SOME SUFFICIENT CONDITIONS FOR CONTINUOUS LINEAR-PROGRAMMING PROBLEMS

by

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INTRODUCTION

The class of optimization problems known as continuous linear programming problems, and sometimes referred to as bottleneck problems, has received considerable attention in the literature [1 - 3]. Because of their formal similarity to ordinary linear programming problems, it appeared that duality considerations might be helpful in searching for solutions. Thus, what to the authors' knowledge was the first sufficient condition for the optimality of a feasible solution to a continuous linear programming problem was obtained by Bellman [1]. He formulated a dual problem and showed that if the dual problem had a solution then the extreme value of the primal and dual functionals was the same. He subsequently used this approach in many other papers, such as [3], to obtain optimal solutions for various problems in economics, engineering, business, etc. An unfortunate characteristic of this duality method, however, is that it takes a great deal of intuition to apply it successfully.

More recently, Tyndall, in his Ph. D. dissertation [2], developed a very useful duality theorem which guarantees that for any problem satisfying readily checked assumptions, not only does an optimal solution exist, but the corresponding dual problem also has an optimal solution. However, he does not provide an algorithm for computing an optimal solution. Tyndall [2] showed that the existence of a solution to the primal problem did not quarantee the existence of a solution to the dual problem and vice versa, a major difference from ordinary linear programming problems. Thus, it should be stressed that the range of duality methods discussed above is restricted to problems whose duals also have solutions.

STATEMENT OF THE PROBLEM

We begin by stating the primal and dual problems to be considered in a form with inequality constraints, partly so as to be able to prove the relation between the primal and dual problems, partly to exhibit the similarities and differences between these problems and the problems treated by Bellman [1] and by Tyndall [2]. We shall then restate these problems in a form, more convenient for us, with differential equation constraints and slack variables.

Let $a(\cdot)$ and $b(\cdot)$ be bounded measurable functions mapping the real line R into a real n-dimensional vector space R^n. Let B be a constant, $n \times n$, nonsingular matrix such that at least one of the following two statements holds: (i) all the elements of B are nonnegative, (ii) all the elements of B^{-1} are nonnegative. Let C and D be constant, $n \times n$ matrices. Finally, let X denote the set of all measurable vector-valued functions $x(\cdot) = (x_1(\cdot), \ldots, x_n(\cdot))$ mapping R into R^n, with bounded components x_i, $i = 1, 2, \ldots, n$, which satisfy the following three conditions: For $t \in [0, T]$

$$x(t) \geqq 0, \tag{1}$$

$$Bx(t) \leqq b(t) + \int_0^t Cx(s)ds + \int_0^t Dx(s-1)ds, \tag{2}$$

$$x(t) = 0 \text{ for } t < 0 \tag{3}$$

Similarly, let Y denote the set of all measurable vector-valued functions $y(\cdot) = (y_1(\cdot), \ldots, y_n(\cdot))$ mapping R into R^n, with bounded components y_i, $i = 1, 2, \ldots, n$, which satisfy a dual set of three conditions: For $t \in [0, T]$

$$y(t) \geqq 0, \tag{4}$$

$$B\,y(t) \geqq a(t) + \int_t^T y(s)Cds + \int_t^T y(s+1)Dds, \tag{5}$$

$$y(t) = 0 \text{ for } t > T. \tag{6}$$

We shall consider the following continuous, linear programming problems:

We begin this paper by reformulating the general primal and dual continuous-linear programming problems with time delays in the constraints into a form that is somewhat more convenient for us. Thus, treating the primal and dual problems separately, we construct functions which are optimal solutions, provided they are feasible and provided some additional simple conditions are satisfied. When the functions we obtain are not optimal, we have no specific method for finding optimal solutions. Thus, the main advantage of the method proposed is that it is constructive and does not depend on the simultaneous existence of solutions to the primal and dual problems. We also give sufficient conditions for the existence of a solution for the dual problem when the primal has a solution and vice versa. When both the primal and dual problems are solvable, we exhibit the related solutions. It will be seen that for the set of problems tractable by both methods, the hypotheses used in this paper are sometimes less restrictive than Tyndall's [2]. We conclude the paper by discussing the relationship between the continuous linear programming problems considered and those treated in optimal control and in the classical theory of the calculus of variations. An example is worked out in the appendix.

NOTATION

For the most part we shall use standard mathematical notation. Matrices are denoted by capital letters such as B, C, D. Small letters such as a, b, δ denote vectors. No distinction is made between row and column vectors; the meaning will always be clear from the context.

Let x be an n component vector, and let x_i denote the ith component of x. $x \geqq 0 \rightarrow x_i \geqq 0 \; \forall \; i$. $x > 0 \rightarrow x_i > 0 \; \forall \; i$.

If x and y are n-vectors then xy will denote the scalar product.

We define the function

$$
\begin{aligned}
1(x_i) &= 1 \text{ if } x_i > 0 \\
&= 0 \text{ if } x_i \leqq 0
\end{aligned}
$$

If A is an n x n matrix then A^i will denote the ith row of A, and A_j will denote the jth column of A.

<u>Primal Problem</u>: Find an $x^0 \in X$ such that

$$\int_0^T a(t)\, x^0(t)\, dt = \max_{x \in X} \int_0^T a(t)\, x(t)\, dt \tag{7}$$

<u>Dual Problem</u>: Find a $y^0 \in Y$ such that

$$\int_0^T y^0(t)\, b(t)\, dt = \min_{y \in Y} \int_0^T y(t)\, b(t)\, dt \tag{8}$$

The relation between the primal and dual problems stated above is directly analogous to the relation between ordinary primal and dual linear programming problems [4] and is summed up by Theorem 1.

<u>Theorem 1</u>: If $x^0 \in X$ and $y^0 \in Y$ satisfy the condition

$$\int_0^T a(t)\, x^0(t)\, dt = \int_0^T y^0(t)\, b(t)\, dt, \tag{9}$$

then x^0 and y^0 are the solutions to the primal and dual problem, respectively.

<u>Proof:</u> The following proof is an extension of the proof given by Bellman [1] for the case $D = 0$. Let $x \in X$ and $y \in Y$, then, by making use of (1), (2), (4), and (5), we obtain the following pair of inequalities:

$$\int_0^T a(t)\, x(t)\, dt \leq \int_0^T \left[y(t)\, B - \int_t^T y(s) C ds - \int_t^T y(s+1) D ds \right] x(t) dt, \tag{10}$$

$$\int_0^T y(t)\, b(t)\, dt \geq \int_0^T y(t) \left[Bx(t) - \int_0^t Cx(s) ds - \int_0^t Dx(s-1)\, ds \right] dt \tag{11}$$

We shall now show that the right hand sides of (10) and (11) are equal. First, a simple change in the order of integration of the second term in the right hand side of (10) yields (after the dummy variables in the right hand side have been renamed):

$$-\int_0^T \Big[\int_t^T y(s)\, C ds\Big]\, x(t)\, dt = -\int_0^T y(t)\Big[\int_0^t Cx(s)\, ds\Big]\, dt. \qquad (12)$$

Thus, the second terms of the right hand sides of (10) and (11) are equal. We now proceed to examine the third terms. Since $y(t) = 0$ for $t > T$, we obtain for the third term of the right hand side of (10):

$$-\int_0^T \Big[\int_t^T y(s+1)\, D\, ds\Big]\, x(t)\, dt = -\int_0^{T-1} \Big[\int_t^{T-1} y(s+1)\, D\, ds\Big]\, x(t)\, dt. \qquad (13)$$

Since $x(t) = 0$ for $t < 0$, we now obtain (after renaming the dummy variables)

$$-\int_0^{T-1} \Big[\int_t^{T-1} y(s+1)\, D\, ds\Big]\, x(t)\, dt = -\int_1^T \Big[\int_t^T y(s)\, ds\Big]\, Dx(t-1)\, dt \qquad (14)$$

Now, interchanging the order of integration, we obtain after renaming the dummy variables [see (11)]

$$-\int_0^T y(t)\Big[\int_0^t Dx(s-1)\, ds\Big]\, dt = -\int_1^T \Big[\int_t^T y(s)\, ds\Big]\, Dx(t-1)\, dt. \qquad (15)$$

Thus the right hand sides of (10) and (11) are equal and hence for all $x \in X$ and $y \in Y$

$$\int_0^T a(t)\, x(t)\, dt \leq \int_0^T y(t)\, b(t)\, dt\,. \qquad (16)$$

Therefore, if there exists a pair for functions $x^0 \in X$, $y^0 \in Y$ satisfying (9), then this pair must be solutions to the primal and dual problems, respectively. This concludes the proof of the theorem.

REFORMULATION OF THE PROBLEM

As is usually the case with standard linear programming problems, we find it more convenient to deal with equality rather than inequality constraints. Consequently, we shall now convert the integral inequalities (2) and (5) into differential equations by introducing slack variables.

Definition: Let K_p be the set of all pairs of functions (z, u) which map R into R^n such that

(i) $u(\cdot)$ is measurable and $z(\cdot)$ is absolutely continuous ;

(ii) $u(t) \geqq 0, \ B^{-1}[u(t) + b(t) + Cz(t) + Dz(t-1)] \geqq 0$

$$\text{for } t \in [0, T], \qquad (17)$$

(iii) $z(0) = 0$; (18)

(iv) $\dot{z}(t) = B^{-1}[-u(t) + b(t) + Cz(t) + Dz(t-1)]$

$$\text{for } t \in [0, T] \qquad (19)$$

$= 0$ otherwise.

The matrices b, B, C, D in (17) and (19) are the same as in (2); Eq. (19) is obtained from (2) by letting $z = \int_0^t x(s)\, ds$ and using condition (3). Because of (19), the second part of condition (17) is equivalent to $\dot{z}(t) \geqq 0$ for $t \in [0, T]$, which corresponds to (1). The pairs (z, u) which are elements of K_p will be called feasible solutions to the primal problem.

In terms of the above definitions, we can restate the primal problem as follows:

Primal Problem I: Let the functional J_p be defined by

$$J_p(z, u) = \int_0^T a(t)\, B^{-1}[-u(t) + b(t) + Cz(t) + Dz(t-1)]\, dt. \qquad (20)$$

Find a pair $(z^o, u^o) \in K_p$ such that

$$J_p(z^o, u^o) = \max_{(z, u) K_p} J_p(z, u). \qquad (21)$$

The function $a(\cdot)$ in (19) is the same as in (7). It is readily seen that (20) defines the same functional as (7) when the constraints (i) through (iv) are taken into account. Any pair $(z, u) \epsilon K_p$ which maximizes J_p will be referred to as an optimal solution to the primal problem.

We now reformulate the dual problem in a similar way.

Definition: Let K_d be the set of all pairs of functions (w, v) which map R into R^n such that

(i) $v(\cdot)$ is measurable and $w(\cdot)$ is absolutely continuous;

(ii) $v(t) \geqq 0, [v(t) + a(t) - w(t) C - w(t+1)D] B^{-1} \geqq 0$

$$\text{for } t \epsilon [0, T]; \tag{22}$$

$$\text{(iii)} \quad w(T) = 0; \tag{23}$$

(iv) $\dot{w}(t) = [v(t) + a(t) - w(t) C - w(t+1) D] B^{-1}$

$$\text{for } t \epsilon [0, T] \tag{24}$$

$= 0$ otherwise.

Again, the matrices a, B, C, D are the same as in (5), and (24) is obtained from (5) by letting $w(t) = -\int_t^T y(s)\, ds$ and using condition (6). Because of (24), the second half of (22) implies that $\dot{w}(t) \geqq 0$ for $t \epsilon [0, T]$, which is equivalent to (4). We can now restate the dual problem.

Dual Problem I: Let J_d be the functional defined by

$$J_d(w, v) = \int_0^T \{v(t) + a(t) - w(t)\ C - w(t+1)D\} B^{-1}\ b(t)\ dt\ . \tag{25}$$

Find a pair of functions $(\overset{o}{w}, \overset{o}{v}) \epsilon K_d$ such that

$$J_d(\overset{o}{w}, \overset{o}{v}) = \max_{(w, v) \epsilon K_d} J_d(w, v)\ . \tag{26}$$

Again, the function $b(\cdot)$ in (25) is the same as in (8) and it is clear that Eq. (25)defines the same functional as Eq. (8) when the constraints are taken into account. Any pair $(w, v) \epsilon K_d$ which minimizes J_d will be referred to as an optimal solution to the dual problem.

PRIMAL APPROACH

We shall now consider the Primal Problem I and, by means of multiplier functions, obtain for it a sufficient condition for the existence of an optimal solution pair (z, u). Then, assuming that an optimal solution exists for the Primal Problem I, we shall make use of Theorem 1 to obtain a sufficient condition for the existence of a solution to the Dual Problem I. In the dual approach we shall reverse the order of this procedure.

<u>Multiplier Functions</u>. We begin by discussing differential equations whose solutions, if they exist, will subsequently be used as multiplier functions. Consider the following two differential equations:

$$\dot{\lambda}(t) = -\left\{ [\lambda(t) - a(t)] B^{-1} \Lambda(t)\, C + [\lambda(t+1) - a(t+1)] B^{-1} \Lambda(t+1) D \right\}, \tag{27}$$

with the final condition $\lambda(T) = 0$; and

$$\dot{\delta}(t) = -\left\{ [\delta(t) - a(t)]\, \Delta(t)\, B^{-1} C + [\delta(t+1) - a(t+1)]\, \Delta(t+1)\, B^{-1} D \right\}, \quad t \in [0, T] \tag{28}$$

also with the final condition $\delta(T) = 0$. In (27) and (28), $\lambda(t)$, $\delta(t) \in R^n$, a, B, C, D are the matrices appearing in the statement of the primal problem, and the diagonal n x n matrices $\Lambda(t)$ and $\Delta(t)$ are defined by

$$\Lambda(t) = \mathrm{diag}(1((a(t) - \lambda(t)) B_1^{-1})), \quad t \in [0, T], \tag{29}$$
$$= 0 \text{ otherwise}$$

$$\Delta(t) = \mathrm{diag}(1(a_i(t) - \delta_i(t))), \quad t \in [0, T] \tag{30}$$
$$= 0 \text{ otherwise}$$

Theorem 2: If the function $a(\cdot)$ is piecewise continuous on the interval $[0, T]$ then the differential equations (27) and (28) have unique absolutely continuous solutions, defined over the interval $[0, T]$.

Proof: We first consider the differential equation (27) over the interval $[T-1, T]$. Over this interval the second term of the right hand side is zero and (27) has the form

$$\dot{\lambda}(t) = -\left[\lambda(t) - a(t)\right] B^{-1} \Lambda(t)\, C \quad , \quad \lambda(t) = 0. \tag{31}$$

The right hand side is piecewise continuous in t and for $t \in [T-1, T]$, satisfies a Lipshitz condition with respect to λ, hence (31) has a unique, absolutely continuous solution over $[T-1, T]$. This solution may now be extended to cover the entire interval $[0, T]$. We proceed similarly for Eq. (28).

Feasible Solutions Determined by Multiplier Functions. Whenever the Lagrange multipliers λ and δ exist, they may be used to define a feasible solution to the primal problem I, provided certain conditions are satisfied.

Theorem 3: If the differential equation (27) has a solution λ, then the differential equation

$$\dot{z} = B^{-1} \Lambda(t)\left(b(t) + Cz(t) + Dz(t-1)\right); \; z(0) = 0 \tag{32}$$

which it defines, has a unique solution which will be denoted by z_λ. Similarly, if the differential equation (28) has a solution δ, then the differential equation,

$$\dot{z} = \Delta(t)\, B^{-1}\left(b(t) + Cz(t) + Dz(t-1)\right); \; z(0) = 0 \tag{33}$$

which it defines, has a unique solution which will be denoted by z_δ. The matrices Λ and Δ are defined by (29) and (30) respectively.

Proof: The right hand sides of (32) and (33) are linear in z, and, since all the elements of Λ and Δ are bounded baire functions of measurable functions, they are measurable in t [4],[5]. Hence (32) and (33) have a unique solution each (see [6] p.97, pb. 1). This completes the proof.

Definition: If (27) has a solution, then we define the n x n diagonal matrix Λ_1 by

$$\Lambda_1 + \Lambda = I, \tag{34}$$

where Λ is defined by (29) and I is the identity matrix. Similarly, if (28) has a solution, then we define the n x n diagonal matrix Δ_1 by

$$\Delta_1 + \Delta = I, \tag{35}$$

where Δ is defined by (30).

Definition: Let z_λ be the solution of (32). We define the function u_λ by the relation

$$u_\lambda(t) = \Lambda_1(t)\left[b(t) + Cz(t) + Dz\ (t-1)\right] \text{ for } t \in [0, T] \tag{36}$$

$$= 0 \text{ otherwise.}$$

Similarly, let z_δ be the solution of (32). We define the function u_δ by the relation

$$u_\delta(t) = B\Delta_1(t)B^{-1}[b(t) + Cz(t) + Dz(t-1)] \text{ for } t \in [0, T] \tag{37}$$

$$= 0 \text{ otherwise.}$$

It is readily seen that if $\dot{z}_\lambda(t) \geqq 0$ and $u_\lambda(t) \geqq 0$ for $t \in [0, T]$, then the pair (z_λ, u_λ) is a feasible solution to the Primal Problem I.

Similarly, if $\dot{z}_\delta(t) \geq 0$ and $u_\delta(t) \geq 0$ for $t \in [0, T]$, then (z_δ, u_δ) is a feasible solution pair to the Primal Problem I.

Formal Solution of the Primal Problem. We shall now examine formally the particular case when the matrix D in (19) is the zero matrix and n= 1, i.e., all quantities are scalars, and $B > 0$. We shall consider the problem rigorously and in full generality in the next section. In this particular case we are required to maximize

$$J_p(z, u) = \int_0^T a(t) B^{-1} [-u(t) + b(t) + Cz(t)] \, dt \tag{38}$$

subject to the constraints that

(i) $u(\cdot)$ be a measurable function and $z(\cdot)$ be an absolutely continuous function;

(ii) $u(t) \geqq 0, \; B^{-1}[-u(t) + b(t) + Cz(t)] \geqq 0$ for $t \in [0, T]$; (39)

(iii) $z(0) = 0$; (40)

(iv) $\dot{z}(t) = B^{-1}[-u(t) + b(t) + Cz(t)]$ for $t \in [0, T]$ (41)

$= 0$ otherwise.

We now introduce a multiplier function $\lambda(\cdot)$ and proceed to impose on it conditions which should lead us to a solution. Adjoining (40) and (41) to J_p by means of λ we get a new functional $\hat{J}_p$ defined by

$$\hat{J}_p(z, u) = \int_0^T \{a(t) B^{-1}[-u(t) + b(t) + Cz(t)] + \lambda(t) [\dot{z}(t) - B^{-1}(-u(t) + b(t) + Cz(t))] \} \, dt. \tag{42}$$

After choosing $\lambda(\cdot)$ we shall maximize the integrand in (42) at each instant $t \in [0, T]$, with respect to (z, u), subject to the constraints (i) and (ii) above. First, introducing the condition $\lambda(T) = 0$ and integrating $\int_0^T \lambda(t) \dot{z}(t) \, dt$ by parts we convert the integrand in (42) to the form:

$$a(t)\, B^{-1}[-u(t) + b(t) + Cz(t)] - \dot{\lambda}(t)\, z(t)$$
$$- \lambda(t)\, B^{-1}[-u(t) + b(t) + Cz(t)] \qquad (43)$$

The terms in the above expression can be grouped together in the following two ways:

$$-[a(t) - \lambda(t)]\, B^{-1} u(t) + [a(t) - \lambda(t)]\, B^{-1} b(t)$$
$$+ [a(t) - \lambda(t)]\, B^{-1} Cz(t) - \dot{\lambda}(t)\, z(t)\,, \qquad (44)$$

or

$$[a(t) - \lambda(t)]\, B^{-1}[-u(t) + b(t) + Cz(t)] - \dot{\lambda}(t)\, z(t) \qquad (45)$$

It may now be seen from (44) that if we set $\dot{\lambda}(t) = [a(t) - \lambda(t)]\, B^{-1} C$ for all those $t \in [0, T]$, for which $[a(t) - \lambda(t)]\, B^{-1} \geqq 0$, then for these t, (44) is maximized by setting $u(t) = 0$ and letting $z(t)$ be an arbitrary function satisfying (i) and (ii). Similarly, since B is positive it follows from (39) that $[-u(t) + b(t) + Cz(t)] \geqq 0$ must be satisfied and if we set $\dot{\lambda}(t) = 0$ for all those $t \in [0, T]$ for which $[a(t) - \lambda(t)] < 0$, the expression (45) is maximized by setting $[-u(t) + b(t) + Cz(t)] = 0$. Summarizing these conclusions we obtain formally, provided (ii) is satisfied, a pair of functions which maximize the integrand of (42) at each $t \in [0, T]$ as solutions of the following set of equations defined on the interval $[0, T]$:

$$\dot{\lambda}(t) = [a(t) - \lambda(t)]\, B^{-1} \Lambda(t)\, C, \quad \lambda(T) = 0\,, \qquad (46)$$

$$B\dot{z}_\lambda(t) = \Lambda(t)\, [b(t) + Cz_\lambda(t)] \quad z_\lambda(0) = 0 \qquad (47)$$

$$u_\lambda(t) = \Lambda_1(t)\, [b(t) + Cz_\lambda(t)] \qquad (48)$$

Thus, if (ii) is satisfied the pair of functions (z_λ, u_λ) belongs to the set K_p and hence it also maximizes the integrand of J_p.

The more general forms (27), (32), and (36), and similarly (28), (33), and (37) can be obtained by pursuing essentially the same line of reasoning. We shall now show rigorously that provided the multiplier functions λ and δ defined by (27) and (28) exist, the solution pairs (z_λ, u_λ) and (z_δ, u_δ) they determine are optimal if they satisfy the constraints (17), (18) and (19), for the matrix cases $B \geqq 0$ and $B^{-1} \geqq 0$ respectively.

Duality Theorem Ia: If the multiplier function λ determined by (27) exists, if the solution pair (z_λ, u_λ), which it determines by means of (32) and (36) is feasible, and if all the elements of the $n \times n$ matrix B are nonnegative, then the pair (z_λ, u_λ) is an optimal solution to the primal problem I.

Proof: We are required to maximize the functional

$$J_p(z, u) = \int_0^T a(t)\,[-u(t) + Cz(t) + Dz(t-1)]\,dt$$

subject to $(z, u) \in K_p$. We repeat here the conditions (17), (18), and (19) for functions in K_p:

(ii) $u(t) \geq 0$ and $B^{-1}\left[-u(t) + b(t) + Cz(t) + Dz(t-1)\right] \geqq 0$ for $t \in [0, T]$

(iii) $z(0) = 0$

(iv) $\dot{z} = B^{-1}\left[-u(t) + b(t) + Cz(t) + Dz(t-1)\right]$ for $t \in [0, T]$

$= 0$ otherwise.

We now use the multiplier function λ given by (27) to adjoin conditions (iii) and (iv) to J_p. We obtain a new functional $\hat{J}_p$ defined by

$$\hat{J}_p(z, u) = \int_0^T \Big\{ a(t)\,B^{-1}\left[-u(t) + b(t) + Cz(t) + Dz(t-1)\right] + \lambda(t)\Big(\dot{z}(t) - B^{-1}\left[-u(t) + b(t) + Cz(t) + Dz(t-1)\right]\Big)\Big\}\,dt \qquad (49)$$

As in the formal approach, we now convert (40) into a more suitable form for interpretation. First,

$$\int_0^T \lambda(t)\,\dot{z}(t)\,dt = -\int_0^T \dot{\lambda}(t)\,z(t)\,dt$$

$$= +\int_0^T \Big\{[\lambda(t) - a(t)]\ B^{-1}\Lambda(t)\ C +$$

$$+ [\lambda(t+1) - a(t+1)]\ B^{-1}\Lambda(t+1)\ D\Big\}\ z(t)\,dt \qquad (50)$$

Since $\Lambda(t) = 0$ for $t \notin [0, T]$, Eq. (50) becomes

$$\int_0^T \lambda(t)\,\dot{z}(t)\,dt = \int_0^T [\lambda(t) - a(t)]\ B^{-1}\Lambda(t)\,[\,Cz(t) + Dz(t-1)]\ dt \qquad (51)$$

Substituting the right hand side of (51) for $\int_0^T \lambda(t)\,\dot{z}(t)$ in (40) and collecting terms we get:

$$\hat{J}_p(z, u) = \int_0^T [\,a(t) - \lambda(t)]\ B^{-1}(-u(t) + b(t) + \Lambda_1(t)\,(Cz(t) + \ \ z(t-1)]\ dt, \qquad (52)$$

where the matrix $\Lambda_1(t)$ was defined in (34). We shall now maximize $\hat{J}_p$ over a set L_p of funtions (z, u) satisfying the conditions (i) and (ii) for the set K_p, but not necessarily the conditions (iii) and (iv), thus $L_p \supset K_p$. We shall then show that a pair (z, u) which maximizes $\hat{J}_p$ over L_p is also in K_p. Consequently, this pair maximizes J_p over K_p since $J_p = \hat{J}_p$ over K_p. Since all the elements of B are nonnegative, it is clear that the second part of condition (ii) implies the condition

$$-u(t) + b(t) + Cz(t) + Dz(t-1) \geqq 0 \text{ for } t \in [0, T] \qquad (53)$$

Rewriting (52) so as to exhibit the effect of this condition we obtain

$$\hat{J}_p(z, u) = \int_0^T \Big\{[a(t) - \lambda(t)]\,B^{-1}\big[\Lambda(t)\,[-u(t) + b(t)]$$

$$+ \Lambda_1\,[-u(t) + b(t) + Cz(t) + Dz(t-1)\,]\big]\Big\}dt \qquad (54)$$

Now, from (29) and (34), the vector $[a(t) - \lambda(t)] B^{-1} \Lambda(t) \geqq 0$ and the vector $[a(t) - \lambda(t)] B^{-1} \Lambda_1(t) \leq 0$ for $t \in [0, T]$. Hence, since $u(t) \geqq 0$ and because of (53), the integrand, and consequently also $\hat{J}_p$, is maximized by any pair of functions $(z, u) \in L_p$ which satisfy the condition

$$u(t) = \Lambda_1(t)(b(t) + Cz(t) + Dz(t-1)) \text{ for } t \in [0, T]. \tag{55}$$

Now the pair $(z_\lambda, u_\lambda) \in L_p$ by assumption and satisfies (55), hence it maximizes $\hat{J}_p$ over L_p. Hoever, the pair $(z_\lambda, u_\lambda) \in K_p$, and $J_p(z,u) = \hat{J}_p(z,u)$ for all $(z, u) \in K_p$, thus (z_λ, u_λ) also maximizes J_p over K_p. This completes the proof.

<u>Remark:</u> It should be pointed out that (z_λ, u_λ) is not necessarily a unique feasible solution pair maximizing the functional J. Thus, suppose that $a(t) - \lambda(t) = 0$ for $t \in I$, a nonzero subinterval of $[0, T]$. Then any solution pair $(z', u') \in K_p$, such $z'(t) = z_\lambda(t)$, $u'(t) = u_\lambda(t)$ for $t \notin I$, will also maximize the integrand of (54) and hence will also be an optimal solution pair.

<u>Duality Theorem IIa</u>: If the Lagrange multiplier δ determined by (28) exists, if the solution pair (z_δ, u_δ) which it determines by means of (33) and (37) is feasible, and if all the elements of the n x n matrix B^{-1} are nonnegative, then the pair (z_δ, u_δ) is an optimal solution to the primal problem I.

<u>Proof:</u> The proof of this theorem is carried out in a manner similar to the one used for the proof of the preceeding theorem and will therefore be omitted.

Whenever the multiplier functions λ or δ or both exist, they may also define a solution to the dual problem I. The manner in which this may happen is summed up in the following two theorems.

<u>Duality Theorem Ib</u>: Suppose the the multiplier function λ defined by (27) exists and that it defines an optimal solution pair (z_λ, u_λ)

by means of (31) and (36) for Primal Problem I. Let the functions (w_λ, v_λ) be determined by λ as solutions of the equations

$$\dot{w}_\lambda(t) = [\,a(t) - w_\lambda(t)\,C - w_\lambda(t+1)\,D\,]\,\Lambda(t)\,B^{-1} \tag{56}$$

$$= 0 \text{ otherwise}$$

$$v_\lambda(t) = -[\,a(t) - w_\lambda(t)\,C - w_\lambda(t+1)\,D\,]\,\Lambda_1(t) \tag{57}$$

$$w_\lambda(T) = 0\,.$$

If the pair (w_λ, v_λ) satisfies the condition

$$\dot{w}_\lambda(t) \geqq 0,\quad v_\lambda(t) \geqq 0 \text{ for all } t \in [0, T]\,, \tag{58}$$

and the $n \times n$ matrix B^{-1} commutes with the matrix $\Lambda(t)$ in multiplication, then the pair (w_λ, v_λ) is an optimal solution to Dual Problem I, i.e., $(w_\lambda, v_\lambda) \in K_d$ and

$$J_d(w_\lambda, v_\lambda) = \min_{(w,\,v) \in K_d} J_d(w, v)\,.$$

Proof: The right hand side of (56) is linear in w_λ and measurable in t and hence (56) has a unique solution which in turn completely defines v_λ by means of (57). Hence, if (58) is satisfied, the pair $(w_\lambda, v_\lambda) \in K_d$. We now make use of Theorem 1 to prove that this pair is optimal for the dual problem by showing that it yields the same cost as the optimal solution pair (z_λ, u_λ) for the primal problem.

Since by assumption the matrices B^{-1} and $\Lambda(t)$ commute, it follows from (32) and (56) that

$$\Lambda(t)\,\dot{z}_\lambda(t) = \dot{z}_\lambda(t),\quad \dot{w}_\lambda(t)\,\Lambda(t) = \dot{w}_\lambda(t)\,. \tag{59}$$

Hence, making use of (20), (32), and (59) we get

$$J_p(z_\lambda, u_\lambda) = \int_0^T a(t)\,\dot{z}_\lambda(t)\,dt = \int_0^T a(t)\,\Lambda(t)\dot{z}_\lambda(t)\,dt\,. \tag{60}$$

Now, substituting for $a(t)\ \Lambda(t)$ from (56) and making use of (59) we get

$$J_p(z_\lambda, u_\lambda) = \int_0^T [\dot{w}_\lambda(t)\, B\dot{z}_\lambda(t) + w_\lambda(t)\, C\dot{z}_\lambda(t) + w_\lambda(t+1)\, D\dot{z}_\lambda(t)]\, dt \quad (61)$$

Similarly, making use of (25) (32), (56), and (59) we get

$$J_d(w_\lambda, v_\lambda) = \int_0^T [\dot{w}_\lambda(t)\, B\dot{z}_\lambda(t) - \dot{w}_\lambda(t)\, Cz_\lambda(t) - \dot{w}_\lambda(t)\, Dz_\lambda(t-1)]\, dt. \quad (62)$$

An integration by parts of the second and third terms in the right hand side of (62) shows that

$$-\int_0^T \dot{w}_\lambda(t)\, Cz_\lambda(t)\, dt = \int_0^T w_\lambda(t)\, C\dot{z}_\lambda(t)\, dt$$

$$-\int_0^T \dot{w}_\lambda(t)\, Dz_\lambda(t-1)\, dt = \int_0^T w_\lambda(t+1)\, D\dot{z}(t)\, dt \ .$$

Hence $J_d(w_\lambda, v_\lambda) = J_p(z_\lambda, u_\lambda)$ which shows that the pair (w_λ, v_λ) is indeed optimal for the dual problem if (58) is satisfied.

<u>Duality Theorem IIb:</u> Suppose that the multiplier function δ defined by (28) exists and that it defines an optimal solution pair (z_δ, u_δ) by means of (33) and (37) for Primal Problem I. Let the functions (w_δ, v_δ) be determined by δ as solutions of the equations

$$\begin{aligned} \dot{w}_\delta(t) &= [a(t) - w_\delta(t)C - w_\delta(t+1)D]\, B^{-1}\, \Delta(t) \\ w_\delta(T) &= 0 \end{aligned} \quad (63)$$

$$v_\delta(t) = -[a(t) - w_\delta(t)\, C - w_\delta(t+1)D]\, B^{-1}\, \Delta(t)\, B \quad (64)$$

If the pair (w_δ, v_δ) satisfy the condition

$$\dot{w}_\delta(t) \geqq 0, \quad v_\delta(t) \geqq 0 \text{ for all } t \in [0, T] \ ,$$

and the matrices B^{-1} and $\Delta(t)$ commute in multiplication, then the pair (w_δ, v_δ) is an optimal solution to Dual Problem I.

The proof of this theorem is completely analogous to that given above and will therefore be omitted.

DUAL APPROACH

Duality Theorems Ib and IIb give constructive sufficient conditions for the existence of an optimal solution to the dual problem when a particular optimal solution to the primal problem exists. Such a solution to the primal problem may not exist (e.g. see [2]), and it is therefore preferable to construct a sufficient condition for the existence of an optimal solution to the dual problem in a manner not predicated upon the behavior of the primal problem.

We begin with a formal treatment of a particular case of the dual problem.

Formal Solution of the Dual Problem. We shall now examine Dual Problem I formally, following the same lines of reasoning that were used for Primal Problem I. We restrict ourselves again to the simpler case $D = 0$, $B > 0$, $n = 1$, i.e., we assume that all quantities are scalar.

Thus we wish to maximize

$$J_d(w, v) = \int_0^T [\, v(t) + a(t) - w(t)\, C \,]\, B^{-1} b(t)\, dt \tag{65}$$

subject to the constraints that

(i) $v(\cdot)$ be measurable and $w(\cdot)$ be absolutely continuous,

$$\text{(ii)} \quad v(t) \geqq 0\,, \quad [\, v(t) + a(t) - w(t)\, C \,]\, B^{-1} \geqq 0, \tag{66}$$

$$\text{(iii)} \quad w(T) = 0, \tag{67}$$

(iv) $\dot{w}(t) = [v(t) + a(t) - w(t)C]B^{-1}$ for $t \in [0, T]$, (68)

$= 0$ otherwise.

We now introduce a multiplier function $\mu(\cdot)$ and proceed to impose on it conditions which should make the problem readily solvable. Adjoining (67) and (68) to J_p by means of the multiplier μ we get the new functional

$$\hat{J}_d(w, v) = \int_0^T \{[v(t) + a(t) - w(t)C]B^{-1}b(t)$$

$$+ \left(\dot{w}(t) - [v(t) + a(t) - w(t)C]B^{-1}\right)\mu(t)\}\, dt \qquad (69)$$

By imposing the condition $\mu(0) = 0$, integrating $\int_0^T \dot{w}(t)\mu(t)\, dt$ by parts, and rearranging terms, we can put the integrand of (69) into either one of the following two forms:

$$v(t)B^{-1}[b(t) - \mu(t)] + a(t)B^{-1}[b(t) - \mu(t)]$$

$$- w(t)CB^{-1}[b(t) - \mu(t)] - w(t)\dot{\mu}(t) \qquad (70)$$

or,

$$[v(t) + a(t) - w(t)C]B^{-1}[b(t) - \mu(t)] - w(t)\dot{\mu}(t) \qquad (71)$$

Now, if we follow the approach used in the primal case and let $\dot{\mu}(t) = -CB^{-1}[b(t) - \mu(t)]$ for all t such that $B^{-1}[b(t) - \mu(t)] \geqq 0$, then (70) is minimized by letting $v = 0$. However, if we examine (71) we see that when $B^{-1}[b(t) - \mu(t)] < 0$, (71) may not have a finite minimum, since $[v(t) + a(t) - w(t)C] \geqq 0$ by assumption ((66) & $B > 0$). Hence we would have to require $B^{-1}[b(t) - \mu(t)] \geqq 0$ for all $t \in [0, T]$. Note however, that this would also make the slack variable $v = 0$ for all $t \in [0, T]$. It appears that a less restrictive assumption can be made as follows. Suppose that $w_\mu(\cdot)$ and $v_\mu(\cdot)$ are the optimal solution pair which we are going to obtain. Now let $\dot{\mu}(t) = -CB^{-1}[b(t) - \mu(t)]$

for all those $t \in [0, T]$ when $[a(t) - w_\mu(t)C] \geqq 0$ and $\dot{\mu}(t) = 0$ for all those $t \in [0, T]$ when $[a(t) - w_\mu(t)C] < 0$. Imposing the additional requirement that $B^{-1}[b(t) - \mu(t)] \geqq 0$, $t \in [0, T]$, the expression (70) is minimized by setting $v(t) = 0$ whenever $[a(t) - w_\mu(t)C] \geqq 0$ and, when $[a(t) - w_\mu(t)C] < 0$ the expression (71) is minimized by setting $v(t) + a(t) + w(t)C = 0$. Thus, provided the conditions (66) are satisfied, the optimal solution pair (w_μ, v_μ) and the corresponding multiplier function μ will be obtained as solutions of the following set of equations

$$\dot{w}_\mu(t) = [a(t) - w_\mu(t)C]\, M(t)B^{-1}; \quad w_\mu(T) = 0 \tag{72}$$

$$v_\mu(t) = -[a(t) - w_\mu(t)C]\, M_1(t) \tag{73}$$

$$\dot{\mu}(t) = -CM(t)B^{-1}[b(t) - \mu(t)]\,; \quad \mu(0) = 0 \tag{74}$$

where $M(t) = 1(a(t) - w_\mu(t)C_1)$ for $t \in [0, T]$ and $M(t) = 0$ otherwise ; $M_1(t)$ is defined by $M(t) + M_1(t) = I$.

Having completed the formal examination we proceed to show rigorously under what conditions the above reasoning does indeed lead to a solution of the matrix dual problem.

Feasible Solutions and Multiplier Functions. It is clear from the preceeding that optimal solution pairs can, possibly, be obtained as solutions of the following set of equations (depending on whether $B \geqq 0$ or $B^{-1} \geqq 0$):

$$\dot{w}_\mu(t) = [\,a(t) - w_\mu(t)\,C - w_\mu(t+1)D]\,M(t)\,B^{-1};\; w_\mu(T) = 0\,. \tag{75}$$

$$\dot{w}_\nu(t) = [\,a(t) - w_\nu(t)\,C - w_\nu(t+1)\,D]\,B^{-1}N(t);\; w_\nu(T) = 0. \tag{76}$$

The matrices a, B, C, D appearing in (75) and (76) are those appearing in the statement of the dual problem; the functions w_μ and w_ν map R into R^n and the n x n diagonal matrices M(t) and N(t) are defined by

$$\begin{aligned} M(t) &= \operatorname{diag} 1\big(a_i(t) - w_\mu(t)\,C_i - w_\mu(t+1)\,D_i\big) \text{ for } t \in [\,0, T\,] \\ &= 0 \text{ otherwise}; \end{aligned} \tag{77}$$

$$\begin{aligned} N(t) &= \operatorname{diag} 1(\,(a(t) - w_\nu(t)\,C - w_\nu(t+1)\,D)\,B_i^{-1}\,) \text{ for } t \in [\,0, T] \\ &= 0 \text{ otherwise}. \end{aligned} \tag{78}$$

Theorem 4: If the function a(·) which maps R into R^n is piecewise continuous on the interval [0, T] then the differential equations (75) and (76) have unique, absolutely continuous solutions.

This theorem may be proved in the same manner as Theorem 2; hence a detailed proof is omitted.

Whenever the functions w_μ and w_ν exist, they can be used to define corresponding functions v_μ , v_ν by means of the following equations:

$$v_\mu(t) = -[\,a(t) - w_\mu\,(t)\,C - w_\mu\,(t+1)\,D]\;\; M_1(t) \tag{79}$$

$$v_\nu(t) = -[\,a(t) - w_\nu(t)\,C - w_\nu(t+1)\,D]\;\; B^{-1}N_1(t)\,B \tag{80}$$

where M_1, N_1 are $n \times n$ matrices defined by $M + M_1 = I$, $N + N_1 = I$. Note that if $\dot{w}_\mu \geqq 0$, $v \geqq 0$ for $t \in [0, T]$ then they form a feasible solution pair. Similarly, if $\dot{w}_\nu \geqq 0$ and $v_\nu \geqq 0$, then (w_ν, v_ν) is a feasible solution pair.

The functions w_μ and w_ν can also be used to define multiplier functions μ and ν by means of the following equations:

$$\dot{\mu}(t) = +CM(t)\, B^{-1}[\mu(t) - b(t)] + DM(t-1)\, B^{-1}[\mu(t-1) - b(t-1)] \tag{81}$$

$$\mu(0) = 0 .$$

$$\dot{\nu}(t) = +CB^{-1}N(t)\,[\nu(t) - b(t)] + DB^{-1}N(t-1)[\nu(t-1) - b(t-1)] \tag{82}$$

$$\nu(0) = 0 .$$

<u>Theorem 5</u>: If the differential equation (75) has a unique solution w_μ then the differential equation (81) which it defines has a unique colution μ. Similarly, if the differential equation (76) has a unique solution w_ν then the differential equation (82) which it defines has a unique solution ν.

This theorem may be proved in the same manner as theorem 3 and hence the proof will be omitted.

We now give a theorem showing under what circumstances the formal treatment of the scalar case leads to a solution of the vector form of Dual Problem I.

<u>Duality Theorem IIIa</u>: If the solution pair (w_μ, v_μ) determined by (75) and (79) exists and is feasible, if all the elements of the $n \times n$ matrix B are nonnegative and if the multiplier function μ determined by (81) satisfies the condition

$$B^{-1}[-\mu(t) + b(t)] \geqq 0 \tag{83}$$

for $t \in [0, T]$, then the solution pair (w_μ, v_μ) is optimal for the Dual Problem I.

Proof: We are required to minimize the functional

$$J_d(w, v) = \int_0^T [v(t) + a(t) - w(t)\,C - w(t+1)\,D]\, B^{-1} b(t)\, dt$$

subject to $(w, v) \in K_d$. We repeat here the last three conditions for functions in K_d.

(ii) $v(t) \geqq 0, [v(t) + a(t) - w(t)\,C - w(t+1)\,D]B^{-1} \geqq 0$ for $t \in [0, T]$

(iii) $w(T) = 0$;

(iv) $\dot{w}(t) = [v(t) + a(t) - w(t)\,C - w(t+1)\,D]\, B^{-1}$ for $t \in [0, T]$
$= 0$ otherwise .

We now use the multiplier function μ determined by (81) to adjoin the conditions (iii) and (iv) to J_d, thus forming the new functional

$$\hat{J}_d(w, v) = \int_0^T \{[v(t) + a(t) - w(t)\,C - w(t+1)\,D]\, B^{-1}\, b(t) + (\dot{w}(t) - [v(t) + a(t) - w(t)\,C - w(t+1)\,D]\, B^{-1})\mu(t)\}\, dt \tag{84}$$

Integrating $\int_0^T \dot{w}(t)\,\mu(t)\,dt$ by parts, substituting for $\dot{\mu}(t)$ from (81) and making use of the assumption that $M(t) = 0$ for t not in $[0, T]$ we finally arrive at the following form for the functional $\hat{J}_d$:

$$\hat{J}_d(w, v) = \int_0^T \{[v(t) + a(t)]\, M(t)\, B^{-1}[b(t) - \mu(t)] + [v(t) + a(t) - w(t)\,C - w(t+1)\,D]\, M_1(t)\, B^{-1}[b(t) - \mu(t)]\}\, dt \tag{85}$$

We now minimize $\hat{J}_d$ over the a L_d of pairs of functions (w, v) which satisfy conditions (i) and (ii) for functions in K_d, but not necessarily the conditions (iii) and (iv), thus $L_d \supset K_d$. Since all the elements of B are nonnegative by assumption, it follows from (ii) that

$$v(t) + a(t) - w(t)\,C - w(t+1)\,D \geqq 0 \text{ for } t \in [0, T] \tag{86}$$

and hence, since by assumption $B^{-1}[b(t) - \mu(t)] \geqq 0$ for $t \in [0, T]$, the integrand, and consequently, $\hat{J}_d$ are minimized by any pair of functions $(w, v) \in L_d$ satisfying

$$v(t) = -[a(t) - w(t)\,C - w(t+1)\,D]\; M_1(t) \text{ for } t \in [0, T].$$

We recognize that the pair $(w_\mu, v_\mu) \in L_d$ by assumption, that it satisfies this condition and hence that it minimizes $\hat{J}_d$ over L_d. However, $(w_\mu, v_\mu) \in K_d$ and, since $\hat{J}_d = J_d$ for all (w,v) in K_d, it follows that (w_μ, v_μ) minimizes J_d over K_d, also, which completes the proof.

We now state without proof the analogous theorem for the case $B^{-1} \geq 0$.

Duality Theorem IVa: If the solution pair (w_ν, v_ν) determined by (76) and (80) exists and is feasible, if all the elements of the matrix B^{-1} are nonnegative and if the multiplier function ν determined by (82) satisfies the condition

$$b(t) - \nu(t) \geqq 0 \tag{87}$$

for all $t \in [0, T]$, then the solution pair (w_ν, v_ν) is optimal for Dual Problem I.

The following two theorems can be proved in the same manner as Duality Theorems Ib and IIb; hence their proofs will be omitted. Their purpose is to establish optimal solutions to the primal problem when optimal solutions to the dual problem are given by the above two theorems.

Duality Theorem IIIb: Suppose that the multiplier function μ defined by (81) exists and that the related solution of the dual problem (w_μ, v_μ) given by (75), (79) is optimal. Let the functions (z_μ, u_μ) be defined as solutions of the following equations:

$$\dot{z}_\mu(t) = B^{-1} M(t)[b(t) + Cz_\mu(t) + Dz_\mu(t-1)] \ ; \ z_\mu(0) = 0 \qquad (88)$$

$$u_\mu(t) = M_1(t)[b(t) + Cz_\mu(t) + Dz_\mu(t-1)] \qquad (89)$$

If the pair (z_μ, u_μ) satisfies $\dot{z}_\mu(t) \geqq 0$, $u_\mu(t) \geqq 0$ for $t \in [0, T]$ and the matrices B^{-1} and $M(t)$ commute in multiplication, then (z_μ, u_μ) is an optimal solution pair of the primal problem.

Duality Theorem IVb: Suppose that the multiplier function ν defined by (82) exists and the related solution to the dual problem (w_ν, v_ν) given by (76) and (80) is optimal. Let the functions (z_ν, u_ν) be defined as the solutions of the following equations:

$$\dot{z}_\nu(t) = N(t) B^{-1}[b(t) + Cz_\nu(t) + Dz_\nu(t-1)] \ ; z_\nu(0) = 0 \ , \qquad (90)$$

$$u_\nu(t) = B\, N(t) B^{-1}[b(t) + Cz_\nu(t) + Dz_\nu(t-1)] \qquad (91)$$

If the pair (z_ν, u_ν) satisfies $\dot{z}_\nu(t) \geqq 0$, $u_\nu(t) \geqq 0$ for $t \in [0, T]$ and the matrices B^{-1} and $N(t)$ commute in multiplication, then (z_ν, u_ν) is an optimal solution pair for the primal problem.

A SPECIAL CASE

For the case where D is the zero matrix, and B the identity matrix, we can readily compare the results obtained here with the results of [2]. For this case, the results of [2] guarantee the existence of solutions to the primal and dual problems if

1) The elements of C and the components of $b(t)$ are non-negative for all $t \epsilon [0, T]$

2) The components of $a(t)$ and $b(t)$ are continuous over $[0, T]$.

We shall show the following:

Theorem 9: Let $B = I$, $D = 0$, and let all the components of $a(\cdot)$ be piece-wise continuous functions of time. If all the components of C and $b(t)$ are non-negative for all t in the interval $[0, T]$, then all the hypotheses of duality theorems III-a and III-b are satisfied and the primal and dual problems have solutions.

Proof: Since all the components of $a(\cdot)$ are peicewise continuous, it follows from theorem 4 that (75) has a solution. The fact that the components of C and of $b(\cdot)$ are non-negative insure that the solution pairs (z_μ, u_μ) and (w_μ, v_μ) are feasible and that (83) holds. (Note $z_\mu = z_\nu$ when $B = I$.)

(i) The above case is a special case of the problem considered by Tyndall [2]. For this special case, however, we have not only relaxed the conditions for the existence of solutions obtained in [2], but we have actually obtained the solutions. It should, however, again be emphasized that the results of [2] apply to a larger class of problems.

We must stress again that even for the case $B = I$, the existence of an optimal solution to the primal problem and the existence of a feasible solution to the dual problem do not guarantee the existence of an optimal solution to the dual problem and vice versa. This fact is contrary to the analagous situation in ordinary linear programming where the mere existence of feasible solutions to the primal and dual problems guarantees the existence of optimal solutions to both problems. This matter, together with an approaprate illustrative example, is further discussed by Tyndall in [2] .

CONCLUDING REMARKS

Computational Aspects. There are no unusual computational difficulties involved in calculating the solutions to the differential equations encountered in this paper. The procedure one would follow, for example, in the application of duality theorem I-a would be to first integrate equation (27) backward in time to obtain $\lambda(t)$. From this (29) is used to obtain $\Lambda(t)$ corresponding to $\lambda(t)$ and the solution to (32) is then obtained by integrating forward in time. If the solutions to (32) and (36) are feasible then they are optimal. A similar procedure is used in applying the other duality theorems. Finally, it should be pointed out that if the solutions obtained are not possible, and hence not optimal, then optimal solutions have to be sought by other means which, mostly, still remain to be discovered.

Pontryagin's Maximum Principle . The reformulated problem is seen to be essentially a variational one. It cannot, however, even for the case without delay, be directly treated by Pontryagin's maximum principle [7], [8] due to the nature of the constraints. In addition, the maximum principle is concerned only with necessary conditions for optimal solutions while we are concerned here with obtaining sufficient conditions.

Calculus of Variations . The reformulated problem can, however, for the non-delay case, be partially treated by the classical calculus of variations, keeping in mind that the conditions one might obtain are only necessary. The constraints such as (17) are handled by the method of Valentine [9] as discussed in [10] . However, to be handled by the classical theory these constraints must satisfy two conditions which we repeat here from [10] in terms of the notation of (17).

We define a 2n component vector valued function of t, $z(t)$, and $u(t)$ (we will assume $D = 0$ and $B = I$, the identity matrix):

$$r(t, z(t), u(t)) = (u_1(t), \ldots, u_n(t), -u_1(t) + b_1(t) + C^1 z(t), \ldots,$$

$$-u_n(t) + b_n(t) + C^n z(t))$$

Then the constraints appear as

$$r(t, z(t), u(t)) \geq 0$$

To be handled by the classical theory these constraints must satisfy the following conditions: (i) At most n components of r can vanish for any $t \in [0, T]$ on any solution (z , u) to the primal problem to which the necessary conditions are applicable; (ii) On any solution (z , u) to the primal problem to which the necessary conditions are applicable, the matrix

$$\left(\frac{\partial r_i}{\partial u_j}\right)$$

where i ranges over those indices where

$$r_i = 0$$

has maximum rank.

It is not difficult to construct examples where the first of these two conditions is violated, and yet the sufficiency conditions of duality theorem I-a are satisfied. To illustrate this consider the problem $(n = 1)$ in the primal problem notation

$$b(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ 1, & 1 < t \leq 2 \end{cases}$$

$$C = 1, \qquad a(t) = 1, \qquad 0 \leq t \leq 2$$

$$D = 0$$

with $[0, 2]$ the interval of interest. The solution to this problem from duality theorem I-a is readily obtained as

$$u(t) = 0, \qquad 0 \leq t \leq 2$$

$$z(t) = 0, \qquad 0 \leq t \leq 1$$

$$= e^{t-1} - 1 \quad 1 < t \leq 2$$

For this problem r is a two component vector, and on the interval $[0, 1]$ both components vanish, violating condition (i)

In summary, we have appraoched the continuous linear programming problem from a point of view rather different from the one found in previous treatments. Furthermore, we have extended the problem to include systems with time delay constraints. Our method is constructive: Whenever optimal solutions can be shown to exist they can also be computed without any difficulty. Finally, for the method presented to be applicable it is not necessary that both the primal and dual problems have solutions simultaneously.

APPENDIX

AN EXAMPLE

Examples of continuous linear programming problems occuring in practice are numerous [1],[2],[3] . Presented here is a variation of a problem treated by [2] .

Consider a steel mill whose rate of steel production at time t is given by $\dot{z}_2(t)$. Let $\dot{z}_1(t)$ denote the rate of stockpiling of produced steel. The rate of production of steel is limited by the initial capacity of the mill b_2 ($= 1$) and the time integral of the rate $\dot{z}_2(t) - \dot{z}_1(t)$ at which produced steel is allocated to increase the capacity of the plant. Since the rate of stockpiling of steel can never be greater than the rate of production we have the constraint that $\dot{z}_1(t) - \dot{z}_2(t)$ can never be positive. Over the time interval $[0, 1]$ it is desired to maximize the net output of steel, the time integral of $\dot{z}_1(t)$. Stating the above as equations we have

$$\dot{z}_1(t) - \dot{z}_2(t) \leqq 0$$

$$\dot{z}_2(t) \leqq 1 + z_2(t) - z_1(t) \ , \quad z_1(0) = 0, \ z_2(0) = 0.$$

Under the assumptions that steel will never be destroyed and that steel once stockpiled cannot be used to increase the capacity of the plant we have

$$\dot{z}_1(t) \geqq 0$$

$$\dot{z}_2(t) \geqq 0$$

In the vector notation of the primal problem I we then have

$$B\dot{z}(t) \leqq b(t) + Cz(t)$$

$$z(0) = 0$$

where

$$\dot{z}(t) = (\dot{z}_1(t), \dot{z}_2(t))$$

$$b(t) = (0, 1), \qquad a(t) = (1, 0)$$

$$C = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

To solve this problem we shall make use of duality theorem II-a.

The inverse of the matrix B is

$$B^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and satisfies the hypothesis of the theorem that all the elements of B^{-1} be non-negative. It is readily verified that

$$\delta_1 = 1 - t \qquad t \in [0,1]$$

$$\delta_2 = t - 1$$

satisfy Eq. (28) where for this case

$$B^{-1}C = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad \Delta(t) = I .$$

The solution to Eq. (33) is verified to be

$$
\left.\begin{aligned} z_1(t) &= t+1 \\ z_2(t) &= t+1 \end{aligned}\right\} \quad t \in [0,1]
$$

and $u^0(t) = 0$.

Also, the solutions are feasible and consequently $(z\ \ , u\ \ )$ is the solution to the problem. The physical interpretation is that at least over the interval $[0,1]$ it is to the greatest advantage to directly stockpile all of the produced steel, and use none of it to increase the capacity of the plant.

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