}{k'}\dot{\theta}_T(t)] + \mu f_2(t). \end{aligned} \tag{33}$$

Now $\theta_T(t)$, $\dot{\theta}_T(t)$, $\lambda_T(t)$ and $\psi_T(t)$ are all elements of $L_2(0,\infty)$, so let $\Lambda_T(j\omega)$ and $\Psi_T(j\omega)$ be the Fourier transforms of $\lambda_T(t)$ and $\psi_T(t)$. Then using Eqs. (25) and (30)

$$\begin{aligned} \Lambda_T(j\omega) &= -(1 + j\omega q)G(j\omega)X_T(j\omega) - \frac{1}{k}\boldsymbol{\Theta}_T(j\omega) + F_1(j\omega) \\ &= -\left\{(1 + j\omega q)G(j\omega) + \frac{1}{k}\right\}\boldsymbol{\Theta}_T(j\omega) - (1 + j\omega q)G(j\omega)\frac{\xi(0)}{j\omega+\alpha} + F_1(j\omega) \end{aligned}$$

and

$$\Psi_T(j\omega) = -\mu j\omega\{G(j\omega)X_T(j\omega) + \frac{1}{k'}\,\boldsymbol{\theta}_T(j\omega)\} + \mu F_2(j\omega)$$

$$= -\mu j\omega\{G(j\omega) + \frac{1}{k'}\}\boldsymbol{\theta}_T(j\omega) - \mu j\omega G(j\omega)\frac{\xi(0)}{j\omega+\alpha} + \mu F_2(j\omega).$$

Let

$$\rho(T) = \int_0^{\infty}[\lambda_T(t)\theta_T(t) + \psi_T(t)\dot{\theta}_T(t)]dt. \tag{34}$$

Then using Parseval's equality,

$$\rho(T) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\Lambda_T(j\omega)\overline{\boldsymbol{\theta}_T(j\omega)} + \Psi_T(j\omega)\overline{j\omega\boldsymbol{\theta}_T(j\omega)}]\,d\omega$$

$$+\mu[\dot{f}(0) - \xi(0)g(0)]\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\boldsymbol{\theta}_T(j\omega)}d\omega^{(5)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[-\{(1+j\omega q)G + \frac{1}{k}\}|\boldsymbol{\theta}_T|^2$$

$$- (1+j\omega q)G\frac{\xi(0)}{j\omega+\alpha}\overline{\boldsymbol{\theta}}_T + F_1\overline{\boldsymbol{\theta}}_T$$

$$+ \mu\Big\{F_2 - j\omega(G + \frac{1}{k'})\boldsymbol{\theta}_T - j\omega G\frac{\xi(0)}{j\omega+\alpha}\Big\}\Big\{-j\omega\overline{\boldsymbol{\theta}}_T\Big\}$$

$$+ \mu\dot{f}(0)\overline{\boldsymbol{\theta}}_T - \mu\xi(0)g(0)\overline{\boldsymbol{\theta}}_T\Big]d\omega$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}[H(j\omega)|\boldsymbol{\theta}_T(j\omega)|^2 - F_3(j\omega)\overline{\boldsymbol{\theta}_T(j\omega)}]d\omega$$

where

$$H(j\omega) = (1+j\omega q)G(j\omega) + \frac{1}{k} + \mu\omega^2\{G(j\omega) + \frac{1}{k'}\}$$

and

$$F_3(j\omega) = F_1(j\omega) - \mu\mathcal{F}[\dot{f}(t)] + \mu\xi(0)\mathcal{F}[\dot{g}(t)] - (q+\mu\alpha)\xi(0)G(j\omega)\frac{j\omega}{j\omega+\alpha}.$$

If $F_3(j\omega) = \mathcal{F}[f_3(t)]$, then from our assumptions, $f_3(t)$ is an element of $L_2(0,\infty)$. Now

$$\mathrm{Re}\,H(j\omega) \geq \delta > 0 \qquad \text{for all } \omega \geq 0$$

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(5) $\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\boldsymbol{\theta}_T(j\omega)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\boldsymbol{\theta}_T(j\omega)d\omega = \theta_T(0) = 0$

and since $H(-j\omega) = \overline{H(j\omega)}$, this implies $\mathrm{Re}\,H(j\omega) \geq \delta$ for all ω.

Also $\rho(T)$ is real, so

$$\rho(T) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}[\mathrm{Re}\,H|\boldsymbol{\theta}_T|^2 - \mathrm{Re}(F_3\overline{\boldsymbol{\theta}}_T)]d\omega$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}[\mathrm{Re}H\,|\boldsymbol{\theta}_T|^2 - \tfrac{1}{2}(F_3\overline{\boldsymbol{\theta}}_T + \overline{F}_3\boldsymbol{\theta}_T)]d\omega$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sqrt{\mathrm{Re}H}\,\boldsymbol{\theta}_T - \frac{F_3}{2\sqrt{\mathrm{Re}H}}\right)\left(\sqrt{\mathrm{Re}H}\,\overline{\boldsymbol{\theta}}_T - \frac{\overline{F}_3}{2\sqrt{\mathrm{Re}H}}\right)d\omega$$

$$+\frac{1}{8\pi}\int_{-\infty}^{\infty}\frac{|F_3|^2}{\mathrm{Re}\,H}d\omega$$

$$= \frac{1}{8\pi}\int_{-\infty}^{\infty}\frac{|F_3|^2}{\mathrm{Re}\,H}d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\sqrt{\mathrm{Re}H}\,\boldsymbol{\theta}_T - \frac{F_3}{2\sqrt{\mathrm{Re}H}}\right|^2 d\omega$$

$$\leq \frac{1}{8\pi\delta}\int_{-\infty}^{\infty}|F_3(j\omega)|^2 d\omega = C, \text{ a finite number independent of } T.$$

Substitute Eqs. (32) and (33) in Eq. (34).

$$\int_0^{\infty}[\sigma_T(t) + q\dot{\sigma}_T(t) - \frac{1}{k}\theta_T(t)]\theta_T(t)\,dt$$

$$+ \mu\int_0^{\infty}[\dot{\sigma}_T(t) - \frac{1}{k'}\dot{\theta}_T(t)]\dot{\theta}_T(t)\,dt \leq C\,. \tag{35}$$

It is shown in appendix 1 that this inequality results in the inequality

$$\int_0^{T}[\sigma - \frac{1}{k}\phi(\sigma)]\phi(\sigma)dt + q\int_0^{\sigma(T)}\phi(\sigma)d\sigma \leq C^* \tag{36}$$

where C^* is finite and independent of T.

From this point on, the proof of the theorem will follow closely the proof of the Popov theorem given in [1] and consequently the steps in the proof will not be given in detail. The following lemma will be used [7] :

Lemma 1. If the real valued function f(t) is bounded and uniformly continuous on $[0, \infty)$, the continuous real valued function $G(x) > 0$ for all $x \neq 0$ and $G(0) = 0$ then

$$\int_0^\infty G[f(t)] < \infty \qquad \text{implies} \qquad \lim_{t\to\infty} f(t) = 0.$$

Because of the assumptions on $\phi(\sigma)$, it follows that both integrals in inequality (36) are nonnegative. Hence

$$\int_0^T [\sigma - \frac{1}{k}\phi(\sigma)]\phi(\sigma)dt \leq C^* \tag{37}$$

and

$$\int_0^{\sigma(T)} \phi(\sigma)d\sigma \leq \frac{1}{q}C^* . \tag{38}$$

We now restrict the sector in which the nonlinearity $\phi(\sigma)$ may lie from $[0, k]$ to $[\epsilon, k)$. This additional restriction is eliminated in appendix 2.

It then follows from (38) that $\sigma(t)$ is bounded on $[0, \infty)$ and consequently $\xi(t)$ and $y(t)$ are also bounded on $[0, \infty)$.

Since C^* is independent of T, using inequality (37) we may write $\int_0^\infty G[\sigma(t)]dt \leq C^* < \infty$

where

$$G(\sigma) = [\sigma - \frac{1}{k}\phi(\sigma)]\phi(\sigma)$$

which satisfies the conditions of Lemma 1.

From Eq. (17) it can be shown that $\sigma(t)$ is uniformly continuous on $[0, \infty)$ [6] and hence from Lemma 1 we conclude that for all initial states

$$\lim_{t\to\infty} \sigma(t) = 0.$$

It then follows that $\lim_{t\to\infty} \xi(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$ which completes the proof of the theorem.

Examples

1. A particular case given by Aizeman and Gantmacher [1].

Consider the system with transfer function

$$G(s) = \frac{s^2 - b}{(s^2 + 1)(s + c)} \qquad \begin{matrix} b > 0, \; c > 0 \\ b < c^2 \end{matrix} \quad . \tag{39}$$

Conditions for stability in the limit are satisfied [1] and the system is stable for linear gains in the sector $(0, c/b)$.

The Popov sector is found to be $[\epsilon, 1/c]$ where $\epsilon > 0$ is arbitrarily small and $1/c < c/b$. From Eq. (39)

$$G(j\omega) = \frac{\omega^2 + b}{(\omega^2 - 1)(j\omega + c)} = \frac{(\omega^2 + b)(c - j\omega)}{(\omega^2 - 1)(\omega^2 + c^2)} .$$

Using Theorem 2, let

$$X = \operatorname{Re} G = \frac{c(\omega^2 + b)}{(\omega^2 - 1)(\omega^2 + c^2)}$$

$$Y = \frac{\omega \operatorname{Im} G}{1 + \mu\omega^2} = - \frac{\omega^2(\omega^2 + b)}{(\omega^2 - 1)(\omega^2 + c^2)(1 + \mu\omega^2)} .$$

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(6) See Reference [7]. In the special formulation of Eq. (5), uniform continuity follows immediately from the boundedness of $\dot{\sigma}(t)$.

Then

$$X - qY = \frac{(u + b)(c + c\mu u + qu)}{(u - 1)(u + c^2)(1 + \mu u)} \qquad \text{where } u = \omega^2.$$

Inequality (10) can be satisfied only for values of q for which $(c + c\mu u + qu)$ is divisible by $(u - 1)$, that is, for $q = -c(1 + \mu)$. So for this value of q

$$X - qY = - \frac{c(u + b)}{(1 + \mu u)(u + c^2)}.$$

Choosing $\mu \geq \frac{1}{b}$

$$X - qY \geq \frac{-bc}{u + c^2} = - \frac{b}{c} + \frac{bu}{c(u + c^2)}.$$

Now $\frac{bu}{c(u + c^2)} \geq 0$ for all $u \geq 0$ and hence

$$X - qY + \frac{1}{k} > 0 \qquad \text{for } \frac{1}{k} > \frac{b}{c}, \text{ that is, for } k < \frac{c}{b}.$$

Thus the system is stable for nonlinearities with slope restriction $[0, c/b - \epsilon]$ in the sector $[\epsilon, c/b - \epsilon]$. If $b = 1$ and $c = 10$ then this sector is 100 times greater than the Popov sector. For the commonly encountered saturation-type nonlinearity, this increase is very significant.

2. Consider the system with transfer function

$$G(s) = \frac{40}{s(s + 1)(s^2 + 0.8s + 16)}.$$

The frequency response $G(j\omega)$ is plotted for $\omega \geq 0$ in Fig. 2. This type of frequency response is common in compensated feedback servo systems. The system is stable for linear gains in the sector (0, 1.76).

In Fig. 3 the modified Nyquist plot ($\mu = 0$) and the Popov line [1] are shown. The Popov sector is found to be $[\epsilon, 0.65]$.

Using Theorem 2, an X, Y plot is made for several values of μ. The plot for $\mu = 1$ is shown in Fig. 3. It is found that for increasing values of μ, the intersection of the tangent line with the negative X axis approaches the point -0.70.

Hence the system is stable for all nonlinearities with slope restriction $[0, 1.43]$ in the sector $[\epsilon, 1.43]$, $\epsilon > 0$, arbitrarily small.

This shows a significant increase over the Popov sector, but not, in this case, up to the linear limit.

Conclusion

It has been shown that in certain cases an improvement over the Popov criterion can be obtained for monotonic nonlinearities with slope restriction $[0, k]$. An example of the use of inequality (9) has not been obtained and Yakubovitch [3] also did not give such an example. Thus the value of this inequality remains questionable.

Many others have tried without success to improve on Popov's criterion without further restricting the nonlinear function. Brockett [9] has proposed a new stability inequality, but it has been shown that it is, at best, equivalent to the Popov criterion.

Theorem 1 remains to be proved for negative q and for the particular cases. Also the connection between these new inequalities and the existence of Lyapunov functions similar to those used by Yakubovitch [3] will be an interesting topic for future research.

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Appendix 1

Substituting for $\theta_T(t)$ in Eq. (35)

$$\int_0^\infty [\sigma_T(t) + q\dot{\sigma}_T(t) - \frac{1}{k}\xi_T(t) + \frac{\xi(0)}{k}e^{-\alpha t}][\xi_T(t) - \xi(0)e^{-\alpha t}]dt$$

$$+ \mu\int_0^\infty [\dot{\sigma}_T(t) - \frac{1}{k'}\dot{\xi}_T(t) - \frac{\alpha\xi(0)}{k'}e^{-\alpha t}][\dot{\xi}_T(t) + \alpha\xi(0)e^{-\alpha t}]dt \leq C$$

that is

$$\int_0^T [\sigma_T(t) - \frac{1}{k}\xi_T(t)]\xi_T(t)dt + \int_T^\infty [\sigma_T(t) - \frac{1}{k}\xi_T(t)]\xi_T(t)dt$$

$$+ q\int_0^T \xi_T(t)\dot{\sigma}_T(t)dt + q\int_T^\infty \xi_T(t)\dot{\sigma}_T(t)dt$$

$$+ \mu\int_0^T [\dot{\sigma}_T(t) - \frac{1}{k'}\dot{\xi}_T(t)]\dot{\xi}_T(t)dt + \mu\int_T^\infty [\dot{\sigma}_T(t) - \frac{1}{k'}\dot{\xi}_T(t)]\dot{\xi}_T(t)dt$$

$$- \frac{\xi(0)}{R}\int_0^\infty e^{-\alpha t}[2\xi_T(t) - \xi(0)e^{-\alpha t}]dt + \xi(0)\int_0^\infty e^{-\alpha t}[\sigma_T(t) + q\dot{\sigma}_T(t)]dt$$

$$+ \frac{\mu\alpha\xi(0)}{k'}\int_0^\infty e^{-\alpha t}[2\dot{\xi}_T(t) + \alpha\xi(0)e^{-\alpha t}]dt - \mu\alpha\xi(0)\int_0^\infty e^{-\alpha t}\dot{\sigma}_T(t)dt$$

$$\leq C$$

that is

$$\int_0^T [\sigma - \tfrac{1}{k}\phi(\sigma)]\phi(\sigma)dt + \int_T^\infty \left(\tfrac{1}{h} - \tfrac{1}{k}\right)\xi^T(t)^2 dt$$

$$+ q\int_0^T \phi(\sigma)\dot{\sigma}(t)dt + q\int_0^\infty \tfrac{1}{h}\xi^T(t)^2 dt$$

$$+ \mu\int_0^T\left(1 - \tfrac{1}{k'}\tfrac{d\phi}{d\sigma}\right)\tfrac{d\phi}{d\sigma}\,\dot{\sigma}(t)^2 dt + \mu\int_0^\infty\left(\tfrac{1}{h} - \tfrac{1}{k'}\right)\dot{\xi}^T(t)^2 dt$$

$$+ \frac{\xi(0)^2}{2\alpha k} + \frac{\mu\alpha\xi(0)^2}{2k'} - \frac{2\xi(0)}{k}\int_0^\infty e^{-\alpha t}\xi_T(t)dt$$

$$+ \xi(0)\int_0^\infty e^{-\alpha t}\sigma_T(t)dt + (q - \mu\alpha)\xi(0)\int_0^\infty e^{-\alpha t}\dot{\sigma}_T(t)\,dt$$

$$+ \frac{2\mu\alpha\xi(0)}{k'}\int_0^\infty e^{-\alpha t}\dot{\xi}_T(t)dt \le C.$$

The second and fourth through eighth terms in this equality are non-negative and hence may be discarded. So adding to both sides the nonnegative quantity

$\int_0^{\sigma(0)} \phi(\sigma)d\sigma$ and putting $C + \int_0^{\sigma(0)} \phi(\sigma)d\sigma = C_1$

we obtain

$$\int_0^T [\sigma - \frac{1}{k}\phi(\sigma)]\phi(\sigma)dt + q\int_0^{\sigma(T)} \phi(\sigma)d\sigma$$

$$\leq C_1 + \frac{2|\xi(0)|}{k}\int_0^\infty e^{-\alpha t}|\xi_T(t)|dt + |\xi(0)|\int_0^\infty e^{-\alpha t}|\sigma_T(t)|dt$$

$$- (q - \mu\alpha)\xi(0)[\sigma(0) + \alpha\int_0^\infty e^{-\alpha t}\sigma_T(t)dt]$$

$$- \frac{2\mu\alpha\xi(0)}{k'}[\xi(0) + \alpha\int_0^\infty e^{-\alpha t}\xi_T(t)dt]$$

$$\leq C_1 + |q - \mu\alpha|\xi(0)\sigma(0)$$

$$+ |\xi(0)|\left(3 + \alpha|q - \mu\alpha| + \frac{2\mu\alpha^2 k}{k'}\right)\int_0^\infty e^{-\alpha t}|\sigma_T(t)|dt$$

$$= C^*.$$

Now $\alpha > 0$ was arbitrary and from Eq. (28)

$$|\sigma_T(t)| \leq a\, e^{kg_m t}, \qquad t \geq 0.$$

Therefore choosing $\infty > \alpha > kg_m$, the last integral is finite and hence we obtain

$$\int_0^T [\sigma - \frac{1}{k}\phi(\sigma)]\phi(\sigma)dt + q\int_0^{\sigma(T)} \phi(\sigma)d\sigma \leq C^* < \infty$$

where C^* is independent of T.

Appendix 2

In the proof of Theorem 1 the sector $[0, k]$ was changed to the sector $[\epsilon, k)$. In order to eliminate this additional restriction, make the transformation

$$\phi_1(\sigma) = \phi(\sigma) + \epsilon\sigma \qquad \text{for } \epsilon > 0 \text{ small.}$$

Then the transfer function of the transformed system becomes

$$G_\epsilon(j\omega) = \frac{G(j\omega)}{1 - \epsilon G(j\omega)} .$$

For $\epsilon > 0$ small, the functions $G_\epsilon(j\omega)$ and $G(j\omega)$ are not very different and hence

$$\mathrm{Re}\left[1 + \frac{j\omega q}{1 + \mu\omega^2}\right] G(j\omega) + \frac{1}{k} \geq \delta > 0$$

implies

$$\mathrm{Re}\left[1 + \frac{j\omega q}{1 + \mu\omega^2}\right] G_\epsilon(j\omega) + \frac{1}{k} \geq \delta_o > 0^{(7)}. \tag{40}$$

In (40) it is possible to change k to $k + 2\epsilon$ by reducing δ_o sufficiently. Now for $\epsilon > 0$ sufficiently small, the transformed system will be stable for $\phi_1(\sigma) = 0$. Hence the transformed system is also

(7) As in [1], it can be shown that the difference between the left hand sides of these inequalities can be made arbitrarily small (uniformly in ω) by choosing a sufficiently small $\epsilon > 0$.

a principal case and from Theorem 1 it is stable for $\phi_1(\sigma)$ with slope restriction $[0, k + 2\epsilon]$ in the sector $[\epsilon, k + 2\epsilon)$. This implies stability for $\phi_1(\sigma)$ in the sector $[\epsilon, k + \epsilon]$ and hence that the original system is stable for $\phi(\sigma)$ with slope restriction $[-\epsilon, k + \epsilon]$ in the sector $[0, k]$. Thus the original system is stable with the simple slope restriction $[0, k]$ which was to be proved.

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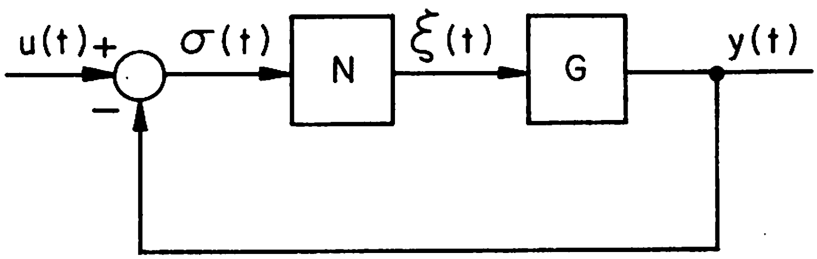

Fig. 1. Nonlinear feedback system.

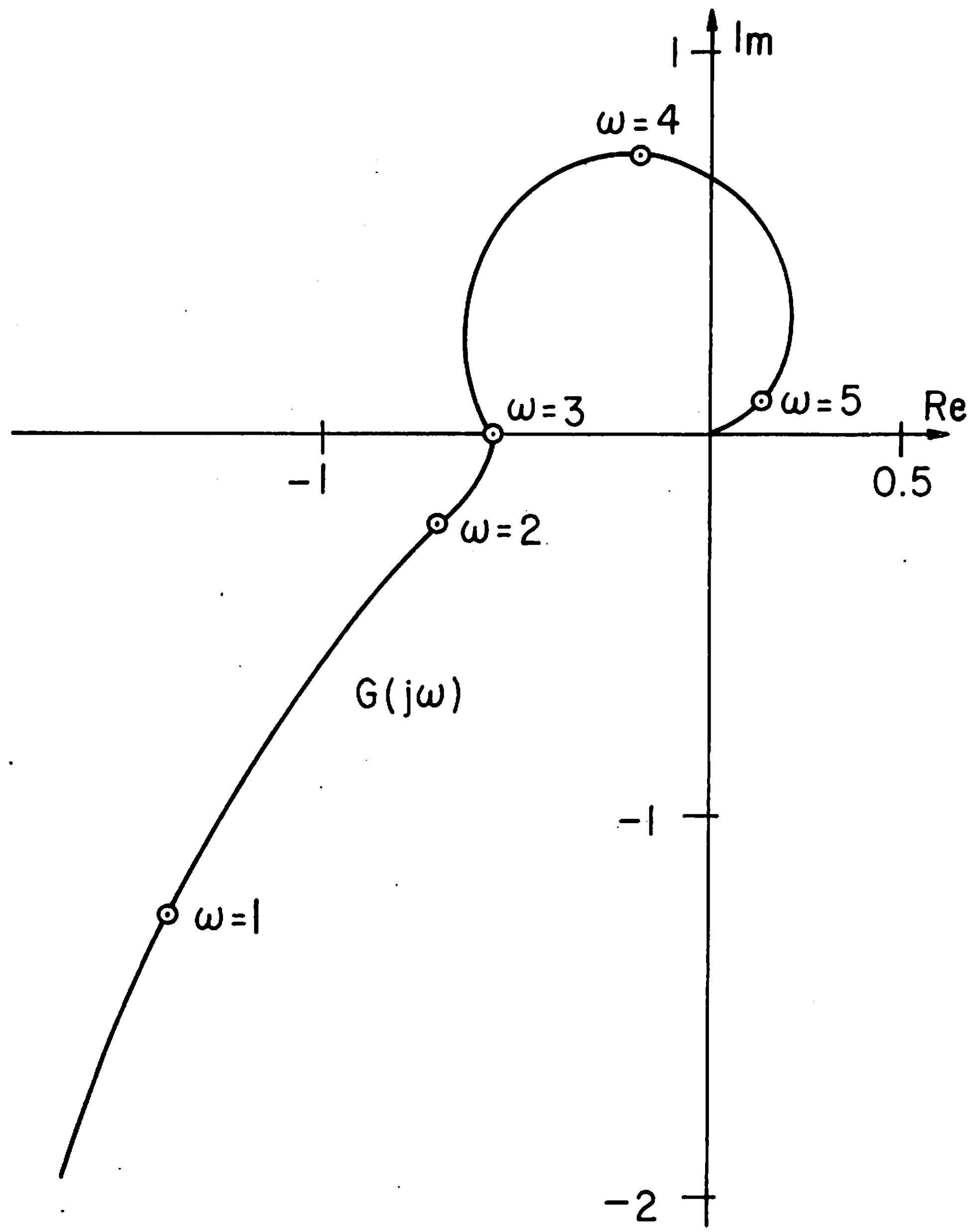

Fig. 2. Frequency response for example 2.

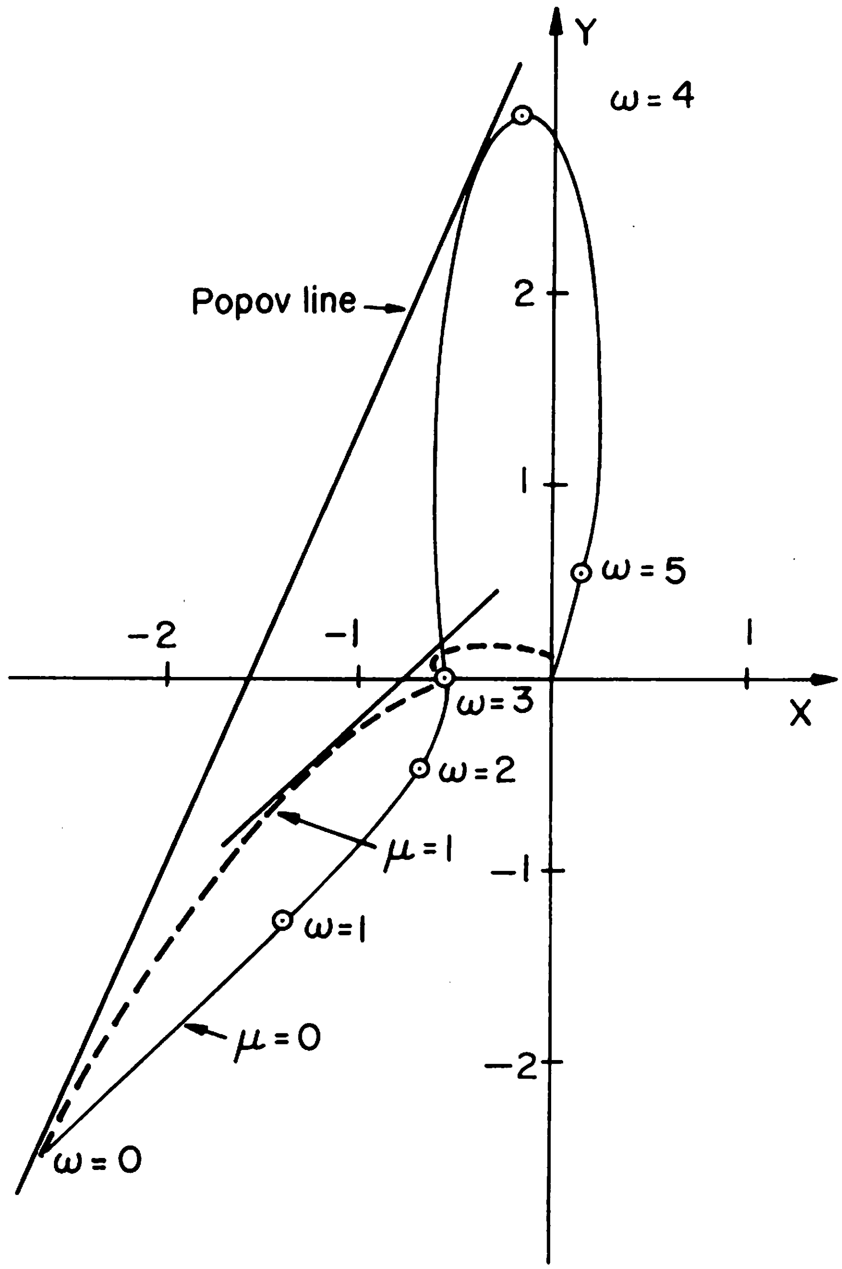

Fig. 3. Frequency plots for $\mu = 0$ and $\mu = 1$.