NONLINEAR PROGRAMMING IN BANACH SPACE

by

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I. INTRODUCTION

The usual nonlinear programming problem is the following:

$$\text{Maximize } \{ f(x) \mid g(x) \geq 0, \; x \geq 0 \} , \tag{1.1}$$

where $x \in R^n$, $g : R^n \longrightarrow R^m$ is a differentiable mapping and $x \geq 0$, $g(x) \geq 0$ means each coordinate of x, respectively, g(x) is nonnegative. f is a real-valued differentiable function which represents the performance index. The first satisfactory necessary conditions that a solution of (1.1) has to satisfy were given by Kuhn and Tucker [1]. Subsequent generalizations include the papers by Arrow <u>et al</u> [2, 3]. The model that we shall consider is the following generalization:

$$\text{Maximize } \{ f(x) \mid g(x) \in A_Y, \; x \in A \} , \tag{1.2}$$

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where X, Y are real B-spaces, [1] $x \in X$, $g: X \longrightarrow Y$ is a differentiable [2] function, A_Y is a convex set in Y and A is an arbitrary subset of X. f is a real-valued differentiable function. In Sec. 2 we introduce some notation and give some useful preliminary results. In Sec. 3 we discuss the Kuhn Tucker constraint qualification [1] and the weak constraint qualification of Arrow, Hurwicz, and Uzawa [6]. In Sec. 4 we give the main results. These are similar to the K. T. necessary conditions. We also exhibit a saddle-value problem related to (1.2) when A_Y is a cone.

II. PRELIMINARY RESULTS

We introduce some terminology and define a pair of sets which we call the local cone (LC) and the local polar (LP) which are essential to our study. The relevance of these sets to (1.2) is given by Theorem 2.1.

Let X be a real B-space and X* its topological dual. Let A be a nonempty subset of X and let $\underline{x} \in A$.

Def. 2.1 By the closed cone of A at $\underline{x}$ we mean the intersection of all closed cones [3] containing the set $A - \underline{x} \triangleq \{a - \underline{x} \mid a \in A\}$. We denote this set by $C(A, \underline{x})$.

Def. 2.2 By the local closed cone of A at $\underline{x}$ we mean the set

$$LC(A, \underline{x}) \triangleq \bigcap_{n \in \mathcal{N}(\underline{x})} C(A \cap N, \underline{x})$$

where $\mathcal{N}(\underline{x})$ is the class of all neighborhoods of $\underline{x}$.

Def. 2.3 By the local polar of A at $\underline{x}$ we mean the set

$$LP(A, \underline{x}) \triangleq \{x^* \in X^* \mid \langle x^*, z \rangle ^{[4]} \leq 0 \text{ for all } z \in LC(A, \underline{x})\} .$$

Remark 2.1 (a) The local cone is always nonempty since it always contains the origin. (b) If A is convex, $C(A, \underline{x}) = LC(A, \underline{x})$ is a closed, convex cone. (c) $LP(A, \underline{x})$ is always a closed, convex cone. The following alternate characterization of $LC(A, \underline{x})$ is straightforward to prove.

Lemma 2.1 A vector $z \in LC(A, \underline{x})$ if and only if there is a sequence of vectors $\{x_n\} \subseteq X$ and a sequence of nonnegative numbers, $\{\lambda_n\}$, such that

$$x_n \rightarrow \underline{x} \quad \text{as } n \rightarrow \infty \quad \text{and,}$$

$$\lambda_n(x_n - \underline{x}) \rightarrow z \quad \text{as } n \rightarrow \infty.$$

Let A be an arbitrary subset of a real B-space X and let f be a real-valued, differentiable function of x. Consider problem (2.1).

$$\text{Maximize } \{f(x) \mid x \in A\} . \tag{2.1}$$

Theorem 2.1 If $\underline{x} \in A$ solves (2.1), then

$$f'(\underline{x})^{[5]} \in LP(A, \underline{x}) . \tag{2.2}$$

Proof Define $S_n = \{x \mid \|x - \underline{x}\| \leq \frac{1}{n}\}$ $n = 1, 2, \ldots$ Then for each $x_n \in A \cap S_n$ we have,

$$f(x_n) \leq f(\underline{x}) \quad n = 1, 2, 3, \ldots \tag{2.3}$$

and since f is differentiable

$$f(x_n) = f(\underline{x}) + \langle f'(\underline{x}), x_n - \underline{x} \rangle^{6]} + o(\| x_n - \underline{x} \|) \leq f(\underline{x}) .$$

$$\therefore \langle f'(\underline{x}), x_n - \underline{x} \rangle = f(x_n) - f(\underline{x}) + o(\| x_n - \underline{x} \|) \leq o(\| x_n - \underline{x} \|)$$

by (2.3)

$$\therefore \left\langle f'(\underline{x}), \frac{x_n - \underline{x}}{\| x_n - \underline{x} \|} \right\rangle \leq \frac{o(\| x_n - \underline{x} \|)}{\| x_n - \underline{x} \|}$$

$$\therefore \lim_{n \to \infty} \sup_{x_n \in A \cap S_n} \left\langle f'(\underline{x}), \frac{x_n - \underline{x}}{\| x_n - \underline{x} \|} \right\rangle \leq 0$$

$$\therefore \lim_{n \to \infty} \sup_{x_n - \underline{x} \in C(A \cap S_n, \underline{x})} \left\langle f'(\underline{x}), \frac{x_n - \underline{x}}{\| x_n - \underline{x} \|} \right\rangle \leq 0$$

$$\therefore \sup \{ \langle f'(\underline{x}), z \rangle \mid z \in \bigcap_{n=1}^{\infty} C(A \cap S_n, \underline{x}) = LC(A, \underline{x}) \} \leq 0$$

$$\therefore \langle f'(\underline{x}), z \rangle \leq 0 \quad \text{for all} \quad z \in LC(A, \underline{x})$$

$$\therefore \quad f'(\underline{x}) \in LP(A, \underline{x}) \quad \text{by Def. 2.3 .}$$ Q. E. D.

<u>Corollary 2.1</u> Let A be convex and f a concave function. Then $\underline{x} \in A$ is a solution of (2.1) if and only if

$$f'(\underline{x}) \in LP(A, \underline{x}) . \tag{2.2}$$

<u>Proof</u> Suppose $\underline{x}$ satisfies (2.2). Since A is convex, by Remark 2.1 (b) $LC(A, \underline{x}) = C(A, \underline{x})$. Therefore by Def. 2.1

$$\langle f'(\underline{x}), x - \underline{x} \rangle \leq 0 \quad \text{for all} \quad x \in A. \tag{2.4}$$

Since f is concave, for $x \in A$,

$$f(x) \leq f(\underline{x}) + \langle f'(\underline{x}), x - \underline{x} \rangle$$

$$\leq f(\underline{x}) \text{ by (2.4).} \qquad \text{Q. E. D.}$$

III. CONSTRAINT QUALIFICATION

The preceding results indicate the relevance of the local cone and the local polar to maximization problems which only consider first variations. Now in most problems of interest, the constraint set (the set A in (2.1)) is not given explicitly but indirectly via some functional constraints. For example, in nonlinear programming this is given as in (1.1); in optimal control problems there are dynamic constraints represented by differential equations and so on. In each of these areas, it can be shown [7] that the main results give sufficient conditions under which the local cone and the local polar can be explicitly determined or approximated. In the case of nonlinear programming these conditions have come to be known as constraint qualifications.

Let X and Y be B-spaces; $g: X \longrightarrow Y$ a differentiable function. Let A_Y be a nonempty subset of Y, and $A_X \triangleq \{x \mid g(x) \in A_Y\} = g^{-1}\{A_Y\}$. (Note: The definitions given below closely parallel Arrow et al [6].) Let $\underline{x} \in A_X$.

Def. 3.1 We say that a vector $z \in X$ is an attainable direction at $\underline{x}$ if there exists an arc $\{x(\theta) \mid 0 \leq \theta \leq 1\} \subseteq A_X$ such that

(1) $x(0) = \underline{x}$

(2) $x(\theta)$ is differentiable from the right at $\theta = 0$ and $\left.\frac{dx}{d\theta}\right|_{\theta=0} \triangleq x'(0) = z$.

Let $AD(\underline{x}) \triangleq \{z \mid z \text{ is an attainable direction at } \underline{x}\}$. Clearly $AD(\underline{x})$ is a cone.

Let $A(\underline{x}) \triangleq$ the closed convex cone generated by $AD(\underline{x})$.

Def. 3.2 We say that a vector $z \in X$ is a locally constrained direction at $\underline{x}$ if $\langle g'(\underline{x}), z \rangle \equiv g'(\underline{x})(z) \in LC(A_Y, g(\underline{x}))$.

Let $L(\underline{x}) \triangleq \{z \mid z$ is a locally constrained direction at $\underline{x}\}$. Clearly $L(\underline{x})$ is a closed cone.

Lemma 3.1 $AD(\underline{x}) \subseteq LC(A_X, \underline{x})$. Hence

$$A(\underline{x}) \subseteq Co(LC(A_X, \underline{x})).$$ [7]

Proof Let $z \in AD(\underline{x})$. Then $z = x'(0)$ where $\{x(\theta) \mid 0 \leq \theta \leq 1\} \subseteq A_X$ and $x(0) = \underline{x}$.

$$x(\theta) = x(0) + \theta x'(0) + o(\theta) = \underline{x} + \theta z + o(\theta). \tag{3.1}$$

Let N be an arbitrary neighborhood of $\underline{x}$ and $\theta(N) > 0$ be sufficiently small so that

$$x(\theta) \in A_X \cap N, \qquad \forall \theta \leq \theta(N).$$

$$\therefore x(\theta) - \underline{x} \in A_X \cap N - \underline{x} \subseteq C(A_X \cap N, \underline{x})$$

$$\therefore z + \frac{o(\theta)}{\theta} \in C(A_X \cap N, \underline{x}) \text{ by (3.1)}$$

$$\therefore z \in C(A_X \cap N, \underline{x}) \text{ since it is closed.}$$

Since N was arbitrary, $z \in \bigcap_{N \in \mathcal{N}(\underline{x})} C(A_X \cap N, \underline{x}) = LC(A_X, \underline{x})$.

Q. E. D.

Lemma 3.2 $LC(A_X, \underline{x}) \subseteq L(\underline{x})$. Hence

$$Co(LC(A_X, \underline{x})) \subseteq Co(L(\underline{x})) .$$

Proof Let $z \in LC(A_X, \underline{x})$. By Lemma 2.1, there is a sequence $\{x_n\} \subseteq A_X$ and a sequence of nonnegative numbers $\{\lambda_n\}$ such that

$$x_n \longrightarrow \underline{x} \quad \text{as} \quad n \longrightarrow \infty \quad \text{and,}$$

$$\lambda_n(x_n - \underline{x}) \longrightarrow z \quad \text{as} \quad n \longrightarrow \infty.$$

Let N be an arbitrary neighborhood of $g(\underline{x})$ in Y. Let n(N) be sufficiently large so that

$$g(x_n) \in A_Y \cap N \text{ for all } n > n(N)$$

$$\therefore g(x_n) - g(\underline{x}) \in \{A_Y \cap N - g(\underline{x})\} \subseteq C(A_Y \cap N, g(\underline{x}))$$

$$\text{for all } n \geq n(N).$$

$$\therefore \frac{g(x_n) - g(\underline{x})}{\|x_n - \underline{x}\|} = \left\langle g'(\underline{x}), \frac{x_n - \underline{x}}{\|x_n - \underline{x}\|} \right\rangle + \frac{o(\|x_n - \underline{x}\|)}{\|x_n - \underline{x}\|}$$

$$\in C(A_Y \cap N, g(\underline{x})) \text{ for } n \geq n(N) . \tag{3.2}$$

Since $g(x_n) = g(\underline{x}) + \langle g'(\underline{x}), x_n - \underline{x} \rangle + o(\|x_n - \underline{x}\|)$.

Also, $z = \lim \lambda_n(x_n - \underline{x})$ so that if $z \neq 0$, we have

$$\frac{z}{\|z\|} = \lim \frac{\lambda_n(x_n - \underline{x})}{\|\lambda_n(x_n - \underline{x})\|} = \lim \frac{x_n - \underline{x}}{\|x_n - \underline{x}\|} . \tag{3.3}$$

In (3.2) let $n \longrightarrow \infty$; then by (3.3) we have

$$\langle g'(\underline{x}),\ z \rangle \in C(A_Y \cap N,\ g(\underline{x})) \text{ for } z \neq 0.$$

Since 0 always belongs to $C(A_Y \cap N,\ g(\underline{x}))$ we have for any neighborhood N of $g(\underline{x})$ that

$$\langle g'(\underline{x}),\ z \rangle \in C(A_Y \cap N,\ g(\underline{x}))$$

$$\langle g'(\underline{x}),\ z \rangle \in \bigcap_{n \in \mathcal{N}(g(\underline{x}))} C(A_Y \cap N,\ g(\underline{x})) = LC(A_Y,\ g(\underline{x})) .$$

$\therefore\ z \in L(\underline{x})$. Q. E. D.

Combining the previous two lemmas, we have

<u>Lemma 3.3</u> (a) $AD(\underline{x}) \subseteq LC(A_X,\ \underline{x}) \subseteq L(\underline{x})$,

(b) $A(\underline{x}) \subseteq Co(LC(A_X,\ \underline{x})) \subseteq Co(L(\underline{x}))$.

<u>Def. 3.3</u> We say that $(g,\ A_X,\ A_Y)$ satisfies <u>the Kahn Tucker constraint qualification (KT)</u>.

$$AD(\underline{x}) \supseteq L(\underline{x}) \text{ for all } \underline{x} \in A_X .$$

<u>Def. 3.4</u> We say that $(g,\ A_X,\ A_Y)$ satisfies <u>the weak constraint qualification (W)</u> if

$$A(\underline{x}) \supseteq L(\underline{x}) \text{ for all } \underline{x} \in A_X .$$

<u>Remark 3.1</u> KT $\Longrightarrow$ W since $AD(\underline{x}) \subseteq A(\underline{x})$.

<u>Corollary 3.1</u> (a) If (g, A_X, A_Y) satisfies KT, then

$$LC(A_X, \underline{x}) = L(\underline{x}) = \{z \mid \langle g'(\underline{x}), z \rangle \in LC(A_Y, g(\underline{x}))\} .$$

(b) If (g, A_X, A_Y) satisfies W, <u>and</u> if A_Y is a convex subset of Y, then

$$Co(LC(A_X, \underline{x})) = L(\underline{x}) = \{z \mid \langle g'(\underline{x}), z \rangle \in LC(A_Y, g(\underline{x}))\} .$$

<u>Proof</u> (a) follows from Lemma 3.3 (a) and Def. 3.3.

(b) follows from Lemma 3.3 (b), Def. 3.4 and the fact that A_Y convex implies $\{A_Y - g(\underline{x})\}$ is convex so that

$$LC(A_Y, g(\underline{x})) = Co(LC(A_Y, g(\underline{x}))$$

by Remark 2.1 (b). Q. E. D.

<u>Remark 3.2</u> It was demonstrated by Theorem 2.1 that the sets unimportant to our discussion are $LC(A_X, \underline{x})$ and $LP(A_X, \underline{x})$. Now usually the constraint set A_X is given indirectly as $g^{-1}\{A_Y\}$ and hence cannot be explicitly determined. The constraint qualifications, introduced above, enable us to determine the unknown set $LC(A_X, \underline{x})$ from the known set $LC(A_Y, g(\underline{x}))$ by Corollary 3.1. In fact, as is shown in the next result, the set $LP(A_X, \underline{x})$ has an even simpler form if a constraint qualification is satisfied.

<u>Theorem 3.1</u> Let A_Y be a convex set in Y and suppose that (g, A_X, A_Y) satisfies W. Let $\underline{x} \in A_X$, then

$$LP(A_X, \underline{x}) = \overline{LP(A_Y, g(\underline{x})) \cdot g'(\underline{x})}$$ [8] where

$$LP(A_Y, g(\underline{x})) \cdot g'(\underline{x}) \triangleq \{y^* \cdot g'(\underline{x}) \mid y^* \in LP(A_Y, g(\underline{x}))\} .$$

Proof (a) Claim: $LP(A_X, \underline{x}) \subseteq \overline{LP(A_Y, g(\underline{x})) \cdot g'(\underline{x})} \triangleq B$.

First notice that B is a closed convex cone in X^*. Suppose $\underline{x}^* \in LP(A_X, \underline{x})$ and $\underline{x}^* \notin B$. Then by the strong separation theorem [4], $\exists z \in X$, α real, and $\epsilon > 0$, such that

$$\langle \underline{x}^*, z \rangle = \alpha > \alpha - \epsilon \geq \langle x^*, z \rangle \; \forall x^* \in B .$$

Since B is a cone we must have,

$$\langle \underline{x}^*, z \rangle > 0 \geq \langle x^*, z \rangle \; \forall x^* \in B . \tag{3.4}$$

It is easy to show that $\underline{x}^* \in LP(A_X, \underline{x})$ and $\langle \underline{x}^*, z \rangle > 0$ implies that $z \notin Co(LC(A_{\underline{X}}, \underline{x}))$. But then, by Corollary 3.1 (b)

$$z \notin Co(LC(A_X, \underline{x})) \Rightarrow \langle g'(\underline{x}), z \rangle \triangleq \underline{y} \notin LC(A_Y, g(\underline{x})) .$$

By hypothesis, A_Y is convex so that $LC(A_Y, g(\underline{x}))$ is a closed convex cone, not containing $\underline{y}$. Once again using the strong separation theorem, $\exists y^* \in Y^*$, β real and $\delta > 0$ such that,

$$\langle y^*, \underline{y} \rangle = \beta > \beta - \delta \geq \langle y^*, y \rangle \; \forall y \in LC(A_Y, g(\underline{x})) .$$

Again since $LC(A_Y, g(\underline{x}))$ is a cone, this gives,

$$\langle y^*, \underline{y} \rangle > 0 \geq \langle y^*, y \rangle \; \forall y \in LC(A_Y, g(\underline{x})) .$$

Then by Def. 2.3 $y^* \in LP(A_Y, g(\underline{x}))$ so that
$y^* \cdot g'(\underline{x}) \in LP(A_Y, g(\underline{x})) \cdot g'(\underline{x}) \subseteq B$.
But $\langle y^* \cdot g'(\underline{x}), z \rangle = \langle y^*, \underline{y} \rangle > 0$ which contradicts (3.4).

(b) Claim $LP(A_X, \underline{x}) \supseteq \overline{LP(A_Y, g(\underline{x})) \cdot g'(\underline{x})}$. First notice that it is sufficient to show that

$$LP(A_X, \underline{x}) \supseteq LP(A_Y, g(\underline{x})) \cdot g'(\underline{x}) .$$

Suppose $\exists y^* \in LP(A_Y, g(\underline{x}))$ such that $y^* \cdot g'(\underline{x}) \notin LP(A_X, \underline{x})$. By the strong separation theorem, there is a $z \in X$, α real, and $\epsilon > 0$ with

$$\langle y^* \cdot g'(\underline{x}), z \rangle \geq \alpha > \alpha - \epsilon \geq \langle x^*, z \rangle \quad \forall x^* \in LP(A_X, \underline{x})$$

$$\langle y^* \cdot g'(\underline{x}), z \rangle > 0 \geq \langle x^*, z \rangle \quad \forall x^* \in LP(A_X, \underline{x}) .$$

It can be shown that the last inequality implies

$$z \in Co(LC(A_X, \underline{x})) . \tag{3.5}$$

Moreover $\langle y^* \cdot g'(\underline{x}), z \rangle = \langle y^*, g'(\underline{x})(z) \rangle > 0$. Since $y^* \in LP(A_Y, g(\underline{x}))$ this implies that $g'(\underline{x})(z) \notin LC(A_Y, g(\underline{x}))$. By Corollary 3.1 (b)

$$z \notin Co(LC(A_X, \underline{x}))$$

which contradicts (3.5). Q. E. D.

IV. MAIN RESULTS

We recall the problem considered by Kuhn and Tucker [1]

$$\text{Maximize}\{ f(x) \mid g(x) \geq 0, \ x \geq 0\} \tag{4.1}$$

where $x \in X(=R^n)$, $g: X \longrightarrow Y(=R^m)$ in a differentiable function and f is a real-valued, differentiable function of x. Equivalently,

$$\text{Maximize}\{ g(x) \mid g(x) \in A_Y, \ x \in A\} \tag{4.2}$$

where A_Y is the nonnegative orthant of Y and A is the nonnegative orthant of X. A saddle-value problem related to this is to find $\underline{x} \geq 0$, $\underline{y} \geq 0$ such that,

$$\phi(x, \underline{y}) \leq \phi(\underline{x}, \underline{y}) \leq \phi(\underline{x}, y) \ \forall x \geq 0, \ \forall y \geq 0$$

where $\phi(x, y) \equiv f(x) + \langle y, g(x) \rangle$. We note that $x \geq 0$ if and only if $x \in A$ and $y \geq 0$ if and only if $y \in -P(A_Y)$. 9]

We consider the following generalization of (4.2):

$$\text{Maximize}\{ f(x) \mid g(x) \in A_Y, \ x \in A\} \tag{4.3}$$

where A_Y is any convex subset of Y and A is an arbitrary subset of X. Also X and Y are arbitrary real B-spaces. This problem however does not have a corresponding saddle-value problem. If however, we restrict A_Y to a closed convex cone we can consider the problem of finding $\underline{x} \in A$, $\underline{y}^* \in -P(A_Y)$ such that

$$\phi(x, \underline{y}^*) \leq \phi(\underline{x}, \underline{y}^*) \leq \phi(\underline{x}, y^*) \ \forall x \in A, \ \forall y^* \in -P(A_Y)$$

where, $\phi(x, y^*) = f(x) + \langle y^*, g(x) \rangle$. Thus, let X, Y be real B-spaces; $g: X \to Y$ a differentiable map and f, a real-valued, differentiable function of x. Let A be an arbitrary subset of X and A_Y a convex subset of Y. We assume that (g, A_X, A_Y) satisfies W.

<u>Theorem 4.1</u> Consider the problem (4.3).

(a) Suppose $\underline{x}$ solves (4.3). Then

$$f'(\underline{x}) \in LP(A \cap A_X, \underline{x}) . \tag{4.4}$$

(b) If in addition, $LP(A \cap A_X, \underline{x}) = LP(A, \underline{x}) + LP(A_X, \underline{x})$ (4.5)

then there is an $\underline{x}^* \in -\overline{LP(A_Y, g(\underline{x})) \cdot g'(\underline{x})}$ such that

$$f'(\underline{x}) + \underline{x}^* \in LP(A, \underline{x}) . \tag{4.6}$$

(c) If in addition $LP(A_Y, g(x)) \cdot g'(\underline{x})$ is a closed set, there is a $y^* \in -LP(A_Y, g(\underline{x}))$ such that

$$f'(\underline{x}) + \underline{y}^* \cdot g'(\underline{x}) \in LP(A, \underline{x}) . \tag{4.7}$$

<u>Proof</u> (a) (4.4) follows from Theorem 2.1.

(b) By (4.4) and (4.5) we have

$$f'(\underline{x}) \in LP(A_X, \underline{x}) + LP(A, \underline{x}) .$$

By Theorem 3.1, $LP(A_X, \underline{x}) = \overline{LP(A_Y, g(\underline{x})) \cdot g'(\underline{x})}$ so that

$$f'(\underline{x}) \in \overline{LP(A_Y, g(\underline{x})) \cdot g'(\underline{x})} + LP(A, \underline{x})$$

which yields (4.6).

(c) By (b) we have

$$f'(\underline{x}) \in \overline{LP(A_Y, g(\underline{x}))\cdot g'(\underline{x})} + LP(A, \underline{x})$$

$$= LP(A_Y, g(\underline{x}))\cdot g'(\underline{x}) + LP(A, \underline{x})$$

by hypothesis of (c). This is equivalent to 4.7.

Q. E. D.

We now specialize to the case when A_Y is a closed, convex cone in Y. Consider the following three problems.

P1. Saddle-value problem.

Find $\underline{x} \in A$, $\underline{y}^* \in -P(A_Y)$ such that

$$\phi(x, \underline{y}^*) \leq \phi(\underline{x}, \underline{y}^*) \leq \phi(\underline{x}, y^*) \ \forall x \in A, \ \forall y^* \in -P(A_Y) \text{ where}$$

$$\phi(x, y^*) = f(x) + \langle y^*, g(x) \rangle .$$

P2. Find $\underline{x}$ which solve

$$\text{Maximize } \{ f(x) \mid g(x) \in A_Y, \ x \in A \} .$$

P3 ($\underline{y}^*$). Find $\underline{x}$ which (for fixed $\underline{y}^*$) solves

$$\text{Maximize } \{ f(x) + \langle \underline{y}^*, g(x) \rangle \mid x \in A \} .$$

The proof of the following three propositions is straightforward and hence omitted. For details see [7].

Theorem 4.2 (a) If $(\underline{x}, \underline{y}^*)$ solves P1, then

(i) $\frac{\partial \phi}{\partial x}(\underline{x}, \underline{y}^*) \equiv f'(\underline{x}) + \underline{y}^* \cdot g'(\underline{x}) \in LP(A, \underline{x})$

(ii) $g(\underline{x}) \in A_Y$ and

(iii) $\langle \underline{y}^*, g(\underline{x}) \rangle = 0.$

(b) If, moreover, A is convex and $\phi(x, \underline{y}^*)$ is concave for $x \in A$ conditions (i) - (iii) are sufficient for $(\underline{x}, \underline{y}^*)$ to solve P1.

(c) If $(\underline{x}, \underline{y}^*)$ solves P1 then $\underline{x}$ solves P2.

Theorem 4.3 If $\underline{x}$ solves P2 and if the hypotheses in Theorem 4.1 (a), (b) and (c) are satisfied then $(\underline{x}, \underline{y}^*)$ solves P1 if $\phi(x, \underline{y}^*)$ is concave over A and $\langle \underline{y}^*, g(\underline{x}) \rangle = 0$.

Theorem 4.4 (a) Suppose $\underline{x}$ solves P3 $(\underline{y}^*)$. Then

$$f'(x) + \underline{y}^* \cdot g'(\underline{x}) \in LP(A, \underline{x}) .$$

(b) If, in addition, $\underline{y}^* \in - P(A_Y)$ then $(\underline{x}, \underline{y}^*)$ solves P1 if and only if

$$g(\underline{x}) \in A_Y \quad \text{and} \quad \langle \underline{y}^*, g(\underline{x}) \rangle = 0.$$

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FOOTNOTES

1| A B-space means a Banach space.

2| Differentiable means Fiéchet-differentiable; see [4, 5].

3| A set C is a cone if and only if all $\alpha \geq 0$, $\alpha C \subseteq C$.

4| $\langle x^*, z \rangle \equiv x^*(z)$.

5| $f'(\underline{x})$ denotes the derivative of f at $\underline{x}$. Note that $f'(\underline{x})$ is an element of X^*. See [4].

6| $\langle f'(\underline{x}), x_n - \underline{x} \rangle \equiv f'(\underline{x})(x_n - \underline{x})$.

7| If C is a cone, Co(C) means the smallest closed convex cone containing C.

8| The overbar denotes closure of the set under it.

9| If C is a closed convex cone we let $P(C) \triangleq LP(C, 0)$.