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A BOUNDARY LAYER MODEL FOR VELOCITY SPACE INSTABILITIES*

by

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ABSTRACT

A model is developed for treating velocity space instabilities in bounded finite-temperature plasmas, in which the wave vector is arbitrary. Density gradients near the plasma boundaries are included. Solutions are obtained in the boundary layer region by solving the Boltzmann equation in the presence of an arbitrarily constructed potential, whose coefficients are determined in a self-consistent manner. The model is used to extend the work of a previous paper, which dealt with two stream instabilities, to a treatment of instabilities arising from temperature anisotropy. A bimaxwellian distribution is considered and the limits of instability are found. The conclusion is reached that the limits of instability can be approximately determined from the unbounded plasma analysis provided the correct transverse separation constant is determined from a simple bounded plasma model.

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I. INTRODUCTION

In a previous paper[1], we considered the problem of bounded plasmas with thermal motion in the direction of the applied magnetic field, B_0. The effects of transverse temperature were neglected because no new effects were expected for the modes being considered with $T_{\parallel} \geq T_{\perp}$, where $T_{\parallel}$ is the "temperature" parallel to B_0, and $T_{\perp}$ is the "temperature" perpendicular to the magnetic field. However, even if this condition were satisfied, there exist an infinite number of modes near the cyclotron harmonies which disappear if $T_{\perp}$ is neglected. Furthermore, if $T_{\perp} > T_{\parallel}$, a mechanism exists for instability due to the anisotropy in temperature.[2-6] The purpose of this paper is to extend the results of

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ref. (1) to examine the stability limits for a plasma with both transverse bounds and finite transverse temperature. More generally, we wish to develop a boundary layer model which enables one to obtain a dispersion relation for a hot plasma slab in the presence of a uniform magnetic field. The model considered in ref. (1) was a homogeneous plasma column in a waveguide, which in the general case did not fill the guide. The boundary conditions applied to the model did not differ from those applied to a cold plasma because the transverse plasma temperature was zero. In the present case, the concept of a surface charge is no longer meaningful. The "surface" is now a boundary layer with a thickness of the order of a Larmor orbit. The first-order charge density at any particular position cannot be computed with the local electric field but must take into account the effect of the electric field experienced by each particle as it travels along its zero-order orbit.

In previous work a number of models were used to describe the particle motion in the boundary layer. In the second part of Landau's paper[7] on the oscillations of an electronic plasma, a semi-infinite plasma was considered with an external longitudinal field of frequency, ω, impinging on the plasma. The problem was to find an expression which described the penetration of the field into the plasma. Among the boundary conditions utilized was the condition of perfect particle reflection at the interface or wall separating the plasma from the vacuum. This condition states that at the wall, i.e., at $x = 0$, $f_1(v, 0) = f_1(-v, 0)$, where f_1 is the first-order distribution function, and v is the velocity in the direction perpendicular to the wall. Although this assumption is an idealization that must be treated with some skepticism, it does give the physically acceptable conclusion that there is no particle current through the wall. A problem similar to Landau's, involving the penetration of an electric field into a metal, was treated by Reuter and Sondheimer.[8] In this situation a semi-infinite configuaration was also assumed, but both spectral and diffuse reflections were permitted at the boundary. In studying the Tonks-Dattner[9] resonances, Weissglas[10] assumed a uniform plasma situated between two perfectly reflecting walls. He found that asymptotically in time, the expressions for the resonance frequencies are the same as for an infinite plasma; the effect of the walls is to restrict the wave number to specific discrete values. Montgomery and Gorman[11] found the same result and stated that the addition of an externally imposed magnetic field would not alter the result. At first glance this conclusion may seem surprising, because a finite hot plasma in a magnetic field is diamagnetic. It would appear therefore that in the zero-order distribution function, a spatially-dependent term must be included for the surface current. However, if the plasma is uniform and bounded by perfectly reflecting walls, the particles that are reflected by the walls result in a current equal and opposite to the surface current from the non-reflected particles and the zero-order distribution is in fact the same as in an infinite medium.[12,13]

It can be concluded from these analyses that the infinite plasma model may, in certain instances, lead to the same results as the finite plasma model with perfectly reflecting walls. This statement is demonstrated in Appendix A. These solutions have been limited, however, to situations in which the propagation is perpendicular to the boundary surface, and the plasma is uniform.

Expansion techniques have also been utilized in bounded plasma problems. These techniques have had some success both for uniform and non-uniform plasmas. In studying the Tonks-Dattner resonances, Parker, Nickel, and Gould[14] used the first two moments of the Boltzmann equation as their basic equations, i.e., the equations of continuity and momentum conservation. The retention of higher terms in the moment equations is equivalent to retaining higher terms in powers of $(k\alpha_T/\omega)^2$, thermal speed over wave speed squared, in the dispersion relation. Retaining terms of the first order in $(k\alpha_T/\omega)^2$ results in the familiar dispersion relation for an infinite uniform plasma,

$$\left(\frac{\omega}{\omega_p}\right)^2 = 1 + \frac{3}{2}\,\frac{\alpha_T^{\,2} k^2}{\omega_p^{\,2}}\,, \tag{1}$$

where ω_p is the plasma frequency, and α_T is the thermal speed. By using the moment equations and the quasi-static approximation, $\nabla \times E = 0$, a fourth order differential equation is obtained for the potential. In a cylindrical configuration, two solutions are discarded since they are singular on the axis. Hence, two boundary conditions are necessary: the first matches the potential across the plasma surface, and the second is a condition on the particle motion. Parker, et al. assumed that the plasma was contained in a glass tube and considered the normal particle current at the walls of the tube as zero. The problem was solved numerically for a non-uniform plasma and the results were in close agreement with the experimental data.[14]

The hydrodynamic equations do not include the resonance effects of the particle motion and hence such phenomena as Landau damping and, in the presence of a magnetic field, cyclotron harmonics, are not accounted for. In addition, solutions have only been obtained for $k_z = 0$. If instead of taking moment equations, the Boltzmann equation is retained and utilized in conjunction with Poisson s equation, an integro-differential equation may be obtained for the potential. For a strong dc magnetic field, this equation may be expanded in powers of $k\alpha_T/\omega_c$, where ω_c is the cyclotron frequency. The resonance nature of the particle is still accounted for and the integro-differential equation is reduced to a differential equation, the order depending upon the order retained in powers of $k\alpha_T/\omega_c$. (The parameter $k\alpha_T/\omega_c$ is the ratio of the Larmor radius to wavelength.) This approach has been taken by several authors.[15,16] Buchsbaum and Hasegawa[16] solved for the perturbed distribution function in the usual manner but carried through an operator, d/dy, rather than the wave number, k_y, of a homogeneous plasma. They expanded in the parameter $(\alpha_\perp/\omega_c)(d/dy)$ and retaining terms of fourth order the result was a fourth degree differential equation for the potential. By taking k_z equal to zero the equation reduces to a second-degree differential equation and for certain density profiles this equation can be solved in terms of known functions. Using the boundary condition of zero-particle current at the wall, they derive a dispersion relation. For a uniform plasma this relation has the form (in our notation),[16]

$$(4\omega_c^2 - \omega^2)(\omega_c^2 + \omega_p^2 - \omega^2) - \frac{3\alpha_\perp^2}{2}\omega_p^2\left(\frac{n\pi}{2a}\right)^2 = 0, \tag{2}$$

where $2a$ is the width of the plasma. The model used to obtain this relation is reminiscent of the model employed in Appendix A for the derivation of Eq. A-10. In that situation the boundary condition was perfect reflectivity at the wall. This boundary condition not only specified the velocity at the wall, but all other moments too. If, however, Eq. A-10 is expanded and only terms through order $(k_y \alpha_\perp / \omega_c)^4$ are retained, the information concerning the higher moments is lost and the equation reduces to Eq. 2. (There is a small numerical difference but this is due to the fact that Eq. A-10 satisfies only the even modes, while Eq. 2 satisfies both odd and even modes.)

If k_z is finite, we must consider the fourth degree differenital equation. For an inhomogeneous plasma this equation has non-constant coefficients and numerical methods must be used to obtain solutions. For the homogeneous plasma, on the other hand, we obtain a fourth-degree polynomial, for the dispersion relation. There are two wave numbers that satisfy this equation, and an attempt was made to obtain its numerical results. However, in the process of searching for solutions it was discovered that the expansion used to derive the two wave numbers fails. One of the wave numbers apparently is always large enough so that $k_y \alpha_\perp / \omega_c$ takes on a value in the neighborhood of unity and the expansion is not valid. Physically this result may be explained as follows: In order to keep the expansion parameter small, one selects a small value of $\alpha_\perp / a\omega_c$. In a cold plasma the first-order velocity is proportional to the local electric field. In the present problem there is a normal electric field at the wall. As $\alpha_\perp \rightarrow 0$ the plasma becomes nonthermal in the perpendicular direction, and the particle, spiraling in a circle of decreasing radius, becomes less dependent upon distant electric fields and more dependent upon the local field. To meet the condition of zero velocity at the wall in the presence of a finite electric field requires that the electric field oscillate rapidly with distance; in the limit of $\alpha_{\perp} = 0$, one of the wave numbers goes to infinity, i.e., its corresponding wave length goes to zero.

In all of the above bounded configurations, in which solutions have been obtained by imposing boundary conditions on the particles, $k_z = 0$. Furthermore, the particle boundary condition in all cases was that of zero current at the plasma boundary. In order to relax these stringent assumptions it appears necessary to consider a plasma boundary in which no explicit assumptions concerning the particle velocity at a boundary need be introduced. One such configuration is clearly an inhomogeneous plasma in which the density falls to zero at the boundary. In the following section we introduce variation in the z direction and consider such an inhomogeneous plasma, thus eliminating the necessity of a boundary surface. In Section III a dispersion relation is derived and numerical results are given.

II. BOUNDARY LAYER MODEL

Consider the model of a homogeneous plasma slab between two conducting walls, but not filling the space between the walls. In an earlier discussion we concluded that the concept of a surface charge on the slab is without meaning if there is thermal motion in the direction perpendicular to the slab surface. Not only is the surface distributed over a distance of the order of Larmor orbit, the perturbed charge at any one point also depends upon the electric field over all space. Computing the charge within this inhomogeneous boundary layer, from Gauss's law we relate the field in the plasma to that in the vacuum. Thus, the boundary layer serves as the equivalent of a surface charge. In Fig. 1, three regions are considered: a homogeneous plasma, a boundary layer region, and an outlying vacuum. We assume the following hierarchy,

$$\frac{\alpha_\perp}{\omega_c} < \ell < \Lambda, \tag{3}$$

where Λ in this instance is the transverse wavelength within the homogeneous plasma, and ℓ is the width of the boundary layer. The boundary layer is taken to be several Larmor diameters in extent. This assumption is physically acceptable, and at the same time, as we shall show in the discussion below, it will enable us to obtain a solution by utilizing the dispersion relation for the infinite homogeneous plasma.

A qualitative discussion is now given of the physical situation. Consider a particle in the homogeneous region in the vicinity of the boundary; the particle's orbit extends into the boundary layer region. Although there is a small class of particles which traverse the entire boundary layer, for all practical purposes the largest depth of penetration is a Lamor diameter. Since the boundary layer is several times greater than this distance, and since we will consider a density profile with a zero gradient at the plasma interface, the density at the deepest point of penetration is still of the same magnitude as the density within the homogeneous plasma. Therefore, as far as all particles within the central region are concerned, the plasma is homogeneous and infinite in extent; hence the dispersion relation for a uniform infinite plasma[12]

$$k_y^2 + k_z^2 = \sum_i \omega_p^2 \sum_{m=-\infty}^{\infty} \exp\left(-\frac{\lambda^2}{2}\right) I_m\left(\frac{\lambda^2}{2}\right) \left[\frac{2m\omega_c}{k_z \alpha_z \alpha_\perp^2} Z(\zeta_m) - \frac{2}{\alpha_z^2}\left(1 + \zeta_m Z(\zeta_m)\right) \right] \tag{4}$$

is valid in the central region for small λ, where

$$\zeta_m = \frac{\omega + m\omega_c}{k_z \alpha_z}$$

$$\lambda = \frac{k_y \alpha_\perp}{\omega_c} .$$

We have introduced the function

$$Z(\zeta) = 2 j e^{-\zeta^2} \int_{-\infty}^{j\zeta} \exp(-t^2)\, dt$$

which is the Fried and Conte plasma dispersion function, and α_z is the thermal velocity in the z direction. We consider a solution in the central core with a single wave number which necessitates retaining terms through λ^2 in an expansion of Eq. 4. By doing so we limit ourselves to a study of only the fundamental mode and neglect the higher harmonics. Since the fundamental mode is the most unstable for anisotropic instabilities, this is the mode of most interest to us.

Equation 4 can be used directly to verify the assumption of ref. (1) that transverse temperature can be neglected in treating two stream instabilities. Assuming an isotropic temperature, and substituting the parameters used in the previous calculations,[1] we find that if the dispersion relation is computed for the two lower modes, the difference in the results obtained from Eq. 22 in ref. (1) is negligible.

We now consider the solution in the boundary layer. The solution in an inhomogeneous plasma cannot, in general, be obtained in terms of known functions. To surmount this difficulty we consider $\Lambda/4 > \ell$.

The potential variation through the boundary layer can then be expected to be a slowly varying and monotonically decreasing function. This being the case, we need not solve for the potential in the boundary layer but can construct it from an arbitrary set of independent functions. If the potential and the normal electric field are matched across both interfaces, the constructed potential should closely approximate the true solutions. Furthermore, the potential will appear only under an integral sign, and hence the solution is rather insensitive to the exact shape of the potential profile.

Using the above prescription we proceed to solve for the perturbed distribution function in the boundary layer region, and from this quantity compute the charge density. The zero-order Boltzmann equation for an inhomogeneous plasma is

$$v \cdot \nabla f_0 + \frac{e}{m}(\underline{v} \times B_0) \cdot \nabla_v f_0 = 0 . \tag{5}$$

If density variation is considered only in the y direction, the general solution to this equation is

$$f_0 = f_0(v_z, v_\perp{}^2 + \frac{2e\varphi_0}{m}, v_x - y\omega_c + C) . \tag{6}$$

where the independent variables are the constants of motion of a particle in the presence of a potential φ_0, and a constant magnetic field, B_0, (a low β plasma is assumed, β being the ratio of particle pressure to magnetic pressure, and therefore the magnetic field is constant throughout the inhomogeneous region), and C is an arbitrary constant. We will consider a neutral plasma, in which case the zero-order potential is constant. Zero-order electric fields may occur in laboratory plasmas but they depend upon the diffusion process. Since we are considering a collisionless plasma, the inclusion of a zero-order electric field would not necessarily reflect the actual physical situation.

Any combination of the independent variables may be used to construct a zero-order distribution function that satisfies Eq. 5. We choose a density variation in the boundary layer of the form,

$$n(y) = n_0\left(1 - \frac{(y - a + \ell)^2}{\ell^2}\right), \tag{7}$$

where n_0 is the density of the homogeneous plasma. Let,

$$f_0 = A \exp\left[-\frac{v_x^2}{\alpha_\perp^2} - \frac{v_y^2}{\alpha_\perp^2} - \frac{v_z^2}{\alpha_z^2}\right]\left(1 - b(v_x - (y - a + \ell)\omega_c)^2\right). \tag{8}$$

The constants A and b are to be evaluated subject to the condition imposed by Eq. 7. We have for the density

$$n(y) = \int f_0 \, d^3v = A\pi^{3/2} \alpha_z \alpha_\perp^2 \left(1 - \frac{b\alpha_\perp^2}{2} - b(y - a + \ell)^2 \omega_c^2\right) \tag{9}$$

and from Eq. 7 we find,

$$b = \frac{1}{\ell^2 \omega_c^2 + \frac{\alpha_\perp^2}{2}} \tag{10}$$

$$A = \frac{n_0}{\pi^{3/2} \alpha_z \alpha_\perp^2} \; \frac{1}{\left(1 - \frac{b\alpha_\perp^2}{2}\right)} = \left(1 + \frac{\alpha_\perp^2}{2\ell^2 \omega_c^2}\right) \frac{n_0}{\pi^{3/2} \alpha_z \alpha_\perp^2} .$$

We note that this distribution function leads to an average macroscopic current density in the x direction. The $\underline{j} \times \underline{B}$ force resulting from this current balances the force due to the pressure gradient.

To find the first order distribution function we use the linearized Boltzmann equation:

$$\frac{\partial f_1}{\partial t} + \underline{v} \cdot \nabla f_1 + \frac{e}{m}(\underline{v} \times \underline{B}_0) \cdot \nabla_v f_1 = \frac{e}{m} \nabla \varphi_1 \cdot \nabla_v f_0 . \tag{11}$$

The left hand side of the equation is the total derivative of f_1 in time along the zero order path of a particle in phase space. The solution to this equation is

$$f_1 = \frac{e}{m} \int_{-\infty}^{t} \nabla \varphi_1(y') \cdot \nabla_v f_0 \, dt' . \tag{12}$$

After applying the equation,

$$\frac{d\varphi}{dt} = \frac{dr}{dt} \frac{\partial \varphi}{\partial r} + \frac{\partial \varphi}{\partial t} , \tag{13}$$

integrating by parts, and using the transformation $s = t - t'$, the perturbed distribution function is,

$$f_1 = -\left(\frac{2e}{m\alpha_\perp^2}\right) \varphi(y) f_0 + \frac{e}{m} \int_0^\infty \left[\left(\frac{-2v_z}{\alpha_z^2}\right) jk_z + \left(\frac{-2}{\alpha_\perp^2}\right)\left(j(\omega - k_z v_z)\right)\right]$$

$$\times e^{j(\omega - k_z v_z)s} f_0 \varphi(y') \, ds . \tag{14}$$

Variation in the x direction has not been included although this variation may also be taken into account in the present theory. There is no zero-order electric field and the zero-order particle orbits are therefore the same as in a homogeneous plasma. From Stix,[17] y' is thus given by

$$y' = \frac{-v_x}{\omega_c}\,[1 - \cos\omega_c s] - \frac{v_y}{\omega_c}\sin\omega_c s + y. \tag{15}$$

The potential $\varphi(y)$ is still to be specified. We will assume that $\varphi(y)$ is a sum of independent exponential functions with a characteristic wave number β, where the coefficients are to be determined by the boundary conditions, and β will be determined from the solution of the dispersion relation. For ease of calculation we assume $\varphi(y)$ is $e^{j\beta y}$ and substitute the full expression into the final results. Following the procedure used for a uniform plasma,[17] we substitute the value of y' into Eq. 14, and integrating over v_x and v_y we obtain,

$$\int f_1\, dv_x\, dv_y = -\frac{2e}{m\alpha_\perp^2}\, e^{j\beta y}\int\int f_0\, dv_x dv_y + \frac{e}{m}\int_0^\infty \left[\frac{-2v_z}{\alpha_z^2}\, jk_z + \left(\frac{-2}{\alpha_\perp^2}\right)\right.$$

$$\left.\times\left(j(\omega - k_z v_z)\right)\right] e^{j(\omega - k_z v_z)s} \exp\left(-\frac{\beta^2\alpha_\perp^2}{2\omega_c^2}\right)$$

$$\times\exp\left(\frac{\beta^2\alpha_\perp^2}{2\omega_c^2}\cos\omega_c s\right)\left[1 + \frac{\alpha_\perp^2}{2\omega_c^2\ell^2} - \frac{\alpha_\perp^2}{2\omega_c^2\ell^2} + \frac{\beta^2\alpha_\perp^4}{4\ell^2\omega_c^4}\right.$$

$$\times(1 + \cos^2\omega_c s - 2\cos\omega_c s) - \frac{j\beta\alpha_\perp^2}{\omega_c^2\ell^2}(y - a + \ell)(1 - \cos\omega_c s)$$

$$\left. - \frac{(y - a + \ell)^2}{\ell^2}\right] \exp\left(\frac{-v_z^2}{\alpha_z^2}\right)\frac{n_0 e^{j\beta y}}{\pi^{\frac{1}{2}}\alpha_z}\, ds. \tag{16}$$

The integration over s may be performed after use of the identity,

$$e^{a\cos\theta} = \sum_{n=-\infty}^{\infty} I_n(a)\, e^{jn\theta} \tag{17}$$

In performing this integration we use the usual assumption that our integrals are defined for growing waves and are analytically continued for damped waves. A final integration over v_z gives the charge density for the nonuniform plasma, assuming that the potential is known,

$$\frac{\rho_1}{\epsilon_0} = \omega_p{}^2 e^{j\beta y}\left\{\frac{-2}{\alpha_\perp{}^2}\left(1 - \frac{(y-a+\ell)^2}{\ell^2}\right) + \exp\left(-\frac{\gamma^2}{2}\right)\sum_{n=-\infty}^{\infty} I_n\left(\frac{\gamma^2}{2}\right)\right.$$

$$\times\left[\left(\frac{1}{\alpha_\perp{}^2} + \frac{3}{8}\,\frac{\gamma^2}{\ell^2\omega_c{}^2} - \frac{j\beta}{\ell^2\omega_c{}^2}(y-a+\ell) - \frac{(y-a+\ell)^2}{\alpha_\perp{}^2\ell^2}\right)(PD_n + 2)\right.$$

$$+ \frac{\gamma^2}{16\ell^2\omega_c{}^2}(PD_{n+2} + PD_{n-2} + 4) + (-1)\left(\frac{\gamma^2}{4\omega_c{}^2\ell^2} - \frac{j\beta}{2\omega_c{}^2\ell^2}(y-a+\ell)\right)$$

$$\left.\left.\times (PD_{n+1} + PD_{n-1} + 4)\right]\right\}, \tag{18}$$

where

$$\gamma = \frac{\beta\alpha_\perp}{\omega_c}$$

and

$$PD_n = -\frac{2\alpha_\perp{}^2}{\alpha_z{}^2}\left(1 + \zeta_n Z(\zeta_n)\right) + \frac{2n\omega_c}{k_z\alpha_z} Z(\zeta_n).$$

If the charge density is integrated over the width of the boundary layer, the result to be utilized in Gauss's law to relate the electric field in the plasma with that in the vacuum, we obtain a dispersion for the plasma waves. Expanding and retaining terms through order γ^2, we obtain

$$a \int_{a-\ell}^{a} \frac{\rho_1}{\epsilon_0}\, dy = a_0 e^{j\beta a} + j\beta a b_0 (e^{j\beta a} - e^{j\beta(a-\ell)}) + j\beta a c_0\, e^{j\beta(a-\ell)}, \quad (19)$$

where

$$a_0 = \frac{a\omega_p^2}{\ell\, \omega_c^2} \left[-\frac{2}{\gamma^2}\, PD_0 \right]$$

$$b_0 = \frac{\omega_p^2}{\omega_c^2} \left(\frac{-2}{\gamma^2 \beta^2 \ell^2}\, PD_0 \right)$$

$$c_0 = \frac{\omega_p^2}{\omega_c^2} \left[\left(\frac{1}{\gamma^2} - \frac{1}{2} \right) PD_0 + \frac{1}{4} (PD_1 + PD_{-1}) \right].$$

Before proceeding further we shall demonstrate that this expression is equivalent to the surface charge found on a homogeneous cold plasma beam. To verify this statement the following limits are taken; first $\alpha_\perp$ is allowed to go to zero, then ℓ is reduced to zero. The resulting expression should be the surface charge found on a beam with thermal motion only along the axis. Letting $\alpha_\perp \to 0$, we have

$$a_0 \underset{\alpha_\perp \to 0}{=} \frac{4a}{\ell} \frac{\omega_p^2}{\beta^2 \alpha_z^2} \left(1 + \zeta_0 Z(\zeta_0)\right) ,$$

$$b_0 \underset{\alpha_\perp \to 0}{=} \frac{4\omega_p^2}{\ell^2 \beta^4 \alpha_z^2} \left(1 + \zeta_0 Z(\zeta_0)\right) ,$$

$$c_0 \underset{\alpha_\perp \to 0}{=} \frac{-2\,\omega_p^2}{\beta^2 \alpha_z^2} \left(1 + \zeta_0 Z(\zeta_0)\right) + \frac{\omega_p^2}{2k_z \alpha_z \omega_c} \left(Z(\zeta_1) - Z(\zeta_{-1}) \right) . \quad (20)$$

To take the limit of $\ell \to 0$, the exponential $e^{-j\ell\beta}$ is first expanded,

$$a \int_{a-\ell}^{a} \frac{\rho_1}{\epsilon_0}\, dy = e^{j\beta a} \left(a_0 + b_0 \left(j \frac{\beta^3 a \ell^2}{2} - \beta^2 \ell a\right) + j\beta a c_0 \right) , \quad (21)$$

and upon collecting terms and allowing $\ell \to 0$ we find that

$$\int_{a-\ell}^{a} \frac{\rho_1}{\epsilon_0}\, dy \underset{\ell \to 0}{=} \frac{j\beta\omega_p^2}{2k_z \alpha_z \omega_c} \left(Z(\zeta_1) - Z(\zeta_{-1}) \right) e^{j\beta a} . \quad (22)$$

Since the potential was taken as $e^{j\beta a}$, $-j\beta e^{i\beta a}$ is the component of the electric field at $y = 0$. In comparison with ref. (1), Eq. 12 reveals that

$\frac{\omega_p^2}{2k_z \alpha_z \omega_c} \left(Z(\zeta_{-1}) - Z(\zeta_1) \right)$, is that part of ϵ_{yy} (ϵ_{rr} is identical to ϵ_{yy}) representing the surface charge. Therefore the expression representing the charge in the boundary layer reduces to the proper value in the limit of a cold plasma with a sharp boundary.

III. DISPERSION RELATION AND NUMERICAL RESULTS

The boundary conditions are now discussed. The normal electric field and the potential are continuous at the boundaries and these properties give four boundary conditions. These conditions assure us of a properly behaved potential in the boundary layer region. A fifth condition relates the field in the plasma to that in the vacuum through Poisson's equation. Consider an element of volume extending through the boundary layer. From Gauss's law we have

$$E_{y_{vac}} - E_{y_{plasma}} + \int_{a-\ell}^{a} \frac{\partial E}{\partial z}\, dy = \int_{a-\ell}^{a} \frac{\rho_1}{\epsilon_0}\, dy \; . \tag{23}$$

The solution in the homogeneous plasma is $A \cos k_y y$ where k_y is given by Eq. 4. The solution in the vacuum region is $Be^{-k_z y}$. Since there are five boundary conditions, the solution within the boundary layer is constructed from three independent functions and is taken as

$$\varphi(y) = d_0 e^{-\alpha y} \cos k_y y + d_1 e^{-\alpha y} \sin k_y y + d_2 e^{-\alpha y} \cos 2k_y y \, . \tag{24}$$

We now have five homogeneous equations and five unknown constants, and a non-trivial solution is found by equating the characteristic determinant of these constants to zero.

$$
\left|
\begin{array}{c:c:c:c:c}
\left.\frac{\partial}{\partial y}\left(e^{-\alpha y}\cos k_y y\right)\right|_a & \left.\frac{\partial}{\partial y}\left(e^{-\alpha y}\sin k_y y\right)\right|_a & \left.\frac{\partial}{\partial y}\left(e^{-\alpha y}\cos 2k_y y\right)\right|_a & 0 & k_z e^{-k_z a} \\
e^{-\alpha a}\cos k_y a & e^{-\alpha a}\sin k_y a & e^{-\alpha a}\cos 2k_y a & 0 & -e^{-k_z a} \\
\left.\frac{\partial}{\partial y}\left(e^{-\alpha y}\cos k_y y\right)\right|_{a-\ell} & \left.\frac{\partial}{\partial y}\left(e^{-\alpha y}\sin k_y y\right)\right|_{a-\ell} & \left.\frac{\partial}{\partial y}\left(e^{-\alpha y}\cos 2k_y y\right)\right|_{a-\ell} & k_y \sin k_y(a-\ell) & 0 \\
e^{-\alpha(a-\ell)}\cos k_y(a-\ell) & e^{-\alpha(a-\ell)}\sin k_y(a-\ell) & e^{-\alpha(a-\ell)}\cos 2k_y(a-\ell) & -\cos k_y(a-\ell) & 0 \\
S_0 & S_1 & S_2 & -a k_y \sin k_y(a-\ell) & a k_z e^{-k_z a}
\end{array}
\right| = 0 \qquad (25)
$$

where

$$S_0 = ak_z^2 \int_{a-\ell}^{a} e^{-\alpha y} \cos k_y y \, dy - \frac{\psi(k_y + j\alpha)}{2} - \frac{\psi(-k_y + j\alpha)}{2}$$

$$S_1 = ak_z^2 \int_{a-\ell}^{a} e^{-\alpha y} \sin k_y y \, dy - \frac{\psi(k_y + j\alpha)}{2j} + \frac{\psi(-k_y + j\alpha)}{2j}$$

$$S_2 = ak_z^2 \int_{a-\ell}^{a} e^{-\alpha y} \cos 2k_y y \, dy - \frac{\psi(2k_y + j\alpha)}{2} - \frac{\psi(-2k_y + j\alpha)}{2}$$

and

$$\psi(\beta) = a \int_{a-\ell}^{a} \frac{\rho_1(\beta)}{\epsilon_0} \, dy \,. \tag{25}$$

If instead of three regions, two regions were considered with the density falling off to zero at a conducting wall, the dispersion relation would be simplified somewhat, the result being a fourth-order determinant rather than fifth order.

We see from the above dispersion relation that the fields in the plasma are related to those in the vacuum by a distributed "surface" charge in a boundary layer region. The potential throughout the boundary layer must be known if the charge is to be computed. This potential is expressed in terms of a set of independent functions, and their coefficients are evaluated by matching the potential and normal electric field at the boundaries. The computed charge retains information concerning the resonance effects of the particles. The zero-order moments in the boundary layer are equal to those in the homogeneous plasma at their common interface. However, the same is not true of first-order moments. To match these moments at the boundary, the potential in the inhomogeneous plasma would have to be expanded in additional functions, and thus the result of matching these moments is a further refinement in the specification of the potential. The resulting dispersion relation is very unwieldy and we choose not to make this refinement.

For an infinite plasma the boundary between stable and unstable regions in parameter space for an anisotropy in temperature has been studied by many authors.[2-6] Ozawa, Kaji and Kito[6] computed curves giving the limits of instability for the fundamental mode in a single species plasma, after first expanding the dispersion relation in powers of the parameter $\alpha_{\perp} k_{\perp}/\omega_c$. The limits of instability were computed for the fundamental mode. However, they only examine the limits of stability for large k_z. This limit is a result of increased Landau damping. For a bi-Maxwellian distribution, growing waves are found only for finite k_z. This statement may be easily verified by allowing k_z to go to zero in Eq. 4 . The dispersion relation then depends upon only $T_{\perp}$ and hence must be valid for a plasma in thermal equilibrium which, from thermodynamic considerations, does not support growing waves. Therefore, before proceding to treat the bounded plasma, we inquire whether an additional region of stability also exists for small k_z for an infinite plasma. A typical curve of the instability limit is extended to smaller values of longitudinal wave number by numerically solving Eq. 4 . (Ozawa, _et al._,[6] arrived at a different form of the dispersion relation, but the two forms are equivalent.) The results are given in Fig. 2. We find that the plasma is stable for small k_z and the region of stability for this particular example occurs at angles of propagation greater than 60° with respect to the magnetic field.

To obtain similar curves for the finite plasma, it is necessary to solve the set of Eqs. 4 and 25 simultaneously. The procedure is as follows: We fix parameters $\alpha_\perp/\omega_c a$, $\alpha_z/\alpha_\perp$, ℓ/a and $\sqrt{2}\omega_c/k_z\alpha_z$. From Eq. 4 we solve for the parameter ω_p^2/ω_c^2 as a function of the fixed parameters and the variables ak_y and ω/ω_c, where ω/ω_c is taken to be real. To satisfy the condition that ω_p/ω_c must be real, we equate the imaginary part of the right hand side of Eq. 4 to zero. From this equation we obtain ω/ω_c as a function of ak_y. Turning our attention to Eq. 25 the only unknown parameters now are the transverse separation constants from Eq. 24, k_y and α. We map the dispersion relation in the complex plane as a function of these variables. Where the map passes over zero we have a solution. The value of ak_y found from this solution is substituted into the expression for ω_p/ω_c and a point is obtained for the instability curve. The value of $\sqrt{2}\omega_c/k_z a_z$ is varied, the process is repeated, and further points are found. In keeping with our approximations, only the lowest order spatially varying mode is investigated. The results are plotted in Fig. 3. The dashed curve gives the limits of stability for a constant wave number as in Fig. 2. For this curve $\alpha_\perp/a\omega_c = 0.1$ and $k_y a = \pi/2$. A direct comparison between these results and the finite plasma cannot be made since in one situation we fix the transverse wave number, while in the other we fix the radius of the plasma. For large values of k_z the value of $k_y a$ for the fundamental mode lies near $\pi/2$; as k_z is decreased the eigenvalue of k_y also decreases, and thus qualitatively the behavior of our dispersion relation may be explained. The effect of the boundary is to quantize the transverse wave number. We may conclude that the dispersion relation for an infinite plasma, in conjunction with a simple model to determine an approximate transverse wave number, would be adequate to predict the regions of instability due to temperature anisotropy. It should also be noted that the instability examined here involves a different wave interaction (4) from that treated previously.[1] Thus a two-stream system could exhibit both types of instability each derived from a separate facet of the departure of the distribution function from equilibrium.

APPENDIX A

It is to be shown that, with certain restrictions, Eq. 4 satisfies the boundary condition $f_1(v, a) = f_1(-v, a)$. It is assumed that there is only propagation across the magnetic field, i.e., $k_z = 0$. In this situation

$$E_y = A \cos k_y y, \quad k_y = \frac{(2n+1)\pi}{2a} \tag{A-1}$$

The perturbed distribution function is,

$$f_1 = -\left(\frac{2e}{m\alpha_\perp^2}\right) f_0 \left[\varphi(y) + \int_0^\infty j\omega e^{j\omega s} \varphi(y')\, ds\right] \tag{A-2}$$

The zero order distribution function f_0, is even in v_y, hence the first term on the RHS of (A-2) is even. We must now show that,

$$I = \int_0^\infty f_0 e^{j\omega s} \varphi(y')\, ds \tag{A-3}$$

is an even function in v_y at $y = \pm a$. To express y' in terms of s we utilize Eq. 15. Taking the form of φ from Eq. A-1 and assuming f_0 to be bi-Maxwellian, upon integrating over v_x we obtain

$$H(v_y, y) = \int_{-\infty}^{\infty} I\, dv_x = \frac{-A}{2ik_y} \int_0^\infty e^{j\omega s} \left(e^{jk_y y} e^{-j\frac{k_y v_y}{\omega_c} \sin \omega_c s} - e^{j\frac{k_y v_y}{\omega_c} \sin \omega_c s} e^{-jk_y y}\right) \exp -\left(\frac{k_y^2 \alpha_\perp^2}{\omega_c^2} (1 + \cos^2 \omega_c s - 2 \cos \omega_c s)\right) ds. \tag{A-4}$$

From the identity,

$$e^{jz\sin\delta} = \sum_{n=-\infty}^{\infty} J_n(z)\, e^{jn\delta}. \qquad \text{(A-5)}$$

We transform Eq. A-4 to

$$H(v_y, y) = \sum_{n=-\infty}^{\infty} \left(e^{jk_y y} J_n\left(\frac{k_y v_y}{\omega_c}\right)(-1)^n - e^{-jk_y y} J_n\left(\frac{k_y v_y}{\omega_c}\right)\right) C_n, \qquad \text{(A-6)}$$

where

$$C_n = \frac{+jA}{2k_y} \int_0^{\infty} e^{j\omega s}\, e^{jn\omega_c s} \exp\left(\frac{-k_y^2 \alpha_\perp^2}{\omega_c^2} \quad (1+\cos^2\omega_c s - 2\cos\omega_c)\right) ds,$$

and we have employed the identity

$$J_n(-z) = (-1)^n J_n(z).$$

For $H(-v_y, y)$ we have,

$$H(-v_y, y) = \sum_{n=-\infty}^{\infty} \left(e^{+jk_y y} J_n\left(\frac{k_y v_y}{\omega_c}\right) - (-1)^n e^{-jk_y y} J_n\left(\frac{k_y v_y}{\omega_c}\right)\right) C_n. \qquad \text{(A-7)}$$

Equations A-6 and A-7 may be written in the form,

$$H(v_y, y) = \sum_{n=-\infty}^{\infty} J_n\left(\frac{k_y v_y}{\omega_c}\right) C_n\, 2j\sin k_y y + \sum_{n=-\infty}^{\infty} J_n\left(\frac{k_y v_y}{\omega_c}\right) C_n\, 2\cos(k_y y).$$

$$H(-v_y, y) = \sum_{n=-\infty}^{\infty} J_n\left(\frac{k_y v_y}{\omega_c}\right) C_n\, 2j \sin k_y y - \sum_{n=-\infty}^{\infty} J_n\left(\frac{k_y v_y}{\omega_c}\right) C_n\, 2 \cos (k_y y) . \tag{A-8}$$

Hence from Eq. A-1,

$$H(v_y, a) = H(-v_y, a) . \tag{A-9}$$

Thus the dispersion relation for the infinite medium may be applied to the case of perfectly reflective walls. Substituting

$$k_y = \frac{(2n+1)}{2a} \pi$$

into Eq. 4 and letting k_z go to zero, we obtain the dispersion relation,

$$\left(\frac{(2n+1)\pi}{2a}\right)^2 = -\omega_p^2 \sum_{n=-\infty}^{\infty} e^{-\frac{\lambda^2}{2}} I_n \frac{\lambda^2}{2} \left(\frac{2n\omega_c}{\alpha_\perp^2 (\omega + n\omega_c)}\right) . \tag{A-10}$$

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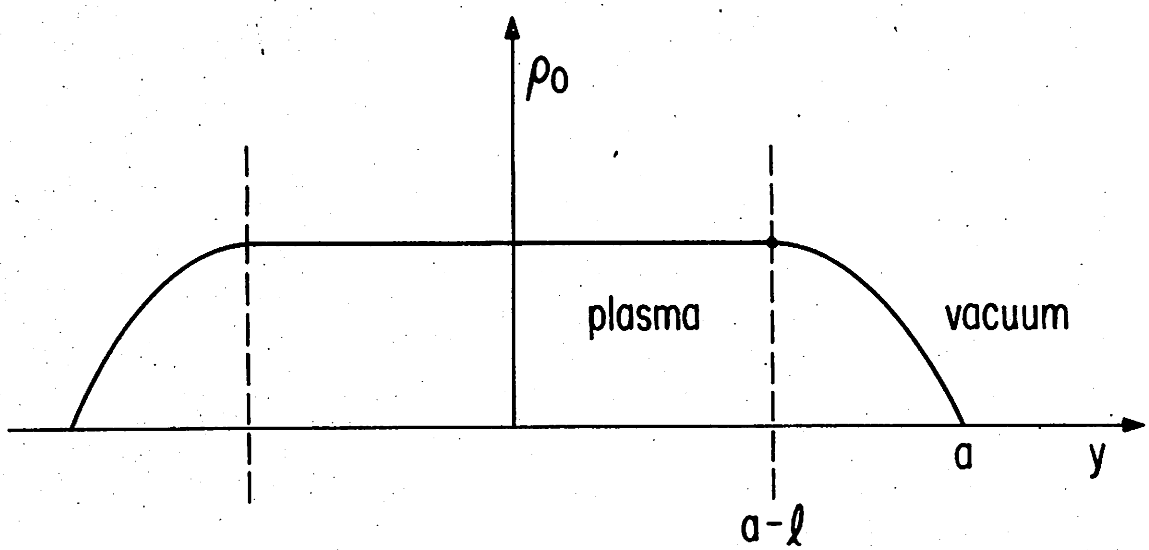

Fig. 1. Density profile of plasma with uniform and boundary layer regions.

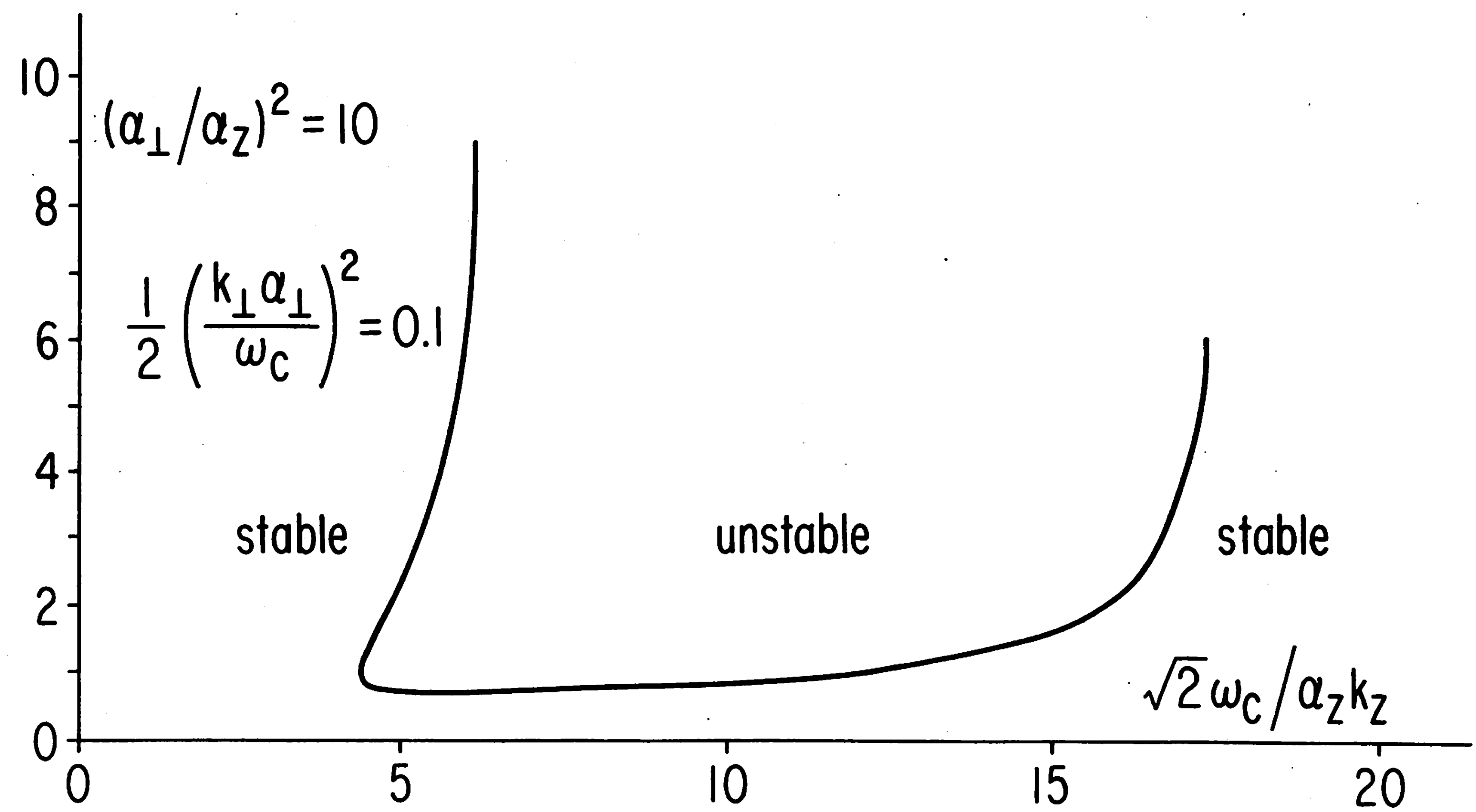

Fig. 2. Limits of stability for a plasma with temperature anisotropy.

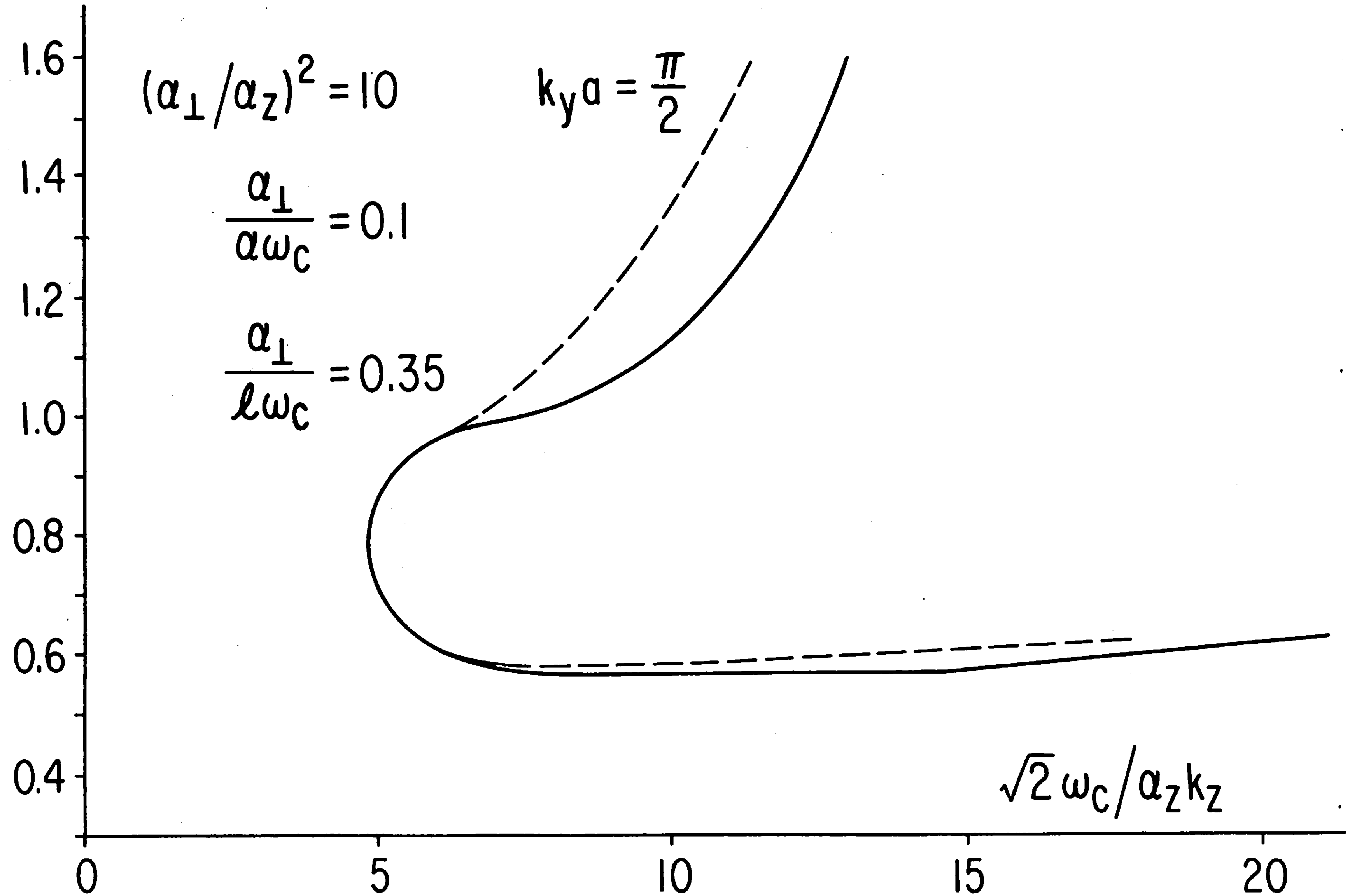

Fig. 3. Limits of stability for a bounded plasma with temperature anisotropy.