THE MINIMAL REALIZATION OF A NONANTICIPATIVE IMPULSE RESPONSE MATRIX

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C. A. Desoer and P. Varaiya

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ELECTRONICS RESEARCH LABORATORY
University of California, Berkeley

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ABSTRACT

This paper is concerned with the problem of obtaining the minimum realization of a linear nonanticipative system characterized by its impulse response matrix: the problem is to find a linear differential system of least order which is zero-state equivalent to the given one.

For the time-varying case, Kalman's decomposition is used to obtain, in some cases, systems of lower order than Youla's globally reduced systems. In special cases, integrators are time-chared and integrators are saved at the cost of relays; from a mathematical point of view, in such cases, the system's matrices will include δ functions.

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INTRODUCTION

This paper is concerned with the problem of obtaining the minimum realization of a linear time-varying nonanticipative system characterized by its impulse response matrix: the problem is to find the linear differential system of least order which is zero-state equivalent to the given one. The key tool is the Kalman[1, 2] decomposition of the impulse response matrix. Our procedure results, in some cases, in a system of lower order than Youla's globally reduced system.[3]

A. Notations

Let $\underline{W}(t, \tau)$ be an $r \times p$ inpulse response matrix of a nonanticipative system. It is assumed that, for each fixed τ, $\underline{W}$ is locally square integrable with respect to t and, for each fixed t, $\underline{W}$ is locally square integrable with respect to τ.

$\underline{W}(t, \tau)$ is said to be realizable[2, 3] if there exists a linear differential system S of finite dimensional state space (say n) which has a zero-state response to any input $\underline{u}(\cdot)$ applied from t_0 and given by

$$(1) \qquad y(t) = \int_{t_0}^{t} \underline{W}(t, \tau)\, \underline{u}(t)\, dt \qquad -\infty > t \geq t_0 > -\infty.$$

More precisely, let the system S be characterized by

$$(2) \qquad \underline{\dot{x}}(t) = \underline{F}(t)\, x(t) + \underline{G}(t)\, \underline{u}(t),$$

$$(3) \qquad \underline{y}(t) = \underline{H}(t)\, \underline{x}(t),$$

where $\underline{F}(\cdot)$, $\underline{G}(\cdot)$, and $\underline{H}(\cdot)$ are, respectively, $n \times n$, $n \times p$ and $r \times n$ matrices whose elements are real-valued functions defined on $(-\infty, \infty)$. Let $\underline{\Phi}(t, t_0)$ be the state transition matrix of (2). Then it is well-known that $\underline{W}(t, \tau)$ is realizable by S if and only if

$$(4) \qquad \underline{W}(t, \tau) = \underline{H}(t)\, \underline{\Phi}(t, \tau)\, \underline{G}(\tau) \text{ for all } t \geq \tau .$$

Since the system S is characterized by $\underline{F}$, $\underline{G}$, and $\underline{H}$, one uses the locution "($\underline{F}$, $\underline{G}$, $\underline{H}$) and realizes $\underline{W}(t, \tau)$."

Under the condition that $\underline{F}$, $\underline{G}$, and $\underline{H}$ are locally square integrable, Kalman has given an interesting characterization of realizability:[2] $\underline{W}(t, \tau)$ is realizable if and only if

$$(5) \qquad \underline{W}(t, \tau) = \underline{\psi}(t)\, \underline{\beta}(\tau) \quad \forall t, \tau \text{ with } t \geq \tau ,$$

where $\underline{\psi}(\cdot)$ and $\underline{\beta}(\cdot)$ are, respectively, $r \times n$ and $n \times p$ matrices which are locally square integrable. We note that this characterization is not valid if $\underline{F}(\cdot)$ is not locally square integrable, The proof is based on the observation that $(\underline{0}, \underline{\beta}, \underline{\psi})$ realizes $\underline{W}(t, \tau)$: thus, under these conditions, it is always possible to simulate any such impulse response matrix using time variable gains and n integrators.

Under the condition that $\underline{F}(\cdot)$ is locally square integrable and that (5) holds for <u>all</u> t and τ, Youla[3] has given an algorithm which, starting from any given factorization of $\underline{W}(t, \tau)$ as $\underline{\psi}(t)\, \beta(\tau)$, arrives at a factorization of $\underline{W}$ of least order. Such a factorization is called a globally reduced realization by Youla. In a nonanticipative system, however, we would require that (5) hold only over the set $t \geq \tau$. We shall give a procedure which obtains a realization of minimum order for this situation. Let us call this problem A.

If, furthermore, we drop the requirement that $\underline{F}(\cdot)$ be locally square integrable, it turns out that we can reduce even further the order of S. We shall call this problem B.

Before we proceed to the reduction algorithms, it may be worthwhile to give an example illustrating the various "minimal" realizations.

Example: Let r = p = 1 and $W(t, \tau) = \psi_1(t)\ \beta_1(\tau) + \psi_2(t)\ \beta_2(\tau) + \psi_3(t)\ \beta_3(\tau)$, where

$$\psi_1(t) = \begin{cases} 1 \text{ if } t \in [-2, -1] \\ 0 \text{ elsewhere,} \end{cases} \quad \psi_2(t) = \begin{cases} 1 \text{ if } t \in [3, 4] \\ 0 \text{ elsewhere,} \end{cases} \quad \psi_3(t) = \begin{cases} 1 \text{ if } t \in [5, 6] \\ 0 \text{ elsewhere,} \end{cases}$$

$$\beta_1(\tau) = \begin{cases} 1 \text{ if } \tau \in [-3, -4] \\ 0 \text{ elsewhere,} \end{cases} \quad \beta_2(\tau) = \begin{cases} 1 \text{ if } \tau \in [1, 2] \\ 0 \text{ elsewhere} \end{cases} \quad \beta_3(\tau) = \begin{cases} 1 \text{ if } t \in [7, 8] \\ 0 \text{ elsewhere} \end{cases}$$

we first note that the functions $\psi_i(\cdot)$ i = 1, 2, 3 are linearly independent over the interval $(-\infty, \infty)$. Similarly, the functions $\beta_i(\cdot)$ i = 1, 2, 3 are linearly independent over $(-\infty, \infty)$. Hence the globally reduced realization of Youla[3] has dimension 3. For the nonanticipative situation however,

$$y(t) = \int_{-\infty}^{t} [\psi_1(t)\,\beta_1(\tau) + \psi_2(t)\,\beta_2(\tau) + \psi_3(t)\,\beta_3(\tau)]\ u(\tau)\ dt$$

$$= \int_{-\infty}^{t} [\psi_1(t)\ \beta_1(\tau) + \psi_2(t)\ \beta_2(\tau)]\ u(\tau)\ dt,$$

since $\psi_3(t)\ \beta_3(\tau) = 0$ for all $t \geq \tau$. Thus we have a realization of dimension 2. Now consider the first order differential system,

$$\dot{\eta} = -\delta(t)\,\eta + [1(-t)\ \beta_1(t) + 1(t)\ \beta_2(t)]\ u(t),$$

$$y = \eta(t)[\psi_1(t) + \psi_2(t)],$$

where $\delta(t)$ is the delta "function," and $1(t)$ is the Heaviside unit step function. It can be verified that this system is zero-state equivalent to the one characterized by $\underline{W}(t,\tau)$. Note that the matrix $F(t)$ which is here $-\delta(t)$, is not locally square integrable.

B. Reduction Algorithm for Problem A

We start with a given factorization of $\underline{W}(t,\tau)$ as $\underline{\psi}(t)\ \underline{\beta}(\tau)$, a product of an $r \times n$ and an $n \times p$ matrix.

Definition 1. (a) For each $t \in R$, define $n \times n$ matrices

(6) $$\underline{B}(t) = \int_{-\infty}^{t} \underline{\beta}(\tau)\ \beta'(\tau)\ dt,$$

and

(7) $$\underline{C}(t) = \int_{t}^{\infty} \underline{\psi}'(\tau)\ \underline{\psi}(\tau)\ dt.$$

(b) Let $\mathcal{R}(t)$ denote the range space of $\underline{B}(t)$ and let $\mathcal{N}(t)$ denote the null space of $\underline{C}(t)$.

Since the integration in (6) and (7) is taken over an infinite interval, the matrices $\underline{B}(t)$ and $\underline{C}(t)$ may not be defined. However, we are only interested in the subspaces $\mathcal{R}(t)$ and $\mathcal{N}(t)$ so that in (6), the lower limit $-\infty$ can be replaced by any sufficiently small number $t_0 < t$ such that the number of linearly independent rows of $\underline{\beta}(\cdot)$ over any interval (t_0', t) with $t_0' < t$ is not greater than the number of linearly independent rows of $\underline{\beta}(\cdot)$ over the interval (t_0, t). Similarly, the upper limit in (7) can be replaced by any sufficiently large number $t_1 > t$ so that the number of linearly independent columns of $\underline{\psi}(\cdot)$ over any interval (t, t_1') with $t_1' > t$ is not greater than the number of linearly independent columns of $\underline{\psi}(\cdot)$ over the interval (t, t_1).

The physical interpretation of the subspaces $\mathcal{R}(t)$ and $\mathcal{N}(t)$ is given by the next definition and lemma.

Definition 2 Let $t \in R$ be fixed.

(a) A vector $\underline{x} \in R^n$ is said to be reachable at time t if there is a square integrable function $\underline{u}(\cdot)$ such that

$$\underline{x} = \int_{-\infty}^{t} \underline{\beta}(\tau)\, \underline{u}(\tau)\, dt.$$

A vector $\underline{x} \in R^n$ is said to be invisible after time t if

$$\underline{\psi}(\tau)\, \underline{x} = 0 \quad \text{for almost all } \tau \geq t.$$

(b) Let $U(t)$ denote the set of vectors reachable at time t and let $V(t)$ denote the set of vectors invisible after time t.

Lemma 1: (a) $U(t) = \mathcal{R}(t)$ for each t. Also $t_1 \leq t_2$ implies that $\mathcal{R}(t_1) \subseteq \mathcal{R}(t_2)$.

(b) $V(t) = \mathcal{N}(t)$ for each t. Also $t_1 \leq t_2$ implies that $\mathcal{N}(t_1) \subseteq \mathcal{N}(t_2)$.

The proof is very similar to the one given by Kalman and Weiss[9] and is therefore omitted. Since $\mathcal{R}(t_1) \subseteq \mathcal{R}(t_2) \subseteq R^n$ for $t_1 \leq t_2$, $\mathcal{R}(\cdot)$ considered as a function of time changes only at finitely many instances. A similar argument is valid for $\mathcal{N}(\cdot)$. Let $t_1 < t_2 \cdots < t_m$ to be the values of time at which either $\mathcal{R}(\cdot)$ or $\mathcal{N}(\cdot)$ changes. Then,

$$
\mathcal{R}(t)\,[\mathcal{N}(t)] = \begin{cases} \mathcal{R}(t_1)\,[\mathcal{N}(t_1)] & \text{for } -\infty < t < t_1 \\ \mathcal{R}(t_2)\,[\mathcal{N}(t_2)] & \text{for } t_1 < t < t_2 \\ \vdots & \\ \mathcal{R}(t_m)[\mathcal{N}(t_m)] & \text{for } t_{m-1} < t < t_m \\ \mathcal{R}(t_{m+1})[\mathcal{N}(t_{m+1})] & \text{for } t_m < t < \infty \end{cases}
$$

where t_{m+1} is any number with $t_{m+1} > t_m$.

We will now decompose $\mathcal{R}(t_i)$ as follows:

Let

$$
\mathcal{R}(t_1) = \mathcal{R}(t_1) \cap \mathcal{N}(t_1) \oplus \mathcal{X}(t_1), \tag{8}
$$

and for $i > 0$,

$$
\mathcal{R}(t_{i+1}) = \mathcal{R}(t_i) + \mathcal{Y}(t_{i+1}) \oplus \mathcal{X}(t_{i+1}), \tag{9}
$$

where $\mathcal{X}(t_1)$ is any arbitrary subspace satisfying (8), and $\mathcal{Y}(t_{i+1})$ is any subspace of $\mathcal{N}(t_{i+1})$ of largest possible dimension which satisfies (9) for some $\mathcal{X}(t_{i+1})$. For symmetry, let us define $\mathcal{Y}(t_1) = \mathcal{R}(t_1) \cap \mathcal{N}(t_1)$. Now let,

$$
\mathcal{X} = \mathcal{X}(t_1) \oplus \cdots \oplus \mathcal{X}(t_{m+1}),
$$

and

$$
\mathcal{Y} = \mathcal{Y}(t_1) \oplus \cdots \oplus \mathcal{Y}(t_{m+1}).
$$

Then we observe that

$$
\mathcal{R}(t_{m+1}) = \mathcal{X} \oplus \mathcal{Y} \quad , \tag{10}
$$

and

$$R^n = \mathcal{R}(t_{m+1}) \oplus \mathcal{R}(t_{m+1})^{\perp} = \mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{R}(t_{m+1})^{\perp}. \tag{11}$$

Remarks: In the above decomposition the subspaces $\mathcal{X}(t_i)$ and $\mathcal{Y}(t_i)$ are not uniquely defined. However, the dimension of each subspace is unique. Therefore, if we let $\bar{n}$ be the dimension of $\mathcal{X}$, $\bar{n}$ is a well-defined number. For an illustration of this decomposition see Fig. 1.

Definition: Let $\underline{P}$ be the matrix representing the projection of R^n onto $\mathcal{X}$ along $\mathcal{Y} \oplus \mathcal{R}(t_{m+1})^{\perp}$. i.e., if $\underline{z} \in R^n$ and $\underline{z} = \underline{x} + \underline{y}$ with $\underline{x} \in \mathcal{X}$ and $\underline{y} \in \mathcal{Y} \oplus \mathcal{R}(t_{m+1})^{\perp}$, we must have

$$\underline{P}(\underline{z}) = \underline{P}(\underline{x} + \underline{y}) = \underline{P}(\underline{x}) + \underline{P}(\underline{y}) = \underline{P}(\underline{x}) = \underline{x}.$$

We again note that although $\underline{P}$ depends upon the particular decomposition chosen, the dimension of the range of $\underline{P}$ is the well-defined number $\bar{n}$. The relationships between this decomposition and the factorization of $\underline{W}(\cdot,\cdot)$ is given by the next lemma.

Lemma 2 (a) Let $1 \leq i \leq m+1$ be fixed and let $t_{i-1} < t < t_i$ be a fixed number. The set of all vectors $x \in R^n$ such that there is a square integrable function $\underline{u}(\cdot)$ with

$$\underline{x} = \int_{t_{i-1}}^{t} \underline{\beta}(\tau)\, \underline{u}(\tau)\, dt$$

contains the set $\mathcal{X}(t_i)$. (Here $t_0 = -\infty$.) Also, $\mathcal{R}(t) \cap \mathcal{X}(t_{i+1}) = \{\underline{0}\}$ for $t < t_i$.

(b) Let $\underline{x}_1, \underline{x}_2 \in \mathcal{X}(t_i)$, and let $t_{i-1} < t < t_i$ be a fixed number.

$$\underline{\psi}(\tau)(\underline{x}_1 - \underline{x}_2) = 0 \qquad \text{for almost all } \tau \geq t$$

implies that $\underline{x}_1 = \underline{x}_2$.

(c) Finally, for almost all (t, τ) with $t \geq \tau$ we have

$$\underline{\psi}(t)\, \underline{\beta}(\tau) = \underline{\psi}(t)\, \underline{P}\, \underline{\beta}(\tau).$$

Proof: (a) Let $\underline{x} \in \mathcal{X}(t_i) \subseteq \mathcal{R}(t_i) = \mathcal{R}(t)$. Therefore there is a function $\underline{u}(\cdot)$ such that

$$x = \int_{-\infty}^{t} \underline{\beta}(\tau)\, \underline{u}(\tau)\, dt$$

$$= \int_{-\infty}^{t_{i-1}} \underline{\beta}(\tau)\, \underline{u}(\tau)\, dt + \int_{t_{i-1}}^{t} \underline{\beta}(\tau)\, \underline{u}(\tau)\, dt$$

$$= \underline{x}_1 + \underline{x}_2 \text{ say.}$$

Obviously $\underline{x}_1 \in \mathcal{R}(t_{i-1})$ so that by the decomposition (9) $\underline{x}_1 = 0$.

(b) By assumption $\underline{\psi}(\tau)(\underline{x}_1 - \underline{x}_2) = 0$ for almost all $\tau > t$, so that $(\underline{x}_1 - \underline{x}_2) \in \mathcal{N}(t_i)$. By the decomposition (9), since $\mathcal{Y}(t_i) \subseteq \mathcal{N}(t_i)$ has maximum dimension we must have

$$\mathcal{X}(t_i) \cap \mathcal{N}(t_i) = \{\underline{0}\}.$$

This implies that $\underline{x}_1 - \underline{x}_2 = \underline{0}$.

(c) It suffices to prove that for all square integrable functions $\underline{u}(\cdot)$, we have

$$\underline{y}(t) = \int_{-\infty}^{t} \underline{\psi}(\tau)\, \underline{\beta}(\tau)\, \underline{u}(\tau)\, dt = \int_{-\infty}^{t} \underline{\psi}(t)\, \underline{P}\,\underline{\beta}(\tau)\, \underline{u}(t)\, dt.$$

Let $\underline{z}(t) = \int_{-\infty}^{t} \underline{\beta}(\tau)\, \underline{u}(\tau)\, dt$. Clearly $\underline{x}(t) \in \mathcal{R}(t) = \mathcal{R}(t_i)$ for some i. By the decomposition (9) we have

$$\underline{z} = \underline{x} + \underline{y},$$

where $\underline{x} \in \mathcal{X}(t_1) + \cdots + \mathcal{X}(t_i)$,

and $\underline{y} \in \mathcal{Y}(t_1) + \cdots + \mathcal{Y}(t_i)$.

By the definition of $\underline{P}$,

$$\underline{P}\,\underline{z} = P(\underline{x} + \underline{y}) = P\underline{x} = \underline{x}.$$

We have to show then that $\underline{\psi}(t)\, \underline{y} = 0$. But this is true because $\mathcal{N}(t) \supseteq \mathcal{Y}(t_1) + \cdots + \mathcal{Y}(t_i)$.

Q.E.D.

Since $P^2 = P$, by lemma 2 we have,

$$\underline{\psi}(t)\ \underline{\beta}(\tau) = \underline{\psi}_1(t)\ \underline{\beta}_1(\tau) \qquad \text{for all } t \geq \tau$$

where

$$\underline{\psi}_1(t) \triangleq \underline{\psi}(t)\, \underline{P} \quad \text{and} \quad \beta_1(\tau) \triangleq \underline{P}\, \underline{\beta}(\tau).$$

Since the range of $\underline{P}$ has dimension $\overline{n}$, there are at most $\overline{n}$ independent rows in the matrix $\underline{\beta}_1(\cdot)$ and at most $\overline{n}$ independent columns of $\underline{\psi}_1(\cdot)$. We start with the factorization of $\underline{W}(t,\tau)$ as $\underline{\psi}_1(t)\,\underline{\beta}_1(\tau)$ and carry out the Youla reduction technique. Let the globally-reduced realization obtained by this method be

$$\underline{W}(t,\tau) = \hat{\underline{\psi}}(t)\,\hat{\underline{\beta}}(\tau) \qquad \text{for } t \geq \tau ,$$

where $\hat{\underline{\psi}}$ and $\hat{\underline{\beta}}$ have dimension $p \times \hat{n}$ and $\hat{n} \times r$, respectively. Clearly $\hat{n} \leq \overline{n}$.

<u>Theorem 1</u>: (a) $\hat{n} = \overline{n}$.

(b) Let $\underline{W}(t,\tau) = \tilde{\underline{\psi}}(t)\,\tilde{\underline{\beta}}(\tau)$, where $t \geq \tau$ be an arbitrary factorization of $\underline{W}$ as a product of $p \times \tilde{n}$ and $\tilde{n} \times r$ matrices respectively. Then $\tilde{n} \geq \overline{n}$.

<u>Proof</u>: It suffices to prove (b). Let $t_1 < t_2 \cdots < t_m$ be the switching times in the definitions (8) and (9). Corresponding to the factorization $\tilde{\psi}$, $\tilde{\beta}$ define the subspaces $\tilde{\mathcal{A}}(t_i)$ $\tilde{\mathcal{N}}(t_i)$, $\tilde{\mathcal{X}}(t_i)$ etc.. Note that these are subspaces of R^n. Let

$$\tilde{\mathcal{X}} = \tilde{\mathcal{X}}(t_1) + \cdots + \tilde{\mathcal{X}}(t_m).$$

Then $\tilde{\mathcal{X}} \subseteq R^{\tilde{n}}$. To show that $\tilde{n} \geq \hat{n}$, we shall in fact show that

$$\tilde{n}_i \triangleq \text{dimension}\left(\tilde{\mathcal{X}}(t_i)\right) = \overline{n}_i \triangleq \text{dimension}\left(\mathcal{X}(t_i)\right) \text{ from}$$

which it follows that $\tilde{n} \geqq \sum \tilde{n}_i = \sum \overline{n}_i = \overline{n}$. Let $t_{i-1} < t < t_i$ be a fixed number. Then (a) and (b) of lemma 2 imply that the impulse response $\underline{\psi}(t)\,\underline{\beta}(\tau)$ gives exactly n_i linearly independent outputs over the interval

(t, ∞). Similarly, the impulse response $\underline{\tilde{\psi}}(t)\ \underline{\tilde{\beta}}(\tau)$ gives exactly $\tilde{n}_i$ linearly independent outputs over the interval (t, ∞). Since these two impulse responses are the same we must have $\bar{n}_i = \tilde{n}_i$.

Q.E.D.

C. Reduction Algorithm for Problem B

As before, we start with a given factorization of $\underline{W}(t,\tau)$ as $\underline{\psi}(t)\ \underline{\beta}(\tau)$, a product of an $r \times n$ matrix and an $n \times p$ matrix. We define the subspaces $\mathcal{R}(t)$ and $\mathcal{N}(t)$ as in problem A. Again let $t_1 < t_2 \cdots < t_m$ be the instants at which either $\mathcal{R}(\cdot)$ or $\mathcal{N}(\cdot)$ changes.

To keep the notation from getting prohibitively complicated we shall illustrate the reduction algorithm for the case when $m = 1$. The extension for $m > 1$ must be clear. Thus, suppose $m = 1$, so that $\mathcal{R}(t) = \mathcal{R}(t_1)$ $[\mathcal{N}(t) = \mathcal{N}(t_1)]$ for $t < t_1$ and $\mathcal{R}(t) = \mathcal{R}(t_2)$ $[\mathcal{N}(t) = \mathcal{N}(t_2)]$ for $t > t_1$, where $t_2 > t_1$ is any number. Let

$$\mathcal{R}(t_i) = \mathcal{R}(t_i) \cap \mathcal{N}(t_i) + \mathcal{X}(t_i) \qquad i = 1, 2,$$

where $\mathcal{X}(t_1)$ and $\mathcal{X}(t_2)$ are chosen in such a manner that they have an intersection of largest possible dimension. This is achieved as follows.

(i) Choose an arbitrary basis B_1 for $\mathcal{R}(t_1) \cap \mathcal{N}(t_1)$.

(ii) Complete the basis to $B_1 \cup B_{21}$ for $\mathcal{R}(t_1) \cap \mathcal{N}(t_2)$.

(iii) Complete the basis to $B_1 \cup B_{21} \cup Q_1$ for $\mathcal{R}(t_1)$. Then $B_{21} \cup Q_1$ is the basis for $\mathcal{X}(t_1)$.

(iv) From (ii) complete the basis to $B_1 \cup B_{21} \cup B_{21}$ for $\mathcal{R}(t_2) \cap \mathcal{M}(t_2)$.

(v) Complete the basis to $B_{21} \cap B_1 \cap B_{21} \cap Q_1 \cup Q_2$ for $\mathcal{R}(t_2)$. Then $Q_1 \cup Q_2$ will be the basis for $x(t_2)$.

(vi) Let N be a basis for $\mathcal{R}(t_2)^{\perp}$.

The decomposition of R^n is illustrated in Fig. 2.

Next we construct a nonsingular $n \times n$ matrix M and its inverse M^{-1} as follows:

$$M = \begin{array}{c} \left[\begin{array}{c|c|c|c|c|c} B_{21} & Q_1 & Q_2 & B_1 & B_{2\bar{1}} & N^r \end{array}\right] \\ \begin{array}{cccccc} \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 \end{array} \end{array} \qquad M^{-1} = \begin{array}{c} \left[\begin{array}{c} B_{21}^r \\ \hline Q_1^r \\ \hline Q_2^r \\ \hline B_1^r \\ \hline B_{2\bar{1}}^r \\ \hline N \end{array}\right] \end{array} \begin{array}{c} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \end{array}$$

Thus the first μ_1 columns of M are the vectors of B_{21}, the next μ_2 columns of M are the vectors of Q_1, and so on. Similarly, the first μ_1 rows of M^{-1} which are denoted by B_{21}^r are the reciprocal basis vectors of B_{21} and so on. The last μ_6 rows of M^{-1} are the vectors of N. Now

$$\underline{\psi}(t)\, \underline{\beta}(\tau) = [\, \underline{\psi}(t)\; M\,][\, M^{-1}\; \underline{\beta}(\tau)\,]$$

$$= [\, \tilde{\underline{\psi}}(t)\,][\, \tilde{\underline{\beta}}(\tau)\,] \quad \text{say.}$$

We can regard $\tilde{\underline{\psi}}(t)$ and $\tilde{\underline{\beta}}(\tau)$ as

$$\tilde{\underline{\psi}}(t) = \begin{array}{c} \left[\begin{array}{c|c|c|c|c|c} \underline{\psi}_1 & \underline{\psi}_2 & \underline{\psi}_3 & \underline{\psi}_4 & \underline{\psi}_5 & \underline{\psi}_6 \end{array}\right] \\ \begin{array}{cccccc} \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 \end{array} \end{array} \qquad \tilde{\underline{\beta}}(\tau) = \begin{array}{cc} \left[\begin{array}{c} \underline{\beta}_1 \\ \hline \underline{\beta}_2 \\ \hline \underline{\beta}_3 \\ \hline \underline{\beta}_4 \\ \hline \underline{\beta}_5 \\ \hline \underline{\beta}_6 \end{array}\right] & \begin{array}{c} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \end{array} \end{array}$$

where for example $\underline{\psi}_4(t) = \underline{\psi}(t)\, B_1$ and $\underline{\beta}_6(\tau) = N\,\underline{\beta}(\tau)$. Since $B_1 \subseteq \mathcal{N}(t_1)$ we must have according to lemma 1a $\underline{\psi}_4(t) = \underline{0}$ for almost all t. Again, as $B_1 \cup B_{21} \cup B_{21} \subseteq \mathcal{N}(t_2)$ we will have $\underline{\psi}_1(t) = \underline{0}$, $\underline{\psi}_4(t) = \underline{0}$ and $\underline{\psi}_5(t) = \underline{0}$ for $t > t_1$. Now $B_1 \cup B_{21} \cup Q_1$ is a basic for $\mathcal{R}(t_1)$ so that $\underline{\beta}_3(\tau) = \underline{0}$, $\underline{\beta}_5(\tau) = \underline{0}$, and $\underline{\beta}_6(\tau) = \underline{0}$ for $\tau < t_1$. Similarly, $\underline{\beta}_6(\tau) = \underline{0}$ for $\tau > t_1$. Taking these facts into account we see that

$$\underline{\psi}(t)\, \underline{\beta}(\tau) = \tilde{\underline{\psi}}(t)\, \tilde{\underline{\beta}}(\tau) = \begin{array}{c} \left[\begin{array}{c|c|c} \underline{\psi}_1 & \underline{\psi}_2 & \underline{\psi}_3 \end{array}\right] \\ \begin{array}{ccc} \mu_1 & \mu_2 & \mu_3 \end{array} \end{array} \quad \begin{array}{cc} \left[\begin{array}{c} \underline{\beta}_1 \\ \hline \underline{\beta}_2 \\ \hline \underline{\beta}_3 \end{array}\right] & \begin{array}{c} \mu_1 \\ \mu_2 \\ \mu_3 \end{array} \end{array},$$

where furthermore $\underline{\beta}_3(\tau) = 0$ for $t < t_1$ and $\underline{\psi}_1(t) = 0$ for $t > t_1$. If $\mu_1 > \mu_3$ we can add $\mu_1 - \mu_3$ identically zero rows to $\underline{\beta}_3$ and $\mu_1 - \mu_3$ identically zero columns to $\underline{\psi}_3$ to make $\mu_1 = \mu_3$. Similarly, if $\mu_3 > \mu_1$ we can add $\mu_3 - \mu_1$ identically zero rows and columns to $\underline{\beta}_1$ and $\underline{\psi}_1$, respectively. Thus we can assume that $\mu_1 = \mu_3$. Let $\underline{\eta}_1$ be a vector of dimension $\mu_1 = \mu_3$ and $\underline{\eta}_2$ be a vector of dimension μ_2 and consider the first order differential system of dimension $\mu_1 + \mu_2$.

$$\dot{\underline{\eta}}_1(t) = -\underline{\delta}(t-t_1)\,\underline{\eta}_1 + [\,\underline{\beta}_1(t)\,\underline{1}\,(t_1-t) + \underline{\beta}_3(t)\,]\,\underline{u}(t)$$

$$\dot{\underline{\eta}}_2(t) = \underline{\beta}_2(t)\,\underline{u}(t)$$

and $\underline{y}(t) = [\,\underline{\psi}_1(t) + \underline{\psi}_3(t)\,\underline{1}\,(t-t_1)\,]\,\underline{\eta}_1(t) + \underline{\psi}_2(t)\,\underline{\eta}_2(t)$, where $\underline{\delta}(t)$ is a $\mu_1 \times \mu_1$ diagonal matrix with $\delta(t)$ as the diagonal elements and $\underline{1}(t)$ is a $\mu_1 \times \mu_1$ matrix with the Heaviside unit function $1(t)$ on the diagonal. It should be clear that the zero-state response of this system is the same as that given by the impulse response matrix $\underline{W}(t,\tau)$. An analog computer setup for this system is given in Fig. 3.

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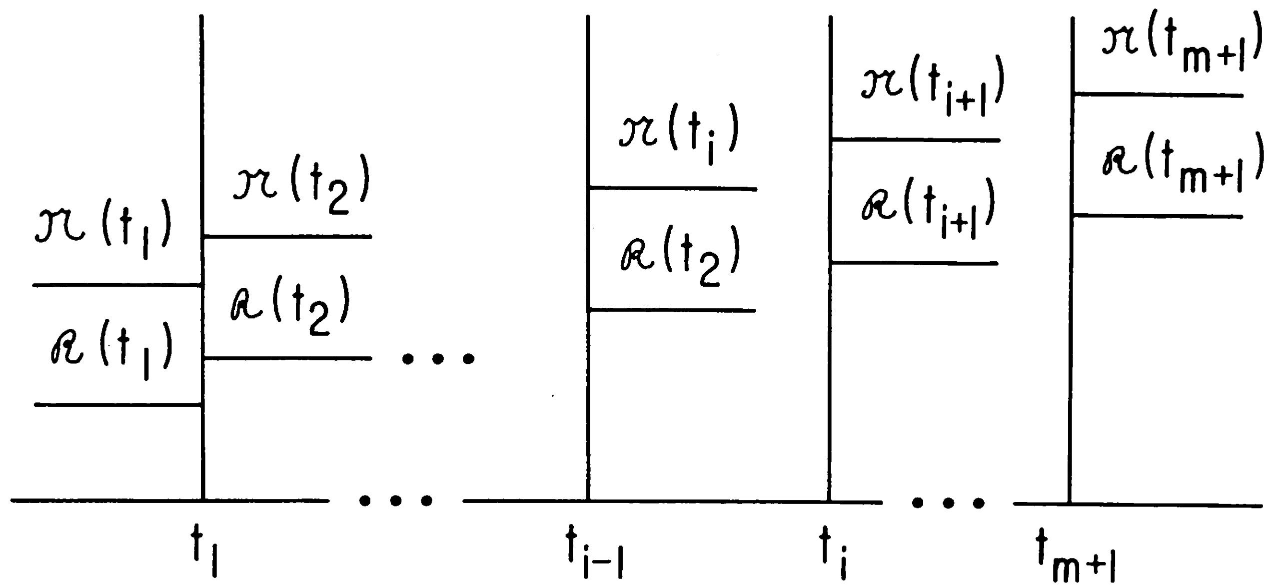

Fig. 1. Decomposition of (9) in Problem A.

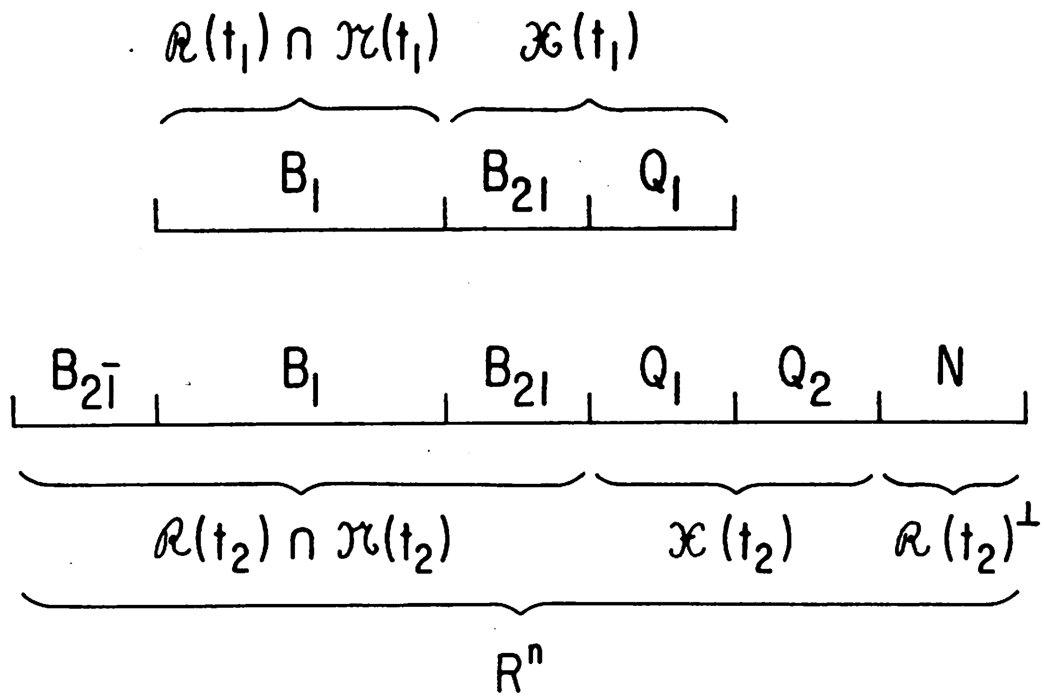

Fig. 2. Decomposition of R^n.

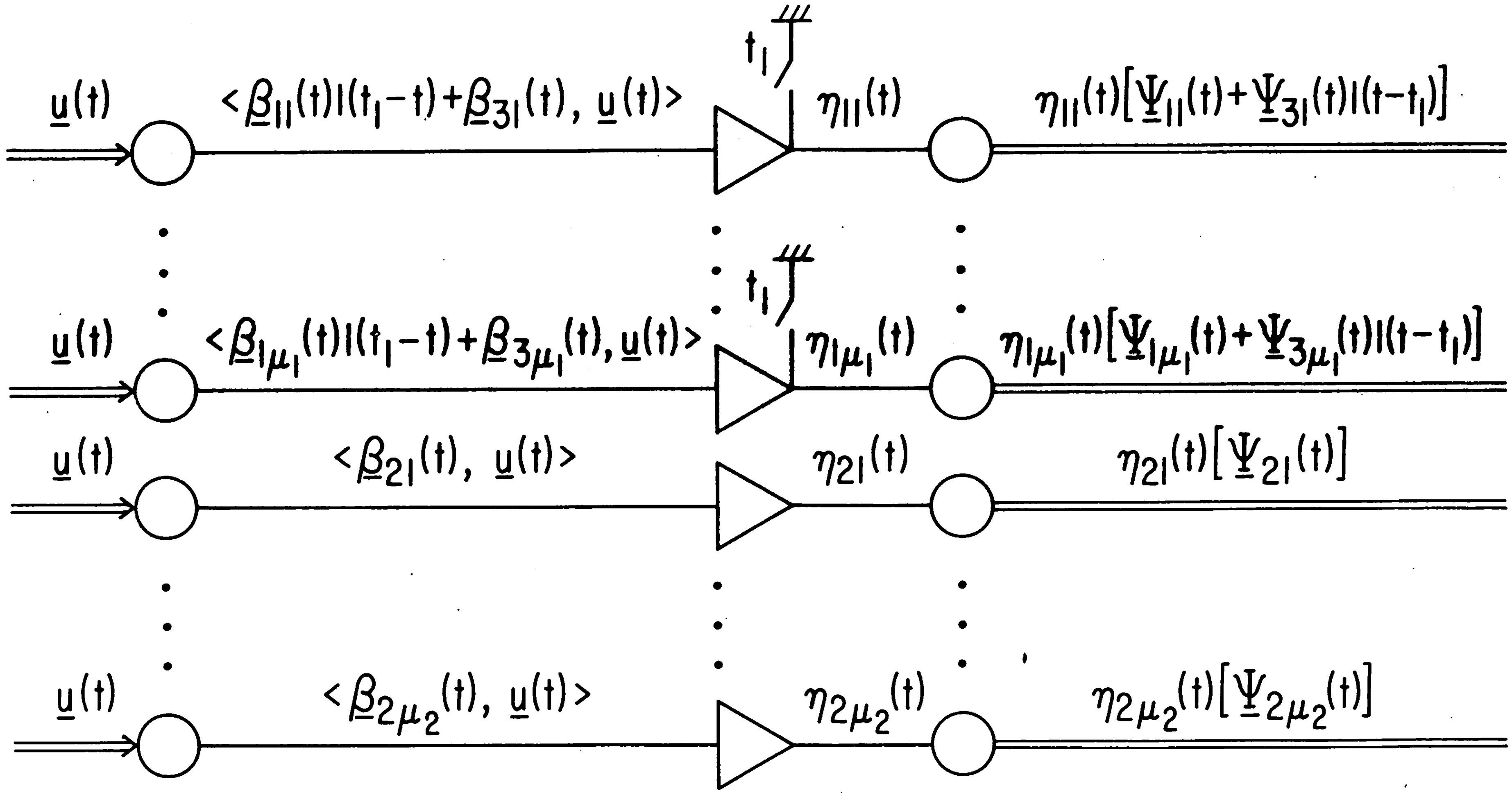

Fig. 3. Analog computer setup of Problem B.