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SUMMARY

The purpose of this paper is to establish the stability of single-loop feedback systems with one differentiable nonlinear element. In those cases where the Popov criterion fails to guarantee stability for the entire sector predicted by Aizerman's conjecture, new results can be obtained by restricting the slope of the nonlinear function. A new frequency domain stability criterion is obtained which, like the Popov criterion, has only one unknown parameter. Thus, a simple graphical interpretation is possible. Examples are given which show a considerable improvement over the Popov criterion.

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INTRODUCTION

Recently, several authors [1-5] have recognized the fact that Aizerman's conjecture is not true in general [6] and that in some cases the V. M. Popov theorem [7] does not guarantee absolute stability for the entire Hurwitz sector as predicted by Aizerman's conjecture. In these papers [1-5], additional restrictions on the slope of the nonlinear element are used to guarantee absolute stability in cases where the Popov criterion is not satisfied.

In [1, 2, and 4] the authors have used a Lyapunov function approach to the problem and have confined their attention to systems whose linear plants are described by a set of ordinary linear differential equations. In [3, 5, and 8], assumptions are made concerning the input-output relation of the linear part of the system but its internal dynamics are not specified. This allows consideration of rather general distributed parameter systems.

In [1, 2, and 5] the stability criteria have two unknown parameters and thus the graphical procedure is more complicated than that for the Popov criterion which has only one parameter. In [3] the criterion has three unknown parameters and a graphical interpretation is virtually impossible. In this paper assumptions are made similar to those in [3] and a stability criterion is obtained with only one unknown parameter. Thus, the criterion has a simple graphical interpretation and it can readily be applied to frequency response data which may be obtained experimentally.

DESCRIPTION OF SYSTEM

The class of nonlinear feedback systems considered will have the configuration of Fig. 1. The block labelled N is a time-invariant memoryless nonlinear gain element whose output $\xi(t)$ is given by

$$\xi(t) = \phi[\sigma(t)] \tag{1}$$

where $\phi(\sigma)$ is a differentiable function of σ,

$$\phi(0) = 0$$

and for all $\sigma \neq 0$,

$$0 \leq \frac{\phi(\sigma)}{\sigma} \leq k \tag{2}$$

or $\frac{\phi(\sigma)}{\sigma} \geq 0$ is the case when k is infinite.

These inequalities restrict the nonlinear function to a sector in the σ, ϕ plane and we will refer to this as a nonlinearity in the sector $[0, k]$.

The further following restrictions are made on N:

a) $\phi(\sigma)$ is uniformly bounded, that is

$$|\phi(\sigma)| \leq M, \quad -\infty < \sigma < \infty, \tag{3}$$

where M is a finite constant independent of σ. In practical examples such a bound will always exist.

b) $$-k_1 < \frac{d\phi}{d\sigma} < k_2 \tag{4}$$

This will be referred to as a slope restriction $(-k_1, k_2)$ and will clearly be consistent with (2) only if

$$k_1 \geq 0 \quad \text{and} \quad k_2 \geq k. \tag{5}$$

The block labelled G is a linear time invariant subsystem described by the equation

$$y(t) = z(t) + \int_0^t g(t - \tau)\ \xi(\tau)\, d\tau, \quad t \geq 0, \tag{6}$$

where z(t) is the zero-input response of G which depends on the initial state, and g(t) is the impulse response of G.

Under the following assumptions[1] the system of Fig. 1 will be referred to as a principal case:

a) for all initial states $z(t)$ is bounded on $[0, \infty)$, $\dot{z}(t)$ is bounded and uniformly continuous on $[0, \infty)$.

b) for all initial states $z(t)$ and $\dot{z}(t)$ are elements of $L_1(0, \infty)$.

The above assumptions guarantee that $z(t)$ and $\dot{z}(t)$ are elements of $L_2(0, \infty)$ and that $g(t)$ and $\dot{g}(t)$ are elements of $L_1(0, \infty)$.

c) the input to the system, $u(t)$ satisfies all the conditions imposed on $z(t)$.

The above assumptions imply that $z(t)$, $\dot{z}(t)$, $u(t)$ and $\dot{u}(t)$ all tend to zero as $t \to \infty$.

Let $G(s)$ be the Laplace transform of $g(t)$, then, in the principal case, $G(s)$ is analytic for $\text{Re}\ s \geq 0$. The theorem is extended to particular cases where $G(s)$ has simple poles on the imaginary axis of the s-plane. In this case it is assumed that the conditions for stability in the limit are satisfied.[2] $z(t)$ and $g(t)$ are modified accordingly but the assumptions on $u(t)$ are the same as for the principal cases. Also, in place of inequalities (2) and (3), the nonlinearity is restricted to the sector $[\epsilon, k]$ and is such that $\phi(\sigma) - \epsilon\sigma$ is uniformly bounded, where $\epsilon > 0$ is arbitrarily small. The case where the only imaginary axis pole is a simple pole at the origin will be called the simplest particular case.

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[1] For an asymptotically stable linear differential subsystem, assumptions (a) and (b) are always satisfied.

[2] These conditions require that the system of Fig. 1 be asymptotically stable for a linear gain $\phi(\sigma) = \epsilon\sigma$ where $\epsilon > 0$ is small. This is a linear problem and graphical conditions for stability in the limit are given by a theorem in [7]. If the residues at the imaginary axis poles have positive real parts, then these conditions are satisfied.

Theorem

For the system shown in Fig. 1, if there exists a finite number q such that for all $\omega \geq 0$,

a) $$H(\omega) = \mathrm{Re}\, j\omega q G(j\omega) + \omega^2 \{1 + (k_2 - k_1)\mathrm{Re}\, G(j\omega) - k_1 k_2 |G(j\omega)|^2\} \geq 0, \tag{7}$$

b) $$G(j\omega) \neq -\frac{1}{k}, \quad G(0) > -\frac{1}{k}, \tag{8}$$

then in the principal case, for all nonlinearities with slope restriction $(-k_1, k_2)$ in the sector $[0, k]$ and for all initial states, the response $y(t)$ is bounded on $[0, \infty)$ and tends to zero as $t \to \infty$.

In the simplest particular case the theorem remains true for all nonlinearities $\phi(\sigma)$ in the sector $[\epsilon, k]$ such that $\phi(\sigma) - \epsilon\sigma$ is bounded on $(-\infty, \infty)$, where $\epsilon > 0$ is arbitrarily small.

When k is infinite, condition (b) becomes

$$G(0) \geq 0. \tag{9}$$

Corollary 1: With the slope restriction $\phi' > -k_1$, condition (a) becomes

$$\mathrm{Re}\, j\omega q G(j\omega) + \omega^2 \{\mathrm{Re}\, G(j\omega) - k_1 |G(j\omega)|^2\} \geq 0. \tag{10}$$

Corollary 2: With the slope restriction $\phi' < k_2$, condition (a) becomes

$$\mathrm{Re}\, j\omega q G(j\omega) - \omega^2 \{\mathrm{Re}\, G(j\omega) + k_2 |G(j\omega)|^2\} \geq 0. \tag{11}$$

Corollary 3: With the slope restriction $(0, k_2)$, condition (a) becomes

$$\mathrm{Re}\, j\omega q G(j\omega) + \omega^2 \{\mathrm{Re}\, G(j\omega) + \frac{1}{k_2}\} \geq 0. \tag{12}$$

In this and the following corollaries, all the particular cases may be considered (see Remark 1).

Corollary 4: With the slope restriction $\phi' > 0$, condition (a) becomes

$$\mathrm{Re}\,(j\omega q + \omega^2)\, G(j\omega) \geq 0. \tag{13}$$

The inequalities may be tested analytically or graphically as is done with the Popov theorem. Inequalities (7), (10), (11), and (12) are useful mainly for analysis problems where at least the values of k_1 and k_2 are known. Inequality (12) may be used to find the maximum value of k for stability with a strictly monotone increasing nonlinearity.

The graphical technique is similar to that used with the Popov criterion. For inequality (7) we plot

$$Y = \omega \; \mathrm{Im}\; G(j\omega) \qquad \omega \geq 0$$

against

$$X = \omega^2 \{ 1 + (k_2 - k_1)\, \mathrm{Re}\; G(j\omega) - k_1 k_2 \, |G(j\omega)|^2 \} .$$

Then (7) becomes

$$X - qY \geq 0.$$

If there exists a straight line of slope $1/q$ through the origin such that the $X - Y$ plot lies to the right of it, then inequality (7) is satisfied.

The other inequalities are handled in a similar manner.

Remark 1. In the particular cases, inequalities (7), (10) and (11) can only be satisfied for the simplest particular case. If $G(s)$ has a pole at $s = j\omega_0$, then for $\omega_0 \neq 0$

$$\lim_{\omega \to \omega_0} \mu \omega^2 \, |G(j\omega)|^2 = +\infty .$$

Since $H(\omega)$ contains the term $-\mu\omega^2 \, |G(j\omega)|^2$, it is impossible to satisfy $H(\omega) \geq 0$.

Remark 2. Conditions (a) and (b) of the Theorem (or inequality (9) for the case $k = \infty$) guarantee the satisfaction of the Nyquist criterion for linear gains in the interval $[0, k]$. Let $\{\omega_i\}$ be the frequencies for which $\text{Im } G(j\omega) = 0$ and let ω_m be that frequency for which $G(j\omega_i)$ is a minimum over i. Then from (7),

$$\omega_m^2 \{1 + (k_2 - k_1) G(j\omega_m) - k_1 k_2 |G(j\omega_m)|^2\} \geq 0,$$

that is,

$$\omega_m^2 k_2 \{G(j\omega_m) + \frac{1}{k_2}\} \{1 - k, G(j\omega_m)\} \geq 0.$$

Now $G(j\omega_m) \leq 0$ and $\frac{1}{k_2} \leq \frac{1}{k}$ so for $\omega_m \neq 0$ this implies that

$$G(j\omega_m) \geq -\frac{1}{k},$$

which, when combined with condition (b) or inequality (9), is the Nyquist criterion.

Remark 3. In the proof of the Theorem we may restrict ourselves to the principal case since the particular cases may be reduced to the principal case by the transformation $\tilde{\xi} = \xi - \epsilon\sigma$. This transformation changes the characteristics of the nonlinear function ϕ to $\tilde{\phi} = \phi - \epsilon\sigma$ and the frequency response of the subsystem $G(j\omega)$ to $\tilde{G}(j\omega)$, where

$$\tilde{G} = \frac{G}{1 + \epsilon G}. \qquad (14)$$

It must be noted that the transformed system differs from the original system only in notation. Now $\tilde{\phi}' = \phi' - \epsilon$ and it can be shown [8] that the transformed system $\tilde{G}$ satisfies all the conditions of a principal case.

Hence, forming the expression

$$\tilde{H}(\omega) = \text{Re } j\omega q\tilde{G}$$

$$+ \omega^2 \{1 + (k_2 - k_1 - 2\epsilon) \text{ Re } \tilde{G} - (k_1 + \epsilon)(k_2 - \epsilon) \, |\tilde{G}|^2 \} \, ,$$

and substituting for $\tilde{G}$ from Eq. (14), we have

$$\tilde{H} \, |1 + \epsilon G|^2 = H, \quad \text{where } H \text{ is given by Eq. (7).}$$

Thus, $H(\omega) \geq 0$ immediately implies $\tilde{H}(\omega) \geq 0$. Also from Eq. (14)

$$G(j\omega) \neq -\frac{1}{k} \text{ implies } \tilde{G}(j\omega) \neq -\frac{1}{k - \epsilon} \, ,$$

and

$$G(0) > -\frac{1}{k} \text{ implies } \tilde{G}(0) > -\frac{1}{k - \epsilon} \, .$$

Once the theorem is proved for the principal case, this implies that for $\tilde{\phi}$ bounded on $(-\infty, \infty)$ with slope restriction $(-k_1 - \epsilon, \, k_2 - \epsilon)$ in the sector $[0, \, k - \epsilon]$, the output of the transformed system is bounded and tends to zero as $t \to \infty$. This then implies that the theorem remains true for the original system G for ϕ with slope restriction $(-k_1, \, k_2)$ in the section $[\epsilon, \, k]$ such that $\phi - \epsilon\sigma$ is bounded on $(-\infty, \infty)$ where $\epsilon > 0$ is arbitrarily small.

On the strength of the above remarks only the proof of the theorem for the principal case will be given.

Preliminaries

From Fig. 1 $\quad \sigma(t) = u(t) - y(t)$,

and using Eq. (6)

$$\sigma(t) = u(t) - z(t) - \int_0^t g(t - \tau) \, \xi(\tau) \, d\tau, \qquad t \geq 0. \tag{15}$$

Let $\sigma(t)$ be a solution of Eq. (15) with an arbirary fixed function $\phi(\sigma)$ which satisfies Eqs. (2), (3) and (6).

Then $\xi(t) = \phi[\sigma(t)]$ is a fixed function of time. Let

$$\xi_T(t) = \begin{cases} \xi(t) & \text{for } 0 \leq t \leq T \\ \xi^T(t) & \text{for } t > T, \end{cases} \tag{16}$$

where T is an arbitrary fixed positive number and $\xi^T(t)$ is the system trajectory obtained by replacing, for all $t > T$, the nonlinear function $\phi(\sigma)$ by the linear function $h\sigma$, where

$$h = \frac{\xi(T)}{\sigma(T)} \text{ and clearly } h \in [0, k] . \tag{17}$$

Then $\xi_T(t)$ is continuous at $t = T$, and from the Nyquist criterion [9], $\xi_T(t) \in L_2(0, \infty)$. Let the Fourier transform of $\xi_T(t)$ be

$$X_T(j\omega) = \int_0^{\infty} \xi_T(t)\, e^{-j\omega t}\, dt . \tag{18}$$

Let

$$\sigma_T(t) = u(t) - z(t) - \int_0^t g(t-\tau)\, \xi_T(\tau)\, d\tau, \quad t \geq 0 \tag{19}$$

$$= u(t) - z(t) + \tilde{\sigma}_T . \tag{20}$$

Then it is clear that

$$\sigma_T(t) = \begin{cases} \sigma(t) & \text{for } 0 \leq t \leq T \\ \frac{1}{h}\, \xi^T(t) & \text{for } t > T. \end{cases} \tag{21}$$

$\sigma_T(t)$ is continuous at $t = T$ and, by the same reasoning as before, $\sigma_T(t) \in L_2(0, \infty)$.

Let

$$\theta_T(t) = \xi_T(t) - \xi(0)\, e^{-\alpha t}, \qquad \alpha > 0, \ t \geq 0, \tag{22}$$

then

$$\theta_T(0) = 0, \quad \text{and}$$

$$\dot{\theta}_T(t) = \frac{d}{dt}\, \theta_T(t) = \dot{\xi}_T(t) + \alpha\, \xi(0)\, e^{-\alpha t}, \qquad t \geq 0. \tag{23}$$

Let

$$\gamma_T(t) = - \int_0^t g(t-\tau)\, \theta_T(\tau)\, d\tau \tag{24}$$

$$= - \int_0^t g(t-\tau)\, \xi_T(\tau) + \xi(0) \int_0^t g(t-\tau)\, e^{-\alpha\tau}\, d\tau$$

$$= \tilde{\sigma}_T(t) + \xi(0) \int_0^t g(t-\tau)\, e^{-\alpha\tau}\, d\tau$$

$$= \sigma_T(t) - u(t) + z(t) + \xi(0) \int_0^t g(t-\tau)\, e^{-\alpha\tau}\, d\tau$$

$$= \sigma_T(t) - f(t), \tag{25}$$

where

$$f(t) = u(t) - z(t) - \xi(0) \int_0^t g(t-\tau)\, e^{-\alpha\tau}\, d\tau. \tag{26}$$

From the assumptions on G, it follows that $\theta_T(t)$, $\dot{\theta}_T(t)$, $\gamma_T(t)$, and $\dot{\gamma}_T(t)$ are all elements of $L_2(0, \infty)$. So let $\Theta_T(j\omega)$ and $\Gamma_T(j\omega)$ be the Fourier transforms of $\theta_T(t)$ and $\gamma_T(t)$, respectively.

From Eq. (26) it follows that

$$\Gamma_T(j\omega) = -G(j\omega)\, \Theta_T(j\omega),$$

and $\gamma_T(0) = 0.$

PROOF OF THE THEOREM

Let

$$\rho(T) = \int_0^{\infty}\Big[q\,\dot{\gamma}_T(t)\,\theta_T(t) + \{k_1\dot{\gamma}_T(t) + \dot{\theta}_T(t)\}\{k_2\dot{\gamma}_T(t) - \dot{\theta}_T(t)\}\Big]\,dt\,. \tag{27}$$

Then using Parseval's equality,

$$\rho(T) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[q\,j\,\omega\,\Gamma_T(j\omega)\,\overline{\Theta_T(j\omega)}$$

$$+ \{k_1\,j\,\omega\,\Gamma_T(j\omega) + j\,\omega\,\Theta_T(j\omega)\}\,\{\overline{k_2\,j\,\omega\,\Gamma_T(j\omega) - j\,\omega\,\Theta_T(j\omega)}\}\Big]\,d\,\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[q\{-j\,\omega\,G(j\omega)\,\Theta_T(j\omega)\}\,\overline{\Theta_T(j\omega)}$$

$$+ \{-k_1\,j\,\omega\,G(j\omega)\,\Theta_T(j\omega) + j\,\omega\,\Theta_T(j\omega)\}\,\{\overline{-k_2\,j\,\omega\,G(j\omega)\,\Theta_T(j\omega) - j\,\omega\,\Theta_T(j\omega)}\}\Big]\,d\,\omega\,.$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[j\,\omega\,q\,G(j\omega) +$$

$$\omega^2\{1 + k_2\,\overline{G(j\omega)} - k_1\,G(j\omega) - k_1 k_2\,|\,G(j\omega)\,|^2\}\Big]\,|\,\Theta_T(j\omega)\,|^2\,d\,\omega\,.$$

Now $\rho(T)$ is real, so

$$\rho(T) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\,|\,\Theta_T(j\omega)\,|^2\,d\,\omega\,,$$

where $H(\omega)$ is given by Eq. (7) and is nonnegative. Hence it follows that $\rho(T) \le 0$.

Thus, from Eq. (27)

$$\int_0^{\infty}\left[q\,\dot{\gamma}_T(t)\,\theta_T(t) + \{k_1\dot{\gamma}_T(t) + \dot{\theta}_T(t)\}\{k_2\dot{\gamma}_T(t) - \dot{\theta}_T(t)\}\right]dt \le 0. \tag{28}$$

It is shown in the Appendix that this inequality results in the inequality

$$\int_0^T (k_1 + \phi')(k_2 - \phi')\,\dot{\sigma}(t)^2\,dt \le C^* < \infty,$$

where the constant C^* is independent of T. Hence, it follows that

$$\int_0^{\infty} (k_1 + \phi')(k_2 - \phi')\,\dot{\sigma}(t)^2\,dt < \infty. \tag{29}$$

Since we have assumed that $|\phi(\sigma)| \le M$, it follows that $|\xi(t)| \le M$. Since $g(t)$ is an element of $L_1(0, \infty)$, it then follows that $y(t)$ and hence $\sigma(t)$ are bounded on $(0, \infty)$. Let $|\sigma(t)| \le M_1$.

Now it was assumed that $-k_1 < \phi'(\sigma) < k_2$, therefore, in the bounded region $|\sigma| < M_1$ this implies that $-(k_1 - \epsilon_1) \le \phi'(\sigma) \le k_2 - \epsilon_1$ where $\epsilon_1 > 0$ is a small number which depends only on M_1.

So from Eq. (29) we have

$$\infty > \int_0^{\infty} (k_1 + \phi')(k_2 - \phi')\,\dot{\sigma}(t)^2\,dt$$

$$\ge \epsilon_1^2 \int_0^{\infty} \dot{\sigma}(t)^2\,dt.$$

That is, $\dot{\sigma}(t)$ is an element of $L_2(0, \infty)$.

Since $g(t)$ is an element of $L_1(0, \infty)$ it follows from Eq. (15) that $\dot{\sigma}(t)$ is uniformly continuous on $[0, \infty)$. From a lemma in [8] it then follows that, for all initial states,

$$\lim_{t \to \infty} \dot{\sigma}(t) = 0.$$

This implies that as $t \to \infty$, $\sigma(t) \to \sigma_0$ and $\xi(t) \to \phi(\sigma_0)$. Now from Eq. (15),

$$\sigma(t) = u(t) - z(t) - \int_0^t \xi(t - \tau)\, g(\tau)\, d\tau.$$

Let $t \to \infty$, then

$$\sigma_0 = - \int_0^\infty \phi(\sigma_0)\, g(\tau)\, d\tau .$$

that is,

$$\sigma_0 + G(0)\, \phi(\sigma_0) = 0 .$$

Since $G(0) > -\frac{1}{k}$ (≥ 0 when k is infinite), this is impossible unless

$$\sigma_0 = \phi(\sigma_0) = 0.$$

Hence

$$\lim_{t \to \infty} \sigma(t) = 0.$$

It then follows that $\lim_{t \to \infty} \xi(t) = 0$ and $\lim_{t \to \infty} y(t) = 0$ which completes the proof of the theorem.

The proof of the corollaries is identical to that of the theorem except that in Eq. (27) for $\rho(T)$, some of the terms are deleted.

EXAMPLES

1. Consider the system shown in Fig. 1 where the linear subsystem G has the transfer function

$$G(s) = \frac{s^2}{s^3 + a s^2 + b s + c} \qquad \begin{matrix} a, b, c, > 0 \\ ab - c > 0 . \end{matrix}$$

The frequency response function $G(j\omega)$ is given by

$$G(j\omega) = \frac{-\omega^2}{(c - a\omega^2) + j\omega(b - \omega^2)} .$$

The system is asymptotically stable for all positive linear gains.

For $a = 2$, $b = c = 1$, the modified Nyquist plot and Popov line are shown in Fig. 2. The Popov sector is found to be $[0, 3 - \epsilon]$ where $\epsilon > 0$ is arbitrarily small.

Using Corollary 4, we first note that $G(0) = 0$.

$$\mathrm{Re}(j\omega q + \omega^2)\, G(j\omega) = -\frac{q\omega^4(b - \omega^2) + \omega^4(a\omega^2 - c)}{(c - a\omega^2)^2 + \omega^2(b - \omega^2)^2}$$

$$= \frac{\omega^4[(a + q)\omega^2 - (c + qb)]}{(c - a\omega^2)^2 + \omega^2(b - \omega^2)^2} .$$

Choose $q = \frac{c}{b}$, then

$$\mathrm{Re}(j\omega q + \omega^2)\, G(j\omega) = \frac{\omega^6\left(a - \frac{c}{b}\right)}{(c - a\omega^2)^2 + \omega^2(b - \omega^2)^2} .$$

$$\geq 0 \text{ for all } \omega$$

The graphical interpretation of this inequality, for the above numerical values, is shown in Fig. 3.

Hence, for all bounded nonlinearities with slope restriction $\phi' > 0$ and for all initial states, the response $y(t)$ is bounded on $[0, \infty)$ and tends to zero as $t \to \infty$.

Actually, for the given numerical example, a stronger result can be obtained using Corollary 1. From inequality (10) with $k_1 = 1$,

$$H(\omega) = \text{Re } j\omega q G(j\omega) + \omega^2 \{\text{Re } G(j\omega) - k_1 |G(j\omega)|^2\}$$

$$= \frac{-q\omega^4(1-\omega^2) + \omega^4(2\omega^2 - 1) - \omega^6}{(1-2\omega^2)^2 + \omega^2(1-\omega^2)^2}$$

$$= \frac{\omega^4[(2+q-1)\omega^2 - (1+q)]}{(1-2\omega^2)^2 + \omega^2(1-\omega^2)^2}$$

Choose $q = -1$, then $H(\omega) \equiv 0$.

Hence, for all bounded nonlinearities with slope restriction $\varphi' > -1$ in the sector $[0, \infty)$ and for all initial states, the response $y(t)$ is bounded on $[0, \infty)$ and tends to zero as $t \to \infty$.

2. A particular case given by Aizerman and Gantmacher [7].

Consider the system shown in Fig. 1 with

$$G(s) = \frac{s^2 - b}{(s^2+1)(s-c)} \qquad b > 0, \; c > 0, \; b > c^2.$$

Conditions for stability in the limit are satisfied [7] and the system is asymptotically stable for linear gains the sector $(0, c/b)$.

The frequency response function $G(j\omega)$ is given by

$$G(j\omega) = \frac{(\omega^2 + b)(c - j\omega)}{(\omega^2 - 1)(\omega^2 + c^2)}.$$

The Popov sector is found to be $[\epsilon, 1/c]$, where $\epsilon > 0$ is arbitrarily small and $1/c < c/b$. In [6] it is shown that for all nonlinearities with slope restriction $[0, c/b]$ in the sector $[\epsilon, c/b - \epsilon]$ which satisfy the condition

$$\overline{\lim_{|\sigma| \to \infty}} \left[\int_0^{\sigma} \phi(\sigma)\, d\sigma - \frac{\sigma \phi(\sigma)}{2} \right] = + \infty$$

and for all initial states, the response y(t) is bounded on $[0, \infty)$ and tends to zero as $t \to \infty$.

Using Corollary 4, we first note that $G(0) = -b/c$. Hence, $k < c/b$

$$\mathrm{Re}(j\omega q + \omega^2)\, G(j\omega) = \frac{q\omega^2(\omega^2 + b) + c\omega^2(\omega^2 + b)}{(\omega^2 - 1)(\omega^2 + c^2)}$$

$$= \frac{\omega^2(q + c)(\omega^2 + b)}{(\omega^2 - 1)(\omega^2 + c^2)} .$$

Choose $q = -c$, then

$$\mathrm{Re}(j\omega q + \omega^2)\, G(j\omega) = 0 \qquad \text{for all } \omega.$$

Hence for all nonlinearities $\phi(\sigma)$ with slope restriction $\phi' > 0$ in the sector $[\epsilon, c/b - \epsilon]$ such that $\phi(\sigma) - \epsilon\sigma$ is bounded on $(-\infty, \infty)$ and for all initial states, the response y(t) is bounded on $[0, \infty)$ and tends to zero as $t \to \infty$.

CONCLUSION

It has been shown that in certain cases stability results can be obtained for systems where the Popov criterion fails to verify Aizerman's conjecture. By bounding the slope of the nonlinear function, new frequency domain stability criteria are obtained. These criteria have a simple graphical interpretation and the theorem can readily be applied to experimental frequency response data. Further examples are required to illustrate these criteria, since only by example can it be shown that an improvement over the Popov theorem is possible.

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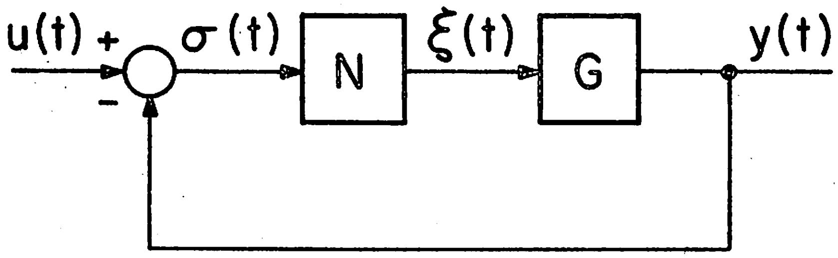

Fig. 1. Nonlinear feedback system.

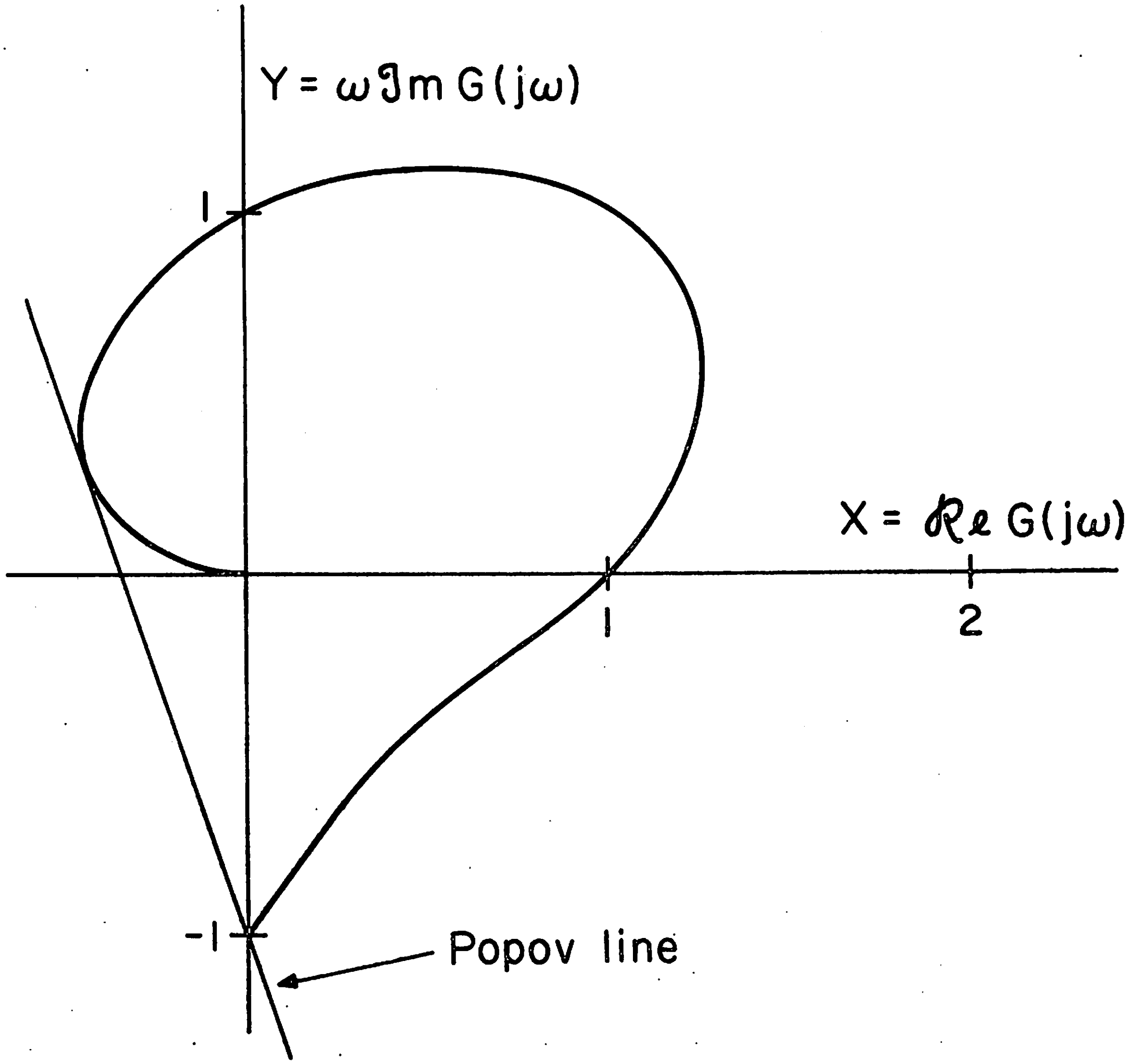

Fig. 2. Modified frequency response for example 1.

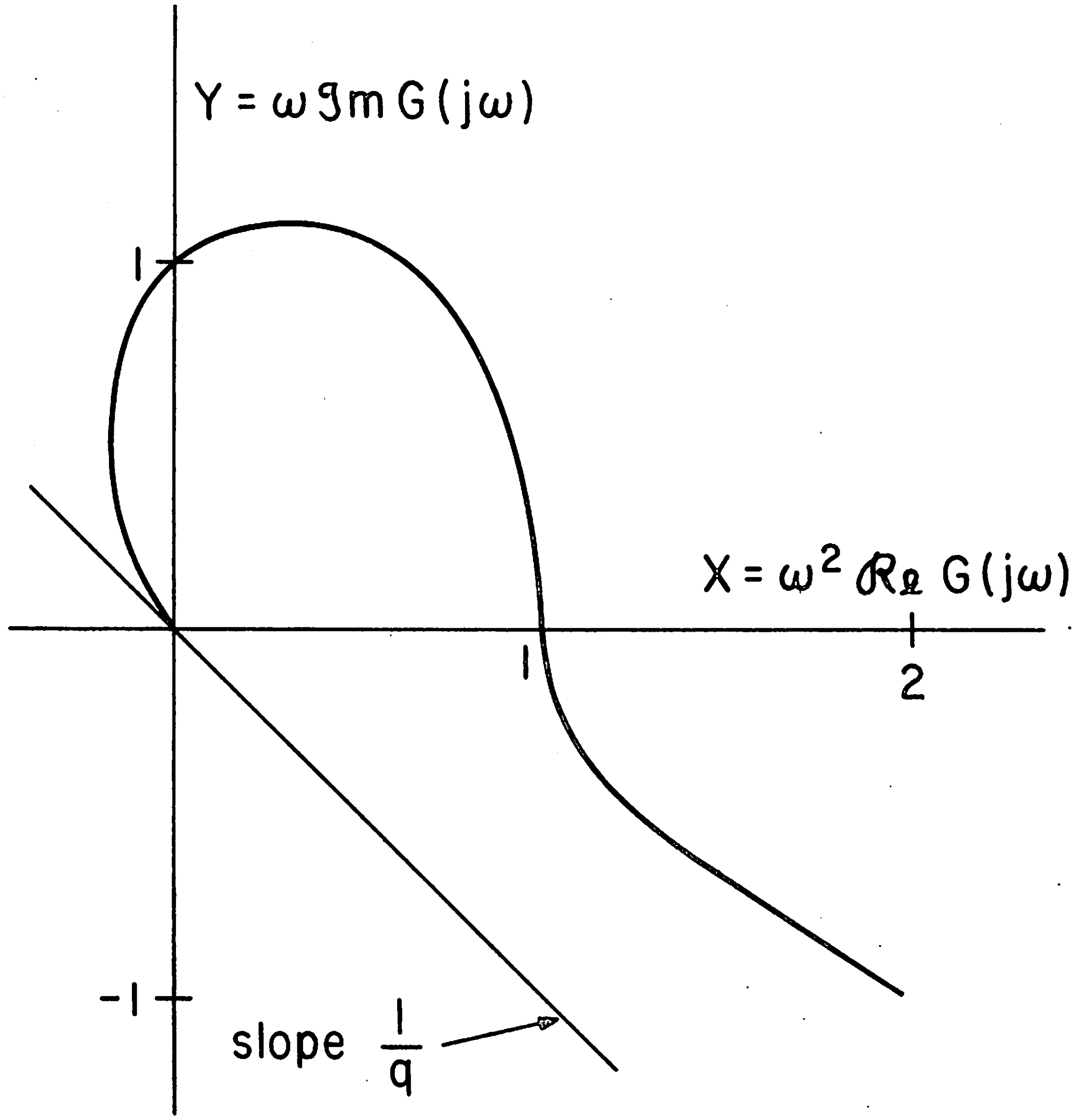

Fig. 3. Graphical interpretation of the new criterion.

APPENDIX

Substituting for $\dot{\gamma}_T(t)$ in Eq. (28) gives

$$q \int_0^\infty \{ \dot{\sigma}_T(t) - \dot{f}(t) \} \; \theta_T(t) \, dt$$

$$+ \int_0^\infty \{k_1 \dot{\sigma}_T(t) - k_1 \dot{f}(t) + \dot{\theta}_T(t)\} \{k_2 \dot{\sigma}_T(t) - k_2 \dot{f}(t) - \dot{\theta}_T(t)\} \, dt \leq 0$$

that is, substituting for $\theta_T(t)$,

$$q \int_0^\infty \dot{\sigma}_T(t) \{ \xi_T(t) - \xi(0) \, e^{-\alpha t} \} \, dt$$

$$+ \int_0^\infty \{k_1 \dot{\sigma}_T(t) + \dot{\xi}_T(t) + \alpha \, \xi(0) \, e^{-\alpha t}\} \{k_2 \dot{\sigma}_T(t) - \dot{\xi}_T(t) - \alpha \, \xi(0) \, e^{-\alpha t}\} \, dt$$

$$\leq q \int_0^\infty \dot{f}(t) \, \theta_T(t) \, dt + 2k_1 \, k_2 \int_0^\infty \dot{f}(t) \, \dot{\sigma}_T(t) \, dt$$

$$+ (k_2 - k_1) \int_0^\infty \dot{f}(t) \, \dot{\theta}_T(t) \, dt - k_1 \, k_2 \int_0^\infty \dot{f}(t)^2 \, dt \qquad (30)$$

Since $| \xi(t) | \leq M$, it follows that $| \xi_T(t) | \leq M_2$ and $| \theta_T(t) | \leq M_3$. From Eq. (15)

$$\dot{\sigma}(t) = \dot{u}(t) - \dot{z}(t) - g(0) \, \xi(t) - \int_0^t \dot{g}(t - \tau) \, \xi(t) \, d\tau$$

so

$$| \dot{\sigma}(t) | \leq | \dot{u}(t) | + | \dot{z}(t) | + M \, | g(0) | + M \int_0^\infty | \dot{g}(t) | \, dt$$

$$\leq M_4$$

and $\quad |\dot{\sigma}_T(t)| \leq M_5$.

Now

$$\dot{\xi}(t) = \phi'(\sigma)\, \dot{\sigma}(t) \quad \text{so} \quad |\dot{\xi}(t)| \leq \max(k_1, k_2)\, M_4,$$

$$|\dot{\xi}_T(t)| \leq M_6 \quad \text{and} \quad |\dot{\theta}_T(t)| = |\dot{\xi}_T(t) + \alpha\, \xi(0)\, e^{-\alpha t}| \leq M_7$$

So from inequality (30),

$$q\int_0^T \dot{\xi}_T(t)\, \dot{\sigma}_T(t)\, dt + q\int_T^\infty \dot{\xi}_T(t)\, \dot{\sigma}_T\, dt$$

$$+ \int_T^\infty \{k_1\, \dot{\sigma}_T(t) + \dot{\xi}_T(t)\}\{k_2\, \dot{\sigma}_T(t) - \dot{\xi}_T(t)\}\, dt$$

$$+ \int_T^\infty \{k_1\, \dot{\sigma}_T(t) + \dot{\xi}_T(t)\}\{k_2\, \dot{\sigma}_T(t) - \dot{\xi}_T(t)\}\, dt$$

$$- q\, \xi(0) \int_0^\infty e^{-\alpha t}\, \dot{\sigma}_T(t)\, dt + (k_2 - k_1)\, \alpha\, \xi(0) \int_0^\infty e^{-\alpha t}\, \dot{\sigma}_T(t)\, dt$$

$$- 2\alpha\, \xi(0) \int_0^\infty e^{-\alpha t}\, \dot{\xi}_T(t)\, dt - \alpha^2\, \xi(0)^2 \int_0^\infty e^{-2\alpha t}\, dt$$

$$\leq |q|\, M_3 \int_0^\infty |\dot{f}(t)|\, dt + 2k_1\, k_2\, M_5 \int_0^\infty |\dot{f}(t)|\, dt$$

$$+ |k_2 - k_1|\, M_7 \int_0^\infty |\dot{f}(t)|\, dt + k_1\, k_2 \int_0^\infty \dot{f}(t)^2\, dt$$

Now $\dot{f}(t)$ is bounded on $(0, \infty)$ and is an element of $L_1(0, \infty)$ and hence is an element of $L_2(0, \infty)$. So

$$q \int_0^T \phi(\sigma)\, \dot{\sigma}(t)\, dt + q \int_T^\infty \frac{1}{h}\, \xi^T(t)\, \dot{\xi}^T\, dt$$

$$+ \int_0^T (k_1 + \phi')(k_2 - \phi')\, \dot{\sigma}(t)^2 dt + \int_T^\infty \left[\frac{k_1}{h} + 1\right]\left[\frac{k_2}{h} - 1\right] \dot{\xi}^T(t)^2\, dt$$

$$+ \{ (k_2 - k_1)\,\alpha - q \}\ \ \xi(0) \int_0^\infty e^{-\alpha t}\, \dot{\sigma}_T(t)\, dt$$

$$- 2\alpha\ \xi(0) \int_0^\infty e^{-\alpha t}\ \dot{\xi}_T(t)\, dt - \frac{\alpha\, \xi(0)^2}{2} \leq C$$

where Ç is finite and independent of T. The fourth time in this inequality is positive and hence may be discarded. So we have

$$q\left[\int_{\sigma(0)}^{\sigma(T)} \phi(\sigma)\, d\sigma - \frac{\sigma(T)\, \phi[\sigma(T)]}{2}\right] + \int_0^T (k_1 + \phi')(k_2 - \phi')\, \dot{\sigma}(t)^2\, dt$$

$$\leq C + q \int_0^{\sigma(0)} \phi)\sigma)\, d\sigma + \left|(k_2 - k_1)\,\alpha - q\right|\, |\xi(0)|\, M_5 \int_0^\infty e^{-\alpha t}\, dt$$

$$+ 2\alpha\ |\xi(0)|\ M_6 \int_0^\infty e^{-\alpha t}\, dt + \frac{\alpha}{2}\ \xi(0)^2$$

that is

$$\int_0^T (k_1 + \phi')\,(k_2 - \phi')\, \dot{\sigma}(t)^2\, dt$$

$$\leq C - q\left[\int_{\sigma(0)}^{\sigma(T)} \phi(\sigma)\, d\sigma - \frac{\sigma(T)\, \phi[\sigma(T)]}{2}\right]$$

$$+ \left|(k_2 - k_1)\,\alpha - q\right|\ |\xi(0)|\, \frac{M_5}{\alpha} + 2\ |\xi(0)|\, M_6 + \frac{\alpha}{2}\ \xi(0)^2$$

Now $\phi(\sigma)$ and $\sigma(t)$ are bounded and hence the right hand side of this inequality is bounded uniformly in T. That is,

$$\int_0^T (k_1 + \phi')(k_2 - \phi')\,\dot{\sigma}(t)^2\,dt \leq C^* < \infty$$

where C^* is independent of T.