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THEORY OF OPTIMUM DISCRETE TIME SYSTEMS

by

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Introduction

In the present paper we shall consider some optimization problems for systems described by difference equations. This paper is divided into three sections: the first by J. B. Rosen, the second by B. W. Jordan and E. Polak and the third by H. Halkin. In each of these sections a different problem is defined and a different set of results is obtained. We have decided to stress the fundamental similarities between our particular problems and to exhibit their real differences in a context which is free of terminological and notational ambiguities.

In each section of this paper the state vector will be an element x of a Euclidean space E^n; the control vector will be an element u of a Euclidean space E^r and the time will assume the discrete values $0,1,2,\dots,k$. The evolution of the system will be described by the difference equations

$$x_{i+1} - x_i = f_i(x_i, u_i) \qquad (0.1)$$

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The problem will be to find sequences $x_o, x_1, \ldots, x_k$ and $u_o, u_1, \ldots, u_{k-1}$ satisfying the relation (0.1) (and possibly some other given constraints) and minimizing (or maximizing) a given function of the variables $x_o, x_1, \ldots, x_k$, $u_o, u_1, \ldots, u_{k-1}$. Corresponding to the different assumptions and constraints which we shall give in each of the three sections of this paper we shall obtain three independent sets of results.

1. State-Constrained Linear System

In this section we consider a discrete problem in which both the control vector and state vector are subject to constraints at each discrete time. The evolution of the system is assumed to be given by means of a linear system of difference equations, and it is desired to minimize a convex function of both the control and state vectors. The approach taken here is closely related to the Kuhn-Tucker theory which shows the equivalence of a constrained minimization problem and the saddle point of a Lagrangian function [1.3]. A computational method based on this approach has previously been described [1.4].

To be specific we wish to determine optimal controls $\hat{u}_i$, $i = 0,1,\ldots k-1$, and the corresponding optimal state vectors $\hat{x}_i$, $i = 0,1\ldots k$, so that

$$\sum_{i=0}^{k-1} \sigma_i(x_i, u_i) = \min, \tag{1.1}$$

where the x_i must satisfy

$$x_{i+1} - x_i = f_i(x_i, u_i), \qquad i = 0,1,\ldots k-1 \tag{1.2}$$

and x_o and the u_i must be selected so that

$$u_i \in U_i, \qquad i = 0,1,\ldots k-1 \qquad (1.3)$$

and

$$x_i \in X_i, \qquad i = 0,1,\ldots k. \qquad (1.4)$$

We will assume that the sets U_i are convex subsets of E^r, and the sets X_i are convex subsets of E^n. We also assume that $\sigma_i(x,u)$ is convex and $f_i(x,u)$ is linear on $X_i \times U_i$, for $i = 0,1,\ldots k-1$.

We can now give the main result of this section as a discrete maximum principle for an optimal solution to a state-constrained problem.

Theorem 1

A necessary and sufficient condition that $\hat{x}_i$, $i = 0,1,\ldots k$, and $\hat{u}_i$, $i = 0,1,\ldots k-1$, are optimal (satisfy (1.1) for all vectors x_i and u_i which satisfy (1.2), (1.3) and (1.4)) is that there exist nonzero vectors $p_i \in E^n$, $i = 0,1,\ldots k$, such that

$$H_i(\hat{x}_i,\hat{u}_i,p_{i+1},p_i) \geq \underset{\substack{x_i \in X_i \\ u_i \in U_i}}{} H_i(x_i,u_i,p_{i+1},p_i), \quad i = 0,1,\ldots k-1 \qquad (1.5)$$

and

$$p_k'(x_k-\hat{x}_k) \underset{x_k \in X_k}{\geq} 0 \qquad (1.6)$$

where the functions H_i are defined by

$$H_i(x,u,p,q) = p'f_i(x,u) + (p-q)'\, x - \sigma_i(x,u) \qquad (1.7)$$

and $p_o = 0$.

The proof requires the following

Lemma

Let Z be a convex subset of E^s, and $\rho(z)$ a function from Z to E^1 which is convex on Z. Let $w(z)$ be a linear function from Z to E^ℓ. We will denote by $W \subset E^s$ the linear manifold determined by $w(z) = 0$. A necessary and sufficient condition that $z^* \in Z \cap W$ satisfies

$$\rho(z^*) = \min_{z \in Z \cap W} \rho(z) \qquad (1.8)$$

is that there exists a vector $\lambda \in E^\ell$ such that

$$\Lambda(z^*) \underset{z \in Z}{\leq} \Lambda(z) \qquad (1.9)$$

where the Lagrangian function $\Lambda(z)$ is given by

$$\Lambda(z) = \rho(z) + \lambda' w(z). \qquad (1.10)$$

The proof of the lemma is similar to that given by Karlin [1.2], for the case where Z is the nonnegative orthant in E^s. To show necessity for the more general case considered here we define the subset $R \subset E^{\ell+1}$ by

$$R = \left\{ \begin{pmatrix} y_o \\ y \end{pmatrix} \middle| \begin{matrix} y_o \geq \rho(z), & z \in Z \\ y = w(z), & z \in Z \end{matrix} \right\} \qquad (1.11)$$

Because of the convexity of $\rho(z)$, the linearity of $w(z)$ and the fact that Z is convex, it is not difficult to show that R is a convex set. Because of (1.8) and the convexity of R, there exists a supporting hyperplane for R at the point $y_o = \rho(z^*)$, $y = 0$, with a normal vector $\frac{1}{\lambda}$ directed into R. That is,

$$\rho(z^*) \leq y_o + \lambda' y \ , \ \begin{pmatrix} y_o \\ y \end{pmatrix} \in R. \qquad (1.12)$$

Choosing $y_o = \rho(z)$ and $y = w(z)$, we get (1.9) from (1.11) and (1.10).

The sufficiency of (1.10) follows immediately from the observation that $w(z) = 0$ for $z \in Z \cap W$, so that (1.9) implies (1.8).

In order to prove the theorem we make use of the lemma by defining the vector $z \in E^s$ in terms of the vectors x_i and u_i. We let

$$z' = \left(x_o', x_1', \ldots x_k', u_o', u_1' \ldots u_{k-1}'\right) \qquad (1.13)$$

so that $s = n(k+1) + rk$. The set $Z \subset E^s$ is taken as the direct product of the sets X_i, $i = 0,1,\ldots k$, and U_i, $i = 0,1\ldots k-1$. The set Z is therefore convex. The linear function $w(z)$ from Z to E^ℓ is given by the linear recursion relations (1.2), so that $w(z) = 0$ is given by

$$x_{i+1} - x_i - f_i(x_i,u_i) = 0, \qquad i = 0,1,\ldots k-1 \qquad (1.14)$$

and $\ell = kn$. While not relevant to this theorem, it is worth noting that because of the structure of (1.2), the (constant) Jacobian matrix of $w(z)$ is of rank ℓ (full row rank).

The function $\rho(z)$ to be minimized is now given by

$$\rho(z) = \sum_{i=0}^{k-1} \sigma(x_i,u_i). \qquad (1.15)$$

To complete the association we let

$$\lambda' = \left(p_1', p_2', \ldots . p_k'\right) \qquad (1.16)$$

and observe that (1.9) is now equivalent to

$$\sum_{i=0}^{k-1} \left\{\sigma_i(\hat{x}_i,\hat{u}_i) + p'_{i+1}\,[\hat{x}_{i+1} - \hat{x}_i - f_i(\hat{x}_i,\hat{u}_i)]\right\}$$

$$\underset{\substack{x_i \,\epsilon\, X_i \\ u_i \,\epsilon\, U_i}}{\leq} \quad \sum_{i=0}^{k-1} \left\{\sigma_i(x_i,u_i) + p'_{i+1}\,[x_{i+1} - x_i - f_i(x_i,u_i)]\right\} . \tag{1.17}$$

To complete the necessity proof we assume the $\hat{x}_i$ and $\hat{u}_i$ are optimal (satisfy (1.1) - (1.4)) and note that this implies that z* satisfies (1.8). By the lemma, (1.9) holds, and therefore (1.17) is satisfied. Combining the terms involving x_i, u_i, we can rewrite (1.17) as

$$\sum_{i=0}^{k-1} H_i(\hat{x}_i,\hat{u}_i,p_{i+1},p_i) \quad \underset{\substack{x_i \,\epsilon\, X_i \\ u_i \,\epsilon\, U}}{\geq} \quad \sum_{i=0}^{k-1} H_i(x_i,u_i,p_{i+1},p_i) \tag{1.18}$$

where the H_i are defined by (1.7), and $p_o = 0$. But (1.18) requires that (1.5) and (1.6) hold since if, say, (1.5) were not satisfied for $i = \beta$, we could get a contradiction to (1.18) by setting $x_i = \hat{x}_i$ and $u_i = \hat{u}_i$ for $i \neq \beta$.

To show sufficiency, we see that (1.5) and (1.6) imply (1.17) and therefore (1.9). By the lemma, (1.8) holds, which insures that the $\hat{x}_i$ and the $\hat{u}_i$ are optimal.

Corollary

Let $\sigma_i(x,u)$ be in C^1 on $X_i \times U_i$. Then for each $i = 0,1,\ldots k-1$, either $\hat{x}_i$ is on the boundary of X_i, or p_i and p_{i+1} satisfy the adjoint equation

$$p_{i+1} - p_i = - f'_{ix}(\hat{x}_i,\hat{u}_i)\; p_{i+1} + \sigma'_{ix}(\hat{x}_i,\hat{u}_i) \qquad (1.19)$$

and either $\hat{x}_k$ is on the boundary of X_k, or $p_k = 0$.

Similarly, for each $i = 0,1,\ldots k-1$, either $\hat{u}_i$ is on the boundary of U_i, or

$$f'_{iu}(\hat{x}_i,\hat{u}_i)\; p_{i+1} - \sigma'_{iu}(\hat{x}_i,\hat{u}_i) = 0 \qquad (1.20)$$

Proof:

The function $H_i(x,u,p,q)$ is concave on $X_i \times U$. Therefore if it attains its maximum at an interior point $\hat{x}_i$ of X_i, the gradient with respect to x of H_i must vanish at $(\hat{x}_i,\hat{u}_i)$. But this is just the requirement (1.19). The corresponding situation for $\hat{x}_k$ follows from (1.6). Similarly, if H_i attains its maximum at an interior point $\hat{u}_i$ of U_i, the gradient with respect to u of H_i must vanish. This requires that (1.20) is satisfied.

A final remark concerning this corollary is in order here. For a continuous problem, the case when $\hat{u}_1$ is interior to U_1 and (1.20) applies, is usually described as a singular arc [1.1]. By analogy we might consider solutions which are not state constrained, and therefore satisfy the adjoint equation (1.19), as singular arcs in the state space.

2. An Optimality Condition for Nonlinear Discrete Time Systems

Consider a system whose state difference equation is the time invariant form of (0. 1), i. e.,

$$x_{i+1} - x_i = f(x_i, u_i),\ i=0, 1, 2, \ldots, k-1 \tag{2.1}$$

In expanded form the vector x_i will be written $(x_i^1, x_i^2, \ldots, x_i^n)$.

The following assumptions will be made:

1) For $i=1, 2, \ldots, k-1$, $u_i \in \Omega \subset E^r$, where Ω is a finite union of disjoint closed and bounded convex

2) The function $f \in C^1$ on $E^n \times \Omega$.

3) The initial state $x_0 = s_o$, where s_o is a given state.

4) The terminal state $x_k \in S \subset E^n$, where S is a subset which will be specified in the conditions of the theorem to follow.

Statement of the Problem:

Find a sequence $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{k-1}$ and a sequence $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{k-1}$ such that

1) $\hat{x}_0 = s_o$;

2) $\hat{x}_{i+1} - \hat{x}_i = f(\hat{x}_i, \hat{u}_i)$, $i=0, 1, \ldots, k-1$;

3) $\hat{u}_i \in \Omega$, $i=0, 1, \ldots, k-1$;

4) $\hat{x}_k \in S$;

5) For any other two sequences $u_0, u_1, \ldots, u_{k-1}$ and $x_0, x_1, \ldots, x_k$ satisfying the conditions 1) to 4) above,

$x_k^1 \le \hat{x}_k^1$.

Definition:

Any two sequences $\hat{u}_0, \ldots, \hat{u}_{k-1}, \hat{x}_0, \ldots, \hat{x}_k$ satisfying the conditions 1) to 5) above will be called optimal.

Remark:

Although this problem formulation deals explicitly only with time invariant systems, time varying systems may be cast into such a form by considering time to be an additional state variable as will later be shown in example 2. Problems in which one is required to maximize a function depending on all or several components of the terminal state x_k and/or on the control sequence $u_0, \ldots, u_{k-1}$ can also be recast into an equivalent standard form in which the first component of x_k only is maximized, see example 1 and Reference 2.2

Theorem:

If $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{k-1}$, and $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_k$ are optimal sequences and if $\Lambda(\hat{u}_i)$ is the set of all vectors δu_i such that $\hat{u}_i + \epsilon\, \delta u_i \in \Omega$, for all ϵ such that $0 \le \epsilon \le \epsilon_1 (\hat{u}_i, \delta u_i)$, where $\epsilon_1 (\hat{u}_i, \delta u_i) > 0$, then there exists a sequence of nonzero vectors $\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_k$ ($\hat{p}_i \in E^n$ for $i=0, 1, \ldots, k$) such that

1) Condition on the Hamiltonian:

$$< \hat{p}_{i+1}, \frac{\partial}{\partial u} f(\hat{x}_i, u)\Big|_{u=\hat{u}_i} \delta u_i > \le 0, \text{ for all } \delta u_i \in \Lambda(\hat{u}_i) \text{ and for all}$$

$i=0, 1, \ldots, k-1.$ (2.2)

2) **Adjoint Equations:**

$$\hat{P}_i - \hat{P}_{i+1} = \left(\frac{\partial}{\partial x} f(x, \hat{u}_i)_{x=\hat{x}_i}\right)^T \hat{P}_{i+1}, \quad i=0, 1, \dots, k-1. \tag{2.3}$$

3) **Transversality Conditions:**

Let $g_1(x), \dots, g_m(x)$, $m \leq n-1$, be continuously differentiable mappings from E^n into E^1 such that for every $x \in E^n$ the vectors $\frac{\partial}{\partial x} g_1(x), \dots, \frac{\partial}{\partial x} g_m(x)$ are linearly independent and $\frac{\partial}{\partial x^1} g_i(x) = 0$, $i=1, 2, \dots, m$. If $S = \{x: g_i(x) = 0, i = 1, 2, \dots, m\}$ then $P_k^{\ 1} \geq 0$ and there exist real numbers $\xi_1, \xi_2, \dots, \xi_m$, such that

$$\hat{P}_k^{\ j} = \sum_{i=1}^{m} \xi_i \frac{\partial}{\partial x_j} g_i(x)\Big|_{x=\hat{x}_k}, \quad j=2, 3, \dots, n; \tag{2.4}$$

Remarks

Special Cases:

(a) If $m=n-1$, $S = \{x = (x = (x^1, \dots, x^n): x^2 = x_d^2, x^3 = x_d^3, \dots, x^n = x_d^n, x_d^j \text{ fixed}, \}$, i.e., S is a line in E^n parallel to the x^1 axis, then $\hat{P}_k^{\ 1} \geq 0$; (2.5)

(b) If $m=0$, $S = E^n$, then

$$\hat{P}_k^{\ 1} \geq 0, \ \hat{P}_k^{\ 2} = \hat{P}_k^{\ 3} = \dots = \hat{P}_k^{\ n} = 0. \tag{2.6}$$

Note that the condition $\frac{\partial}{\partial x^1} g_i(x) = 0$ insures that the "cost" variable $x_k^{\ 1}$ is constrained only through its dependence on the other state variables $x_k^{\ 2}, \dots, x_k^{\ n}$.

Remarks:

If S is a convex n-dimensional subset of E^n with a s ooth boundary and if x_k belongs to the boundary of S the normal transversality condition applies, otherwise, x_k is in the interior of S and special case (b) applies.

The proof of the above theorem may be found in [2.5] and it proceeds as follows. First the optimal sequences $\hat{u}_0, \ldots, \hat{u}_{k-1}$, and $\hat{x}_0, \ldots, \hat{x}_k$ are assumed to exist and to be known. These are then perturbed to obtain sequences $\hat{u}_0 + \epsilon\, \delta u_0, \ldots, \hat{u}_{k-1} + \epsilon\, \delta u_{k-1}$ and the corresponding sequences $\hat{x}_o + \delta x_0, \ldots, \hat{x}_k + \delta x_k$ are computed by means of (2.1). It is then shown that

$$x_k = (\hat{x}_k + \delta x_k) = \hat{x}_k + \epsilon \sum_{i=0}^{k-1} \Phi_{i+1} \frac{\partial}{\partial u} f(\hat{x}_i, u)\Big|_{u=\hat{u}_i} \delta u_i + o(\epsilon),$$

where

$$\Phi_i = \prod_{j=i}^{k-1} \left(I + \frac{\partial f(x, \hat{u}_j)}{\partial x}\Big|_{\hat{x}_j}\right)$$

and that the set

$$K_k = \{\hat{x}_k + y_k : y_k = \sum_{i=0}^{k-1} \Phi_{i+1} \frac{\partial}{\partial u} f(\hat{x}_i, u)\Big|_{u=\hat{u}_i} \delta u_i,\ \delta u_i \in \Lambda(\hat{u}_i),\ i = 0,$$

$$1, \ldots, k-1\}$$

is a convex cone with the property that there exists a hyperplane through $\hat{x}_k$ which separates K_k from the half-line, $\{x \mid x^1 \geq \hat{x}_k^1,\ x^j = \hat{x}_k^j\ j = 2, 3, \ldots, n\}$ and whose normal, p_k, satisfies the transverality conditions previously stated. This fact is expressed by the condition on the Hamiltonian for $i = k - 1$. By invoking the adjoint difference equation, it is shown that this condition on the Hamiltonian must hold for all $i=0, 1, 2, \ldots, k-1$.

Example 1:

Consider the system

$$z_{i+1} - z_i = Az_i + u_i d, \ i=0, 1, 2, \ldots, k-1,$$

where, for $i=0, 1, \ldots, k$, $z_i \in E^{n-1}$, A is a constant matrix, $d \in E^{n-1}$ is a constant vector and the scalars $|u_i| \leq 1$, $i=0, 1, \ldots, k-1$. Given an initial state e_0 and a terminal state e_k, find a sequence $\hat{u}_0, \ldots, \hat{u}_{k-1}$ and a sequence $\hat{z}_0, \ldots, \hat{z}_k$ such that

(i) $z_{i+1} - z_i = Az_i + u_i d$, $i=0, 1, \ldots, k-1$

(ii) $z_0 = e_o$, $z_k = e_k$

(iii) for all sequences $u_0, u_1, \ldots, u_{k-1}$, ($|u_i| \leq 1$), $z_0, z_1, \ldots, z_k$, satisfying (i) and (ii) above

$$-\sum_{i=0}^{k-1} (\hat{u}_i)^2 \geq -\sum_{i=0}^{k-1} (u_i)^2 .$$

To convert this problem to the standard form given, we introduce the following substitutions. Let $x_0^1 = 0$ and $x_i^1 = \sum_{j=0}^{i-1} (u_j)^2$, $i=1, 2, \ldots, k$, $x_i^{j+1} = z_i^j$, $j=1, 2, \ldots, n-1$, $i=0, 1, \ldots, k$. We then obtain the equation

$$x_{i+1} - x_i = Bx_i + q(u_i), \ i = 0, 1, \ldots, k,$$

$$x_0 = (0, e_o^1, \ldots, e_o^{n-1}), \ x_k = (x_k^1, e_k^1, \ldots, e_k^{n-1}),$$

where

$$B = \left(\begin{array}{c|c} 0 & 0 \ldots 0 \\ \hline 0 & \\ \vdots & A \\ 0 & \end{array}\right),$$

$$q(u_i) = ((u_i)^2, u_i d^1, \ldots, u_i d^{n-1}).$$

Thus, this reduces to the standard problem with S a line as in transversality condition (ii). Assuming that a solution exists, (in which case it is unique), we obtain from the condition on the Hamiltonian,

$$(2\hat{u}_i \hat{p}^1_{i=1} + \sum_{j=1}^{n-1} \hat{p}^{j+1}_{i+1} d^j) \delta u_i \leq 0, \text{ for all } \delta u_i \in \Lambda(u_i).$$

By examining the above expression, we conclude that if $(2\hat{u}_i \hat{p}^1_{i+1} + \sum_{j\cdot 1}^{n-1} \hat{p}^{1+1}_{i+1} d^j) > 0$, then $\Lambda(u_i)$ must be the set $\{\delta u_i: \delta u_i < 0\}$ and hence $\hat{u}_i = +1$. Similarly, if this expression is negative, $\hat{u}_i = -1$. Otherwise, $2\hat{u}_i \hat{p}^1_{i+1} + \sum_{j=1}^{n-1} \hat{p}^{j+1}_{i+1} d^j = 0$. This enables us to express the sequence $\hat{u}_0, \ldots, \hat{u}_{k-1}$ in terms of the sequence $\hat{p}_0, \ldots, \hat{p}_k$ as follows:

$$\hat{u}_i = -\text{sat} \sum_{j=1}^{n-1} \frac{\hat{p}^{j+1}_{i+1} d^j}{\hat{p}^1_{i+1}} .$$

As a result, the problem is reduced to a two point boundary value problem:

$$\hat{x}_{i+1} - \hat{x}_i = B\hat{x}_i + q\,(-\text{sat} \sum_{j=1}^{n-1} \frac{\hat{p}^{j+1}_{l+1} d^j}{\hat{p}^1_{i+1}}),$$

$$\hat{p}_i - \hat{p}_{i+1} = B^T \hat{p}_{i+1},$$

$$\hat{x}_0 = (0, e^1_o, \ldots, e^{n-1}_o), \; \hat{x}_k = (x^1_k, e^1_k, \ldots, e^{n-1}_k), \; p^0_k \geq 0.$$

The last condition is derived from transversality considerations.

Example 2: (Reference [2.7]

Consider the time varying discrete time system described by the scalar difference equations,

$$x_{i+1}^1 - x_i^1 = -(x_i^2)^2 + \frac{1}{2}(-2)^i (2u_i + (u_i)^2) - \frac{1}{2}$$

$$x_{i+1}^2 - x_i^2 = u_i - 1$$

with $x_0^1 = 0$, $x_0^2 = 1$, and the scalars $|u_i| \leq 2$. To remove the functional dependence on i, let $x_i^3 = i$, $i=0, 1, 2, \ldots, k$. Thus, the following standard problem is obtained.

Given $x_0 = (0, 1, 0)$ and $k = 2$, find a sequence $\hat{u}_0$, $\hat{u}_1$ such that

$$\text{(i)}\quad \begin{aligned} x_{i+1}^1 - x_i^1 &= -(x_i^2)^2 + \frac{1}{2}(-2)^{x_i^3} (2u_i + (u_i)^2) - \frac{1}{2} \\ x_{i+1}^2 - x_i^2 &= u_i - 1 \\ x_{i+1}^3 - x_i^3 &= 1 \end{aligned} \qquad (2.7)$$

(ii) $x_0 = (0, 1, 0)$

(iii) For all sequences u_0, u_1, ($|u_i| \leq 2$) and x_0, x_1, x_2 satisfying (i) and (ii), $x_2^1 \leq \hat{x}_2^1$.

Solution:

If the system equations are solved, it is found that $x_2^1 = -\frac{1}{2}(1-u_0)^2 - (1+u_1)^2 - \frac{1}{2}$ and thus the optimal sequence $\hat{u}_0$, $\hat{u}_1$ is given by $\hat{u}_0 = 1$, $\hat{u}_1 = -1$. Now consider the condition on the Hamiltonian (2.2). Since we know that the optimal sequence lies inside the control constraint set, δu in (2.2) can be positive or negative. Thus, the condition on the Hamiltonian is expressed as,

$$\left\langle \hat{p}_{i+1}, \frac{\partial}{\partial u} f(\hat{x}_i, u)\Big|_{u=\hat{u}_i} \right\rangle = 0, \quad i=0, 1.$$

The optimal control is thus given by

$$\hat{u}_i = \frac{-\hat{p}^2_{i+1}}{\hat{p}^1_{i+1}\,(-2)\,\hat{x}^3_i} - 1\,. \tag{2.8}$$

Since $S = E^3$, the transversality condition permits us to set $\hat{p}_2 = (1, 0, 0)$. Thus, from (2.8) $\hat{u}_1 = -1$. From the adjoint equations (2.3)

$$\hat{p}^1_{i-1} = \hat{p}^1_i\,, \quad \hat{p}^2_{i-1} = \hat{p}^2_i - 2\hat{x}^2_{i-1}\,\hat{p}^1_i$$

giving $\hat{p}^1_0 = \hat{p}^1_1 = \hat{p}^1_2 = 1$ and $\hat{p}^2_1 = -2\hat{x}^2_1$. Inserting these values into (2.8) we find that $\hat{u}_0 = 2\hat{x}^2_1 - 1$. Substituting this into the system equations (2.7) we obtain $\hat{x}^2_1 = 2\hat{x}^2_1 - 1$, hence, $\hat{x}^2_1 = 1$. Thus, $\hat{u}_0 = 1$. Using these values, we observe that at $x = \hat{x}_0$, $u = \hat{u}_0$, $p = \hat{p}_1$, the Hamiltonian $\langle p, f(x, u) \rangle$ has a maximum with respect to u and at $x = \hat{x}_1$, $u = \hat{u}_1$, $p = \hat{p}_2$ the Hamiltonian has a minimum.

This illustrates the local nature of the optimality conditions (2.2), as well as the major difference between these conditions and the conditions given in the other sections. Note, however, that the class of problems to which the conditions (2.2) can be applied is correspondingly larger than the ones described in the other sections of this paper.

3. A Maximum Principle for Nonlinear Discrete Time Systems

3.1 Problem Statement

In this section the evolution of the system will be described by the difference equations

$$x_{i+1} - x_i = f_i(x_i,u_i) \qquad i = 0,1,2,\ldots,k-1 \tag{3.1}$$

A certain set Ω is given and all the control vectors will be required to belong to this set Ω. For every $i = 0,1,2,\ldots k-1$ the vector valued function $f_i(x,u)$ is given and satisfies the following conditions:

(α) the vector valued function $f_i(x,u)$ is defined for all $(x,u) \in E^n \times \Omega$.

(β) for every $u \in \Omega$ the vector valued function $f_i(x,u)$ is twice continuously differentiable with respect to x.

(γ) the function $f_i(x,u)$ and all its first and second partial derivatives with respect to x are uniformly bounded over $A \times \Omega$ for any bounded set $A \subset E^n$.

(δ) the matrix $I + \frac{\partial}{\partial x} f_1(x,u)$ is not singular on $E^n \times \Omega$.

(ε) the set $\{f_1(x,u) : u \in \Omega\}$ is convex for every $x \in E^n$.

The conditions (α), (β) and (γ) correspond to the usual "smoothness" assumptions. The conditions (δ) and (ε) are of another nature: they are always justified in the case of a system of difference equations which approximates a system of differential equations (Halkin [3.1]) but they are not necessarily justified in the case of a system of difference equations describing a control process which is basically discrete.

We shall now define an initial set

$$\{x : h_i(x) = 0, \ i = 1,2,\ldots,\ell\} \tag{3.2}$$

a terminal set

$$\{x : g_i(x) = 0, \ i = 1,2,\ldots,m\} \tag{3.3}$$

and an objective function $g_o(x)$. The functions $h_1(x),h_2(x),\ldots,h_\ell(x),g_o(x),g_1(x),\ldots,g_m(x)$ are given continuously differentiable mappings from E^n into E^1 such that for every $x \in E^n$ the vectors $\frac{\partial}{\partial x} h_1(x)$, $\frac{\partial}{\partial x} h_2(x),\ldots,$ $\frac{\partial}{\partial x} h_\ell(x)$ are linearly independent and the vectors $\frac{\partial}{\partial x} g_o(x)$, $\frac{\partial}{\partial x} g_1(x),\ldots,$ $\frac{\partial}{\partial x} g_m(x)$ are linearly independent.

Two sequences $\hat{u}_o, \hat{u}_1, \ldots, \hat{u}_{k-1}$ and $\hat{x}_o, \hat{x}_1, \ldots, \hat{x}_k$ are said to be optimal if they satisfy the conditions

(1) $h_i(x_o) = 0$ for $i = 1,2,\ldots,\ell$ (3.4)

(2) $x_{i+1} - x_i = f_i(x_i, u_i)$

for all $i = 0,1,2,\ldots,k-1$ (3.5)

(3) $u_i \in \Omega$ for all $i = 0,1,2,\ldots,k-1$ (3.6)

(4) $g_i(x_k) = 0$ for $i = 1,2,\ldots,m$ (3.7)

and if $g_o(\hat{x}_k)$ is the maximum value of $g_o(x_k)$ subject to these constraints.

3.2 Maximum Principle

If the sequences $\hat{u}_o, \hat{u}_1, \ldots \hat{u}_{k-1}$ and $\hat{x}_o, \hat{x}_1, \ldots, \hat{x}_k$ are optimal then there exists a sequence of nonzero vectors $\hat{p}_o, \hat{p}_1, \ldots, \hat{p}_k$ such that

(1) Maximization of the Hamiltonian

$$f_i(\hat{x}_i, \hat{u}_i) \cdot \hat{p}_{i+1} \geq f_i(\hat{x}_i, u) \cdot \hat{p}_{i+1} \tag{3.8}$$

for all $i = 0,1,2,\ldots,k-1$ and all $u \epsilon \Omega$

(2) Adjoint Equations

$$\hat{p}_i - \hat{p}_{i+1} = \left(\frac{\partial}{\partial x} f_i(x, \hat{u}_i) \Big|_{x=\hat{x}_i}\right)^T \hat{p}_{i+1} \tag{3.9}$$

for all $i = 0,1,2,\ldots,k-1$

(3) Transversality Conditions

There exists real numbers $\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_o, \beta_1, \ldots, \beta_m$ such that

(i) $$\hat{p}_o = \sum_{i=1}^{\ell} \alpha_i \frac{\partial}{\partial x} h_i(x) \Big|_{x=\hat{x}_o} \tag{3.10}$$

(ii) $$\hat{p}_k = \sum_{i=0}^{m} \beta_i \frac{\partial}{\partial x} g_i(x) \Big|_{x=\hat{x}_k} \tag{3.11}$$

(iii) $$\beta_o \geq 0 \tag{3.12}$$

3.3 Outline of the Proof of the Maximum Principle

The proof outlined here is similar to the proof of the Maximum Principle for systems described by differential equations given in Halkin [3.2].

Let us assume that $\hat{x}_o,\hat{x}_1,\ldots,\hat{x}_k;\hat{u}_o,\hat{u}_1,\ldots\hat{u}_{k-1}$ is an optimal solution. We shall prove that the Maximum Principle holds for that optimal solution.

We define the set W of all states x_k corresponding to all sequences $x_o,x_1,\ldots,x_k;\ u_o,u_1,\ldots,u_{k-1}$ satisfying conditions 3.4, 3.5 and 3.6. The set W is called the set of reachable states at time k. Next we define the set $S(\hat{x}_k)$ as the set of all states satisfying the conditions 3.7 and for which the objective function takes a greater value than at $\hat{x}_k$. Formally we have

$$S(\hat{x}_k) = \{x : g_i(x) = 0,\ i = 1,\ldots,m; g_o(x) > g_o(\hat{x}_k)\}. \tag{3.13}$$

We remark immediately that the sets W and $S(\hat{x}_k)$ are disjoint (i.e., have no point in common). Indeed if the sets W and $S(\hat{x}_k)$ had a point in common then the solution $\hat{u}_o,\hat{u}_1,\ldots,\hat{u}_{k-1};\hat{x}_o,\hat{x}_1,\ldots,\hat{x}_k$ would not be optimal and we would have a contradiction. If we knew also that the sets W and

$S(\hat{x}_k)$ were convex then the Maximum Principle would follow immediately, see Halkin [3.1], since two disjoint convex sets are always separable.*

For a nonlinear problem of the type considered in this section the sets W and $S(x_k)$ are not necessarily convex, and hence not necessarily separable. The difficulty is turned by considering a certain linearized problem around the solution $\hat{u}_o,\hat{u}_1,\ldots,\hat{u}_{k-1};\hat{x}_o,\hat{x}_1,\ldots,\hat{x}_{k-1}$. This linearized problem is defined as follows:

(α) the functions $h_i(x)$ are replaced by the functions

$$h_i(\hat{x}_o) + \frac{\partial}{\partial x} h_i(\hat{x}_o)\cdot(x-\hat{x}_o)$$

(β) the functions $g_i(x)$ are replaced by the functions

$$g_i(\hat{x}_k) + \frac{\partial}{\partial x} g_i(\hat{x}_k)\cdot(x-\hat{x}_k)$$

(γ) the functions $f_i(x,u)$ are replaced by the functions

$$f_i(\hat{x}_i,u) + \frac{\partial}{\partial x} f(\hat{x}_i,\hat{u}_i)\cdot(x-\hat{x}_i)$$

---

*Two sets A and B of E^n are separable if there exists a hyperplane P such that A is contained in one of the closed half space determined by P and B is contained in the other closed half space determined by P. There exist disjoint sets which are not separable and separable sets which are not disjoint.

We note immediately that the sequences $\hat{u}_o, \hat{u}_1, \ldots, \hat{u}_{k-1}; \hat{x}_o, \hat{x}_1, \ldots \hat{x}_k$ constitute also a solution (but not necessarily an optimal solution) for the linearized problem defined above.

We define now the sets $\overset{+}{W}(\hat{x}_k)$ and $\overset{+}{S}(\hat{x}_k)$ in the same way as the sets W and $S(\hat{x}_k)$ defined earlier but with respect to the linearized problem defined above and not with respect to the initial nonlinear problem which was used in the definition of W and $S(\hat{x}_k)$. It is easy to prove that the sets $\overset{+}{W}(\hat{x}_k)$ and $\overset{+}{S}(\hat{x}_k)$ are convex. We shall now state a result which is intuitively obvious but which is nevertheless long to prove (see Halkin [3.3]).

<u>Linearization Lemma</u>

If the sets W and $S(\hat{x}_k)$ are disjoint then the sets $\overset{+}{W}(\hat{x}_k)$ and $\overset{+}{S}(\hat{x}_k)$ are separable.

With the help of this Linearization Lemma the proof of the Maximum Principle follows easily.

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