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A NOTE ON AIZERMAN'S CONJECTURE

by

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The following system of three differential equations will be considered:

$$\begin{aligned} \dot{x} &= -cx + y - \varphi(x) \\ \dot{y} &= -x + z \\ \dot{z} &= -cx + b\varphi(x) \end{aligned} \tag{1}$$

where $b > 0$, $c > 0$, $b < c^2$ and the single-valued, piecewise-continuous, nonlinear function $\varphi(x)$ satisfies the conditions

$$\phi(0) = 0 \text{ and } 0 < \frac{\varphi(x)}{x} < k \text{ for all } x \neq 0 ,$$

where k is a finite number. This will be referred to as a nonlinearity in the sector $(0, k)$. In the language of feedback control systems, the transfer function of the linear plant G relating its input $\varphi[x(t)]$ to its output $-x(t)$ is

$$G(s) = \frac{s^2 - b}{(s^2 + 1)(s + c)} . \tag{2}$$

When $\varphi(x)$ is replaced by the linear function hx where h is a constant, we will refer to (1) as the linear system with gain h. If a

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system is stable for linear gains h in the interval (k_1, k_2) , then the open sector bounded by the lines $\varphi(x) = k_1 x$, $\varphi(x) = k_2 x$ will be called the Hurwitz sector for that system.

In 1949, Aizerman [1] made the conjecture that a regulator system with a single nonlinear element will be asymptotically stable in the large for all nonlinearities contained in the Hurwitz sector.

Equation (1) could describe a particular third-order regulator system and the validity of Aizerman's conjecture will be investigated for this system.

It has been shown that this conjecture is true for second order systems with the exception of particular cases when the nonlinear function approaches one side of the Hurwitz sector asymptotically. Bergen and Williams [2] verified the conjecture for certain third order systems with no zeros. However in 1958, Pliss [3] , using the example of system (1) gave sufficient conditions on $\varphi(x)$, lying within the Hurwitz sector, for the system to admit a periodic solution. Thus Aizerman's conjecture was disproved in general.

Aizerman and Gantmacher [4] used system (1) to illustrate the V. M. Popov theorem. The Hurwitz sector for this example is found to be the sector $(0, c/b)$. Using the Popov theorem, absolute stability is guaranteed for all nonlinearities in the sector $[\epsilon , 1/c]$ where $\epsilon > 0$ is arbitrarily small. It is stated in [4] that this example serves to disprove Aizerman's conjecture since $1/c < c/b$. However, since Popov's theorem gives only sufficient and not necessary conditions for stability, the statement is not precise, and only by exhibiting the presence of a periodic solution can Aizerman's conjecture be disproved.

The work of Pliss [3] is often quoted but is not well known. His monograph is not readily available outside the U.S.S.R. and has not been translated from the original Russian. Also it does not contain any numerical examples. This correspondence serves to remove any doubts on the validity of Aizerman's conjecture by presenting a specific numerical counter-example.

Example

In (1) let $b = 1/2$ and $c = 1$. Then the Hurwitz sector is the sector $(0, 2)$ and the Popov sector is the sector $[\epsilon, 1]$. The sufficient conditions given by Pliss for the existence of a periodic solution are very restrictive, so much so that it was not possible to simulate such a nonlinear function on an analog computer. However, it was found experimentally that these conditions are far from necessary. System (1) and the nonlinear function shown in Fig. 1 were simulated on an analog computer and the projections of some system trajectories on the plane $z = 0$ are shown in Figs. 2 and 3. A stable limit cycle is seen to exist, thus contradicting Aizerman's conjecture of asymptotic stability in the large.

The nonlinear function in Fig. 1 has the basic shape required by Pliss' theorem but does not satisfy many of his sufficient conditions. The important factors in the shape of this function were found to be (a) the slope of CD, (b) the negative slope of BC and DE, and (c) the magnitude of $\varphi(x)$ in the regions AB and EF. It was noticed that either an increase in (a) or (b), or a decrease in (c) made the existence of a limit cycle more likely. In the example given, the slope of CD was 1.6 , the slopes of BC and DE were approximately -20 and the value of $\varphi(x)$ at the point E(x = 25) was 2.0.

For this example the state space can be divided into two unbounded regions; from the first region all trajectories are asymptotically stable at the origin and from the complement of this region, all trajectories are asymptotic to the stable limit cycle.

CONCLUSION

The contribution of this note has been to present a numerical counter-example to Aizerman's conjecture. Such an example for continuous systems has not previously appeared in the literature even though the work of Pliss has been known for many years. Since Aizerman's conjecture is not true in general, the approach to the problem of absolute stability must be that of Bergen and Williams, that is, to single out those classes of systems for which the conjecture can be verified.

ACKNOWLEDGMENT

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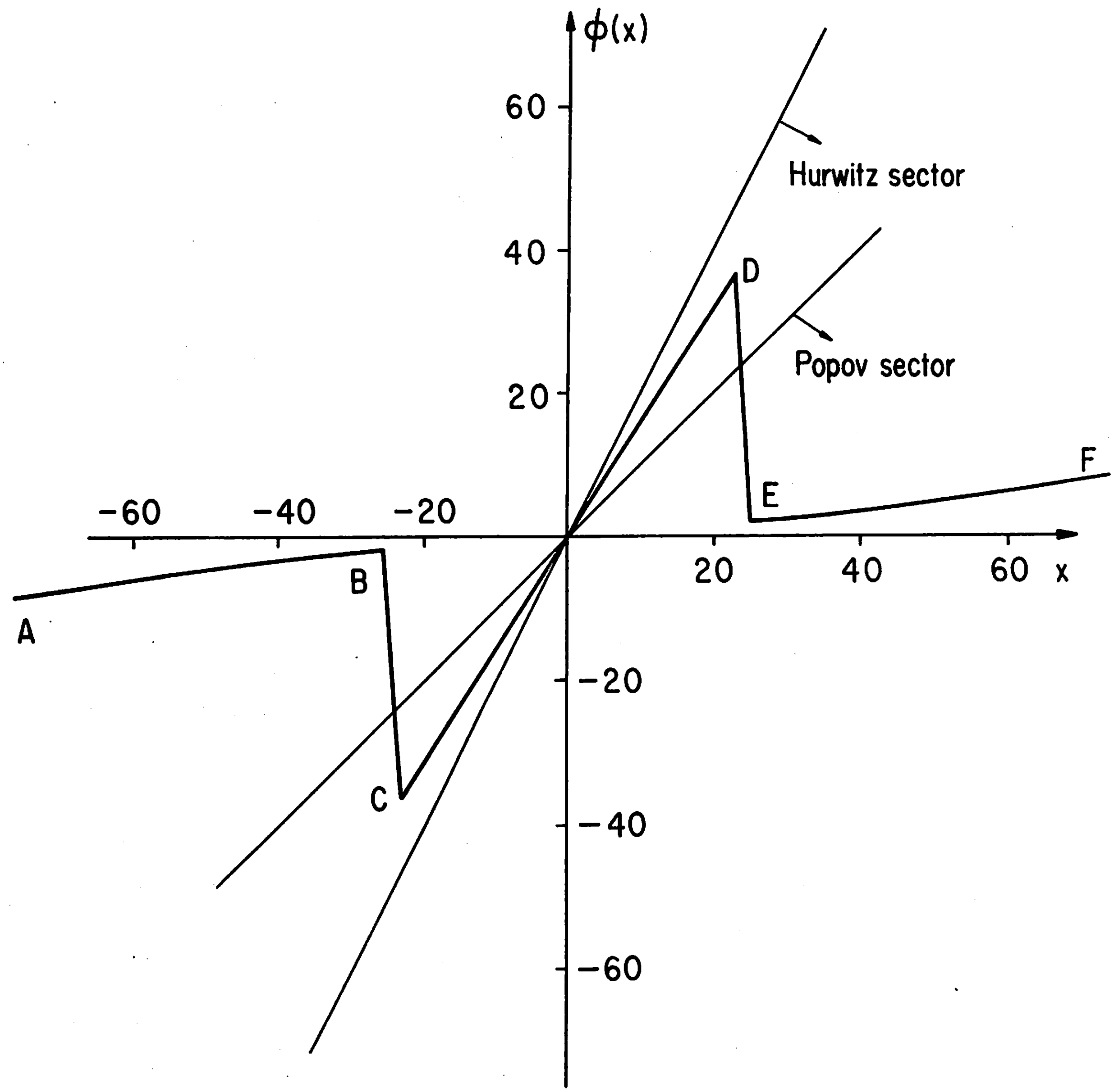

Fig. 1. Nonlinear function.

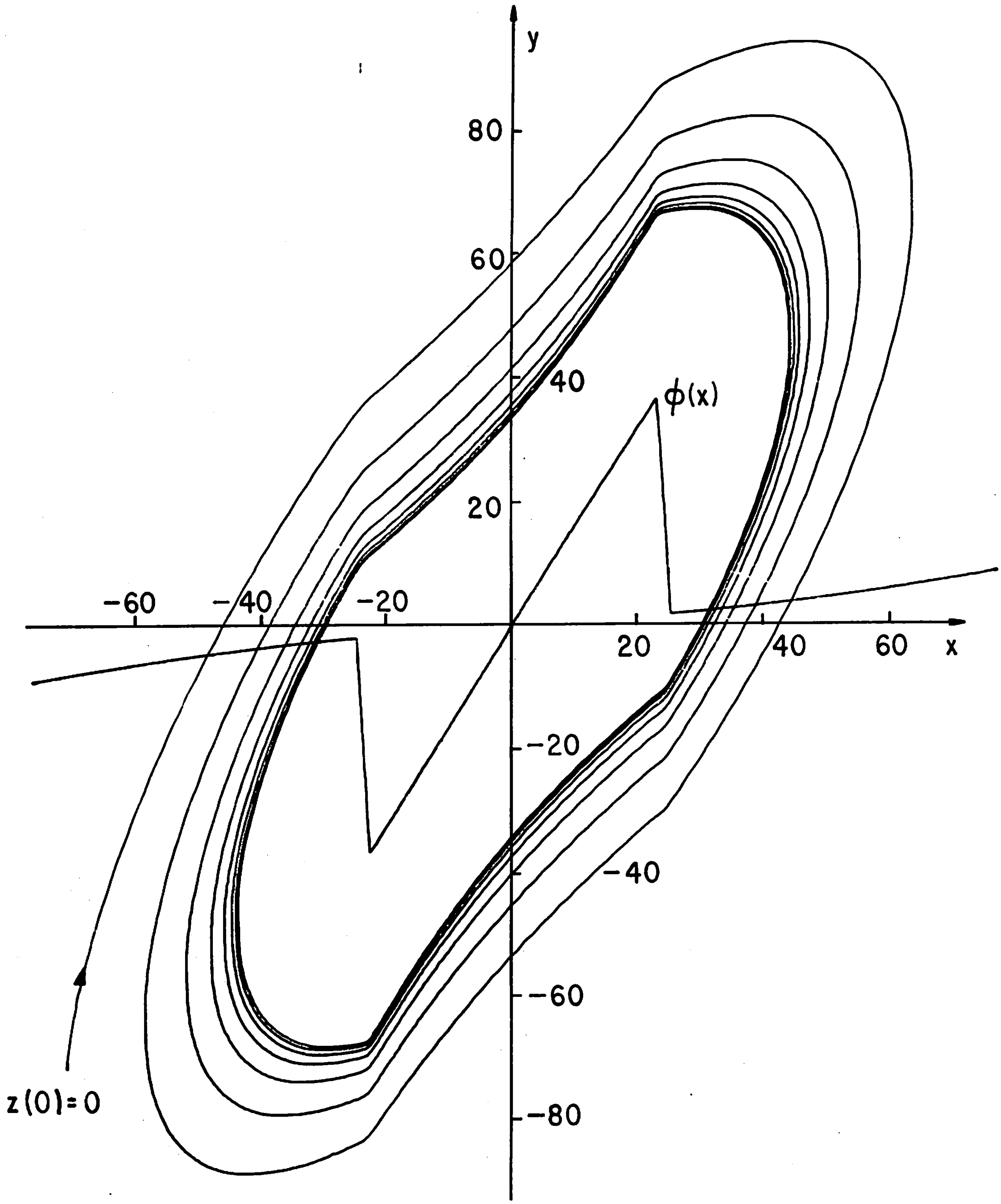

Fig. 2. Projection on the plane $z = 0$ of a trajectory approaching the stable limit cycle.

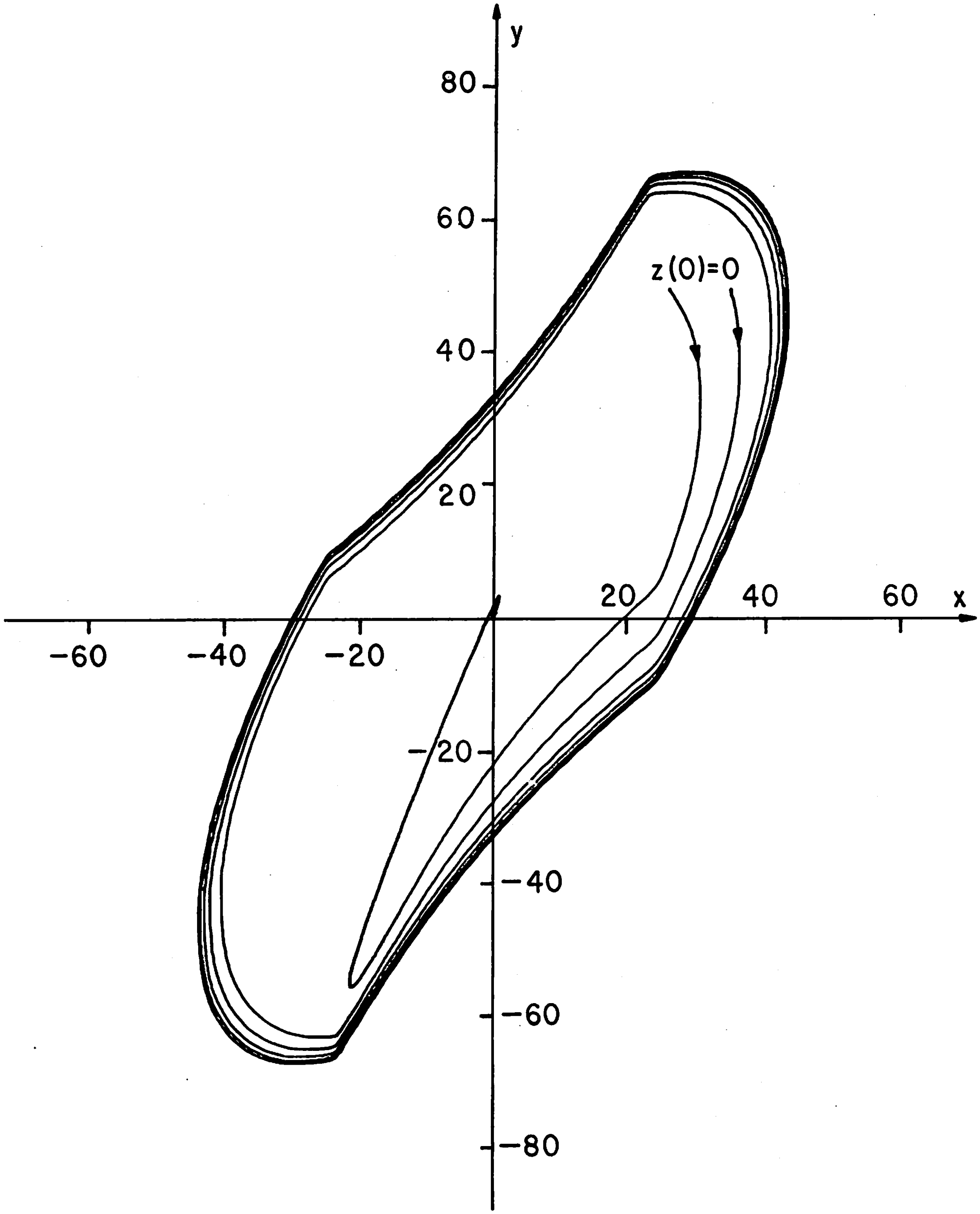

Fig. 3. Projection on the plane z = 0 of the stable and unstable trajectories.