A FREQUENCY-DOMAIN STABILITY CRITERION FOR FEEDBACK SYSTEMS CONTAINING A SINGLE LINEAR TIME-VARYING ELEMENT

by

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SUMMARY

Sufficient conditions for the stability of feedback systems containing a single linear time-varying element are obtained by using the method of Popov. A frequency-domain criterion, which utilizes information on the time derivative of the time-varying element, is developed. In this paper, the linear time-invariant subsystem is described by a convolution integral, that is, no assumptions are made concerning the internal dynamics of the linear time-invariant part. An application of the main result to a parametrically excited system is given to illustrate the improvement of the new stability criterion over the existing results.[11]

INTRODUCTION

With the advent of parametric devices and the control problem encountered in the design of space vehicles, the need has arisen for stability criteria which can be applied easily in the design of linear and nonlinear time-varying systems. Consequently, in the past few years, the development of sufficient stability conditions has been of great interest in the study of feedback systems containing a single nonlinear and/or time-varying element.[1-5, 7, 14, 15] These stability conditions have been sought in terms of the real-frequency characteristics of the linear time-invariant subsystem and the bounds on the nonlinear and/or time-varying element in a manner analogus to the Nyquist stability criterion for linear time-invariant feedback systems.

Frequently, in the practical analysis of nonlinear time-varying feedback systems, more information is known about the time-varying nonlinearity than the fact that it lies in some finite sector. When more information is available, it is expected that better stability results will be obtained. Rekasius and Rowland[6] have investigated this problem by taking into account the rate at which the characteristic of time-varying nonlinearity varies. In this paper, a different approach is made to the stability problem of linear time-varying systems using the method of the Popov theorem. A new stability criterion is derived which shows the improvement over the existing results.[4, 7]

In this paper, the linear time-invariant subsystem is described by a convolution (i.e., input-output relation), no assumptions are made concerning the internal dynamics of the time-invariant part.

Description of the System

Consider the following system:

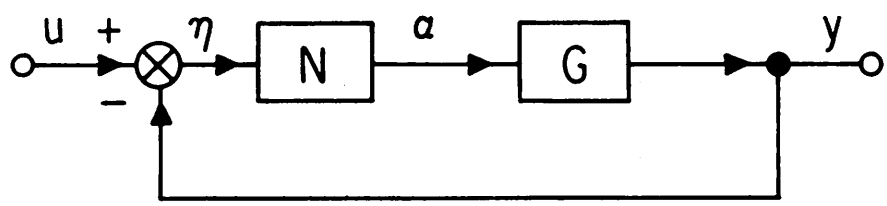

Fig. 1. Linear time-varying feedback system.

where N is a linear time-varying memoryless element, G is a non-anticipative, linear-time invariant subsystem. N is assumed to satisfy the following conditions:

(N.1) $\alpha(t) = c(t)\ \eta(t)$, $c(t)$ is continuous and differentiable over $[0, \infty)$ such that there exist positive real numbers ϵ and k with the property that

$$0 < \epsilon \leq c(t) \leq k - \epsilon \qquad \text{for all } t \geq 0 . \tag{1}$$

G is characterized by the following:

(G.1) If α is its input, its output y is given by

$$y(t) = z(t) + \int_0^t g(t - \tau)\ \alpha(\tau)\ d\tau \qquad \text{for } t \geq 0 \tag{2}$$

where g is the unit impulse response of G, z is the zero-input response of G.

(G.2) For all initial state, $z(0)$ is finite, $\dot{z}$ exists on $[0, \infty)$ and z, $\dot{z}$ are elements of $L_2(0, \infty)$ and $z(t) \to 0$ as $t \to \infty$.

(G.3) $g \in L_1(0, \infty) \cap L_2(0, \infty)$, $\dot{g}$ exists on $[0, \infty)$ and belongs to $L_2(0, \infty)$.

The input u is assumed to satisfy the same conditions imposed on z. Note that from (G.2) it is readily seen that z is bounded on $[0, \infty)$.

From Fig. 1

$$\begin{aligned} \eta(t) &= u(t) - y(t) \\ &= u(t) - z(t) - \int_0^t g(t-\tau)\,\alpha(\tau)\,d\tau \\ &= -z_1(t) - \int_0^t g(t-\tau)\,\alpha(\tau)\,d\tau \quad \text{for } t \geq 0 \end{aligned} \tag{3}$$

where $z_1 = -u + z$.

Define $G(s)$, the Laplace transform of g, by

$$G(s) = \int_0^\infty g(t)\,e^{-st}\,dt .$$

<u>Theorem 1:</u> Let Fig. 1 be the single-input, single-output linear time-varying feedback system under consideration, where N is a linear time-varying memoryless element which satisfies (N.1), and G is a nonanticipative linear time-invariant subsystem satisfying (G.1), (G.2) and (G.3). If there exist real numbers δ, q and m such that

$$\text{(i)} \quad c(t)\left[1 - \frac{c(t)}{k}\right] - \frac{q}{2}\,c(t) \geq m > 0 \quad \text{for all } t \geq 0 \tag{4}$$

and

$$\text{(ii)} \quad \mathrm{Re}\,(1 + i\omega q)\, G(i\omega) + \frac{1}{k} \geq \delta > 0 \quad \text{for all } \omega \in (-\infty, \infty). \tag{5}$$

Then for any initial state, the output

$$y \in L_2(0, \infty) \cap L_\infty(0, \infty), \text{ and } y(t) \to 0 \text{ as } t \to \infty \tag{6}$$

<u>Proof</u>: By Lemma 2 (see Appendix), in the proof of the theorem, q may be assumed to be nonnegative.

The system is characterized by (3), i.e.,

$$\eta(t) = -z_1(t) - \int_0^t g(t - \tau)\, \alpha(\tau)\, d\tau \quad \text{for } t \geq 0 .$$

Let α_T be defined by

$$\alpha_T(t) = \begin{cases} \alpha(t) & \text{for } 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where T is a finite positive number. Denote $\eta_T(t)$ by

$$\eta_T(t) = -z_1(t) - \int_0^t g(t - \tau)\, \alpha_T(\tau)\, d\tau \quad \text{for } t \geq 0 \tag{8}$$

then

$$\dot{\eta}_T(t) = -\dot{z}_1(t) - g(0^+)\, \alpha_T(t) - \int_0^t \dot{g}(t - \tau)\, \alpha_T(\tau)\, d\tau \quad \text{for } t \geq 0 . \tag{9}$$

Define

$$\lambda_T(t) = \eta_T(t) + q\, \dot{\eta}_T(t) - \frac{\alpha_T(t)}{k} \tag{10}$$

$$r(t) = z_1(t) + q\, \dot{z}_1(t) . \tag{11}$$

Our assumptions (G.2) and (G.3) on the differentiability of z_1, and q imply that they are continuous and hence bounded for finite t. By taking absolute values of both sides of (3) and using (1); the Gronwall-Bellman lemma implies that

$$|\eta(\tau)| \leq \max_{0 \leq t \leq T} |z_1(t)| \exp\left[\max_{0 \leq t \leq T} |g(t)| \int_0^T c(\tau)\, d\tau\right]. \tag{12}$$

which is finite for finite T, therefore, α_T is bounded and belongs to $L_2(0, \infty)$. Thus from (8), (9) and (11) we see that η_T, $\dot{\eta}_T$ as well as r are elements of $L_2(0, \infty)$. Let $A_T(i\omega)$, $\Lambda_T(i\omega)$ and $R(i\omega)$ be the Fourier transforms of α_T, λ_T and r respectively.

Then from Eqs. (8) - (11)

$$\Lambda_T(i\omega) = - \left[(1 + i\omega q)\, G(i\omega) + \frac{1}{k}\right] A_T(i\omega) - R(i\omega) \tag{13}$$

Let

$$\rho(T) = \int_0^\infty \lambda_T(t)\, \alpha_T(t)\, dt. \tag{14}$$

Since $\operatorname{Re}(1 + i\omega q)\, G(i\omega) + \frac{1}{k} \geq \delta < 0$, by Lemma 3 (in Appendix) we have

$$\rho(T) = \int_0^T \left\{\eta(t) - \frac{\alpha(t)}{k}\right\} \alpha(t)\, dt + q \int_0^T \alpha(t)\, \dot{\eta}(t)\, dt \leq C_1 \tag{15}$$

where $C_1 = \frac{1}{4\delta} \int_0^\infty |r(t)|^2\, dt = \frac{1}{4\delta} \int_0^\infty |z_1(t) + q\, \dot{z}_1(t)|^2\, dt$, is finite and independent of T. Since $\alpha(t) = c(t)\, \eta(t)$, by adding and substracting $q \int_0^T \frac{\dot{c}(t)\, \eta^2(t)}{2}\, dt$ in the left-hand side of (15), we have

$$\int_0^T \left\{ \eta(t) - \frac{c(t)\ \eta(t)}{k} \right\} c(t)\ \eta(t)\ dt - \frac{q}{2} \int_0^T \dot{c}(t)\ \eta^2(t)\ dt$$

$$+ \frac{q}{2} \quad (T)\ \eta^2(T) \leq C_1 + \frac{q}{2}\ c(0)\ \eta^2(0) \tag{16}$$

or

$$\int_0^T \left\{ c(t) \left[1 - \frac{c(t)}{k} \right] - \frac{q}{2}\ \dot{c}(t) \right\} \eta^2(t)\ dt + \frac{q}{2}\ c(T)\ \eta^2(T) \leq C_1^* \tag{17}$$

where $C_1^* = C_1 + \frac{q}{2}\ c(0)\ \eta^2(0)$ is finite and independent of T, since $\eta(0) = - z_1(0)$, is a finite number.

Since $c(t) \left[1 - \frac{c(t)}{k} \right] - \frac{q}{2}\ \dot{c}(t) \geq m > 0$ and q is nonnegative, each integral in the left-hand side of (17) is nonnegative. Therefore

$$\int_0^T \left\{ c(t) \left[1 - \frac{c(t)}{k} \right] - \frac{q}{2}\ \dot{c}(t) \right\} \eta^2(t)\ dt \leq C_1^* \tag{18}$$

and

$$\frac{q}{2}\ c(T)\ \eta^2(T) \leq C_1^* . \tag{19}$$

From (18) and from the fact that C_1^* is independent of T we have

$$\int_0^\infty \eta^2(t)\ dt \leq \frac{C_1^*}{m}, \text{ a finite number,} \tag{20}$$

i.e., $\eta \in L_2(0, \infty)$. Hence $\alpha(t) \in L_2(0, \infty)$. From (3), as $t \to \infty$

$$\lim_{t \to \infty} \eta(t) = - \lim_{t \to \infty} \int_0^t g(t - \tau)\ \alpha(\tau)\ d\tau . \tag{21}$$

By Lemma 1 (in Appendix)

$$\lim_{t \to \infty} \eta(t) = 0 . \tag{22}$$

Hence $\eta \in L_2(0, \infty)$ and $\eta(t) \to 0$ as $t \to \infty$. But $\eta(t)$ is bounded for finite t, it is readily seen that $\eta(t)$ is bounded for all $t \geq 0$. Thus, in view of (3)

$$y \in L_2(0, \infty) \cap L_\infty(0, \infty) \text{ and } y(t) \to 0 \text{ as } t \to \infty.$$

Note that for $q \neq 0$, $\eta(t)$ is bounded by $\sqrt{\frac{2C_1^*}{\epsilon q}}$ (from (19)). Q. E. D.

In the theorem we observe that as $|\dot{c}(t)|$ approaches zero for all $t \geq 0$, the Popov condition[5] is obtained. Moreover, if ϵ is chosen to be arbitrarily small, q must be chosen small accordingly. In the limit, Theorem 1 coincides with the results obtained by Rozenvasser[4] and Sandberg.[7]

Corollary 1: Let m_1 and m_2 be two positive real numbers such that $-m_1 \leq \dot{c}(t) \leq m_2$ for $t \geq 0$. Then if, in the theorem, the condition (i) is replaced by (i)':

$$\text{(i)}' \; -\frac{2\epsilon}{m_1}\left(1 - \frac{\epsilon}{k}\right) < q < \frac{2\epsilon}{m_2}\left(1 - \frac{\epsilon}{k}\right) \tag{23}$$

the same conclusion holds.

Proof: Since $0 < \epsilon \leq c(t) \leq k - \epsilon$, we see that $c(t)\left[1 - \frac{c(t)}{k}\right] \geq \epsilon\left(1 - \frac{\epsilon}{k}\right)$ for all $t \geq 0$. Hence for q such that $-\frac{2\epsilon}{m_1}\left(1 - \frac{\epsilon}{k}\right) < q < \frac{2\epsilon}{m_2}\left(1 - \frac{\epsilon}{k}\right)$ the function $c(t)\left[1 - \frac{c(t)}{k}\right] - \frac{q}{2}\dot{c}(t)$ is positive for all $t \geq 0$. By Theorem 1, the same conclusion holds.

Q. E. D.

Frequently, in the study of parametrically excited dynamical systems, the variation of the time-varying gain is sinusoidal. We have the following:

Corollary 2:* Consider the same feedback system as shown in Fig. 1, in which the time-varying gain is of the form:

$$c(t) = \frac{k}{2}\,[1 + a \sin(\omega_0\, t + \theta)] \tag{24}$$

for all $t \geq 0$ with $\omega_0 > 0$, $0 \leq a < 1$.

Let the hypotheses of Theorem 1 be satisfied with

$$|q| < \frac{2\sqrt{1 - a^2}}{\omega_0} \quad \text{if} \quad \frac{1}{\sqrt{2}} < a \;\; 1$$

or (25)

$$|q| < \frac{1}{a\,\omega_0} \quad \text{if} \quad 0 \leq a \leq \frac{1}{\sqrt{2}}$$

If

$$\operatorname{Re}(1 + i\,\omega\, q)\; G(i\omega) + \frac{1}{k} \geq \delta > 0 \tag{26}$$

is satisfied for all $\omega \in (-\infty, \infty)$, then for any initial state,

$$y \in L_2(0, \infty) \cap L_\infty(0, \infty) \text{ and } y(t) \to 0 \text{ as } t \to \infty .$$

Proof: Let

$$h(t) = c(t)\left[1 - \frac{c(t)}{k}\right] - \frac{q}{2}\,\dot{c}(t) \tag{27}$$

then, by putting (24) in (27), we have

$$h(t) = \frac{k}{4}\left[1 - a^2 \sin^2(\omega_0\, t + \theta) - q\, a\, \omega_0 \cos(\omega_0 t + \theta)\right] . \tag{28}$$

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* After the general result stated in Theorem 1 had been obtained, the author noticed that a similar result for sinusoidal case (stated in Corollary 2) was obtained by Rekasius and Rowland[13] for lumped systems only in an unpublished report. However, our approach and results are different from theirs and our results hold for lumped as well as distributed systems.

It is shown in Lemma 4 (in Appendix) that $h(t) > 0$ for all $t \geq 0$ if and only if (25) is true. Hence by Theorem 1, we conclude that

$$y \in L_2(0, \infty) \cap L_\infty(0, \infty) \text{ and } y(t) \to 0 \text{ as } t \to \infty . \qquad \text{Q. E. D.}$$

Suppose that the linear time-invariant subsystem G includes an integrator. Then with slight modification on the system, we have

Corollary 3: Consider the same feedback system as shown in Fig. 1. Suppose that 1) $0 < \epsilon < c(t) \leq k - \epsilon$ for all $t \geq 0$, 2) G has the unit impulse response of the form $g(t) = d + g_1(t)$ for $t \geq 0$, where d is a positive real constant and g_1 satisfies the condition (G.3), 3) $z_1(0)$ is finite, $z_1(t) \to z_{1\infty}$ as $t \to \infty$ and $(z_1 - z_{1\infty})$, $\dot{z}_1$ are elements of $L_2(0, \infty)$ where $z_{1\infty}$ is a finite number. Then the conclusions of Theorem 1 still hold.

Proof: As shown in Lemma 2, the original system is equivalent to the system with α, G replaced by $\hat{\alpha}$, $\hat{G}$ respectively, where

$$\hat{\alpha}(t) = \hat{c}(t)\,\eta(t), \qquad \hat{c}(t) = k - c(t) \tag{29}$$

and
$$\hat{G}(s) = \frac{-G(s)}{1 + k\,G(s)} . \tag{30}$$

It can be shown[9] that the equivalent system satisfies all the conditions of Theorem 1. A direct calculation shows that for all ω

$$\mathrm{Re}(1 - i\,\omega\,q)\,\hat{G}(i\omega) + \frac{1}{k - \epsilon_0}$$

$$= \frac{1}{|1 + k\,G(i\omega)|^2}\left\{\mathrm{Re}\,(1 + i\,\omega\,q)\,G(i\omega) + \frac{1}{k}\right\} + \frac{\epsilon_0}{k(k - \epsilon_0)} \tag{31}$$

where $\epsilon_0 > 0$ and is sufficiently small. We see that, for all ω

$$\mathrm{Re}\,(1 + i\,\omega q)\; G(i\omega) + \frac{1}{k} > 0 \tag{32}$$

implies that $$\mathrm{Re}\,(1 - i\,\omega q)\; \hat{G}(i\omega) + \frac{1}{k - \epsilon_0} > 0\,. \tag{33}$$

Now, we observe that

$$\hat{c}\,(t) \left[1 - \frac{\hat{c}(t)}{k}\right] - \frac{q}{2}\; \dot{\hat{c}}(t) = c\,(t) \left[1 - \frac{c\,(t)}{k}\right] - \frac{q}{2}\; \dot{c}\,(t) \tag{34}$$

for all $t \geq 0$.

Since $\hat{c}(t)$ is a bounded function, it follows that, for all $t > 0$, $c\,(t)\left[1 - \frac{c\,(t)}{k}\right] - \frac{q}{2}\;\dot{c}\,(t) \geq m > 0$ implies that

$$\hat{c}\,(t) \left[1 - \frac{\hat{c}\,(t)}{k - \epsilon_0}\right] - \frac{q}{2}\; \dot{\hat{c}}\,(t) \geq \hat{m} > 0. \tag{35}$$

By Theorem 1, the equivalent system $\hat{G}$ is stable with $\epsilon \leq \hat{c}(t) \leq (k - \epsilon_0) - \epsilon$ for all $t \geq 0$. This then implies the stability of the original system with $\epsilon < C(t) \leq k - \epsilon$, which is precisely what was to be proved. Q. E. D.

Note that from (31) we can weaken the inequality (32) to $\mathrm{Re}\,(1 + i\,\omega q)\; G(i\omega) + \frac{1}{k} \geq 0$ provided $G(i\omega) \neq -\frac{1}{k}$ for all ω.

Application: We now apply the main result to the following parametric circuit (Fig. 2):

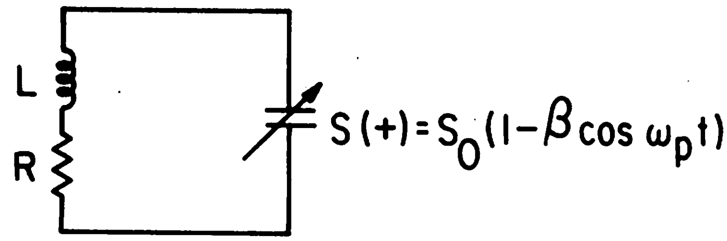

Fig. 2. A simple time-varying circuit.

Let η be the charge on the time-varying capacitor. Applying Kirchhoff's laws to the circuit, we have

$$\frac{d^2\eta}{dt^2} + \frac{R}{L}\frac{d\eta}{dt} + \frac{S(t)}{L}\eta = 0 \quad \text{for } t \geq 0 \tag{36}$$

where $S(t) = S_0(1 - \beta \cos \omega_p t)$ for $t \geq 0$, and R, L, S_0, β, ω_p are positive. ω_p and $\beta(0 \leq \beta < 1)$ are the pump-frequency and the index of modulation respectively. Let

$$A = \frac{R}{L}; \quad b_0 = \frac{S_0}{L}.$$

Then (36) can be rewritten, by adding $D\, b_0\, \eta$ to both sides, as follows:

$$\frac{d^2\eta}{dt^2} + A\frac{d\eta}{dt} + D\, b_o\, \eta = -(1 - D)\, b_0 \left[1 - \frac{\beta}{(1 - D)} \cos \omega_p t\right] \eta \quad \text{for } t \geq 0 \tag{37}$$

where $0 < D < 1$ such that $\frac{\beta}{1 - D} < 1$.

From (37) we obtain

$$\eta(t) = z(t) - \int_0^t g(t - \tau) \left[c(\tau)\, \eta(\tau) \right] d\tau \quad \text{for } t \geq 0 \tag{38}$$

in which z is the solution of

$$\frac{d^2 z}{dt^2} + A \frac{dz}{dt} + D\, b_0\, z = 0 \tag{39}$$

and

$$c(t) = (1 - D)\, b_0 \left[1 - \frac{\beta}{(1 - D)} \cos \omega_p t \right] \tag{40}$$

$$g(t) = \mathcal{L}^{-1} \{ G(s) \} = \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + A s + D\, b_0\, s} \right\} . \tag{41}$$

In accordance with our assumption that A and $D b_0$ are positive, it follows that z, g satisfy the conditions of Theorem 1. Thus, by Corollary 2, if there exist δ and q such that

$$\text{(i)} \quad |q| < \frac{2\sqrt{1 - \left(\frac{\beta}{1 - D}\right)^2}}{\omega_p} \quad \text{if} \quad \frac{1}{\sqrt{2}} < \frac{\beta}{1 - D} < 1 \tag{42}$$

$$\text{or} \quad |q| < \frac{1 - D}{\beta\, \omega_p} \quad \text{if} \quad 0 \leq \frac{\beta}{1 - D} \leq \frac{1}{\sqrt{2}} \tag{43}$$

and

$$\text{(ii)} \quad \mathrm{Re} \left\{ \frac{1 + i \omega q}{(-\omega^2 + D b_0) + i A \omega} \right\} + \frac{1}{2(1 - D)\, b_0} \geq \delta > 0 \quad \text{for all } \omega . \tag{44}$$

Then for any initial state, η is bounded and approaches zero as $t \rightarrow \infty$.

Let $Q = \sqrt{b_0}/A$, the quality factor of the resonant circuit with S(t) replaced by S_0. By a straight-forward calculation, it can be shown that if

$$q > q_0 = \frac{2Q^2\left[1 - \sqrt{D(2 - D)}\right] - 1}{2\sqrt{b_0}\, Q(1 - D)} \tag{45}$$

then condition (ii) is satisfied for all ω. Let us take $\beta = 0.16$, $\omega_p = 2\sqrt{b_0}$, $D = 0.8$ as an example. Using (42) (since $\beta/1 - D = 0.8$) and incorporating with (45) we find that if $Q < Q_0 = 8.68$, then there exist q and δ such that both (i) and (ii) are satisfied. For values of D other than 0.8, the corresponding values of Q_0 are obtained and plotted in Fig. 3.

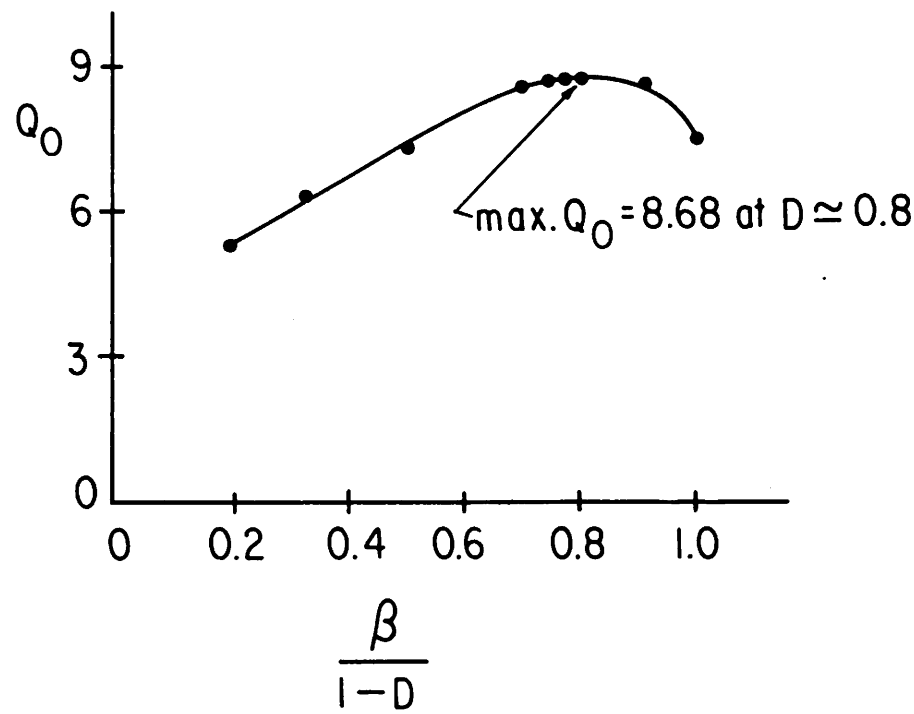

Fig. 3. $\beta = 0.16$, $\omega_p = 2\sqrt{b_0}$.

Figure 3 shows that, with $\beta = 0.16$ and $\omega_p = 2\sqrt{b_0}$, Q_0 attains its maximum for $D \doteq 0.8$. In other words, if $Q = \sqrt{b_0}/A$ is less than 8.68, then the circuit is stable.

Table 1: Q_0 is computed for $\omega_p = 2\sqrt{b_0}$

β	0.024	0.04	0.06	0.08	0.12	0.16	0.20
Q_0	59	35	23	17.8	11.8	8.68	6.8
Q_1	41.7	25	16.7	12.5	8.3	6.3	5
Q_2	100	50	33	25	17	12.5	10

The values of Q_0 given in Table 1 are obtained in a similar manner with $\omega_p = 2\sqrt{b_0}$ and β ranging from 0.024 to 0.20. It says that if, for a given β, $\sqrt{b_0}/A$ does not exceed the corresponding value of Q_0, then for any initial state the solution of (36) is bounded and approaches zero as $t \to \infty$. For comparison, it is of interest to consider the recent results of Sandberg[11] concerning the solutions of second order differential equations similar to (36) with the exception that S(t) is no longer a periodic function of time. Sandberg finds that if, with a given β, $\sqrt{b_0}/A$ does not exceed the corresponding value of Q_1 in Table 1, then for any S(t) (not necessarily periodic) such that $S_0(1 - \beta) \leq S(t) \leq S_0(1 + \beta)$ for $t \geq 0$, all solutions of (36) approach zero as $t \to \infty$. We observe that, for the special case of (36) with $\omega_p = 2\sqrt{b_0}$, our results show the improvement over Sandberg's results.

In the above example we have chosen ω_p to be $2\sqrt{b_0}$. This is because we also want to make comparison with the results of Phillips[12] concerning the determination of the values of reactance variation in order that parametric oscillations can just be maintained in the same circuit of Fig. 2. Using a semigraphical technique and the results of McLachlan concerning the Mathieu equation, Phillips finds that if, with a given $\beta \leq 0.2$, $\sqrt{b_0}/A$ exceeds the corresponding value of Q_2 in Table 1, then there exists ω_p, in the neighborhood of $2\sqrt{b_0}$, for which all solutions of (36) do not approach zero as $t \to \infty$. Observe that the values of Q_2 are only roughly 1.5 times the corresponding values of Q_0.

It must be noted that if ω_p is reduced, then, with a given β, the corresponding value of Q_0 will be increased. For example, let $\beta = 0.16$, $\omega_p = \frac{1}{2}\sqrt{b_0}$, the corresponding Q_0 is obtained to be 21.5, which is much larger than 6.3 as was obtained by Sandberg for the general case. Moreover, as ω_p approaches infinity, values of Q_0 approach the corresponding values of Q_1.

CONCLUSION

In this paper we have obtained sufficient conditions for the stability of feedback systems containing a single linear time-varying element whose input-output characteristic lies within a finite sector. An improved frequency-domain criterion, which utilizes information on the time derivative of the time-varying gain, is developed. The main result stated in Theorem 1 together with its corollaries holds for general distributed systems as well as lumped systems. A restricted class of inputs, which satisfies all the conditions imposed on the zero-input response, is also allowed.

An application of the main result to a parametrically excited system is given to illustrate the improvement of the new stability criterion over the existing results. In general, when more information about the time-varying element is available, better stability results can be obtained.

APPENDIX

Lemma I: Let f_1, f_2 be real-valued functions and elements of $L_2(0, \infty)$. If $h(t) = \int_0^t f_1(t - \tau) f_2(\tau) d\tau$ for $t \geq 0$, then

$$\lim_{t \to \infty} h(t) = 0 .$$

Proof: Let $F_1(i\omega)$, $F_2(i\omega)$ be the Fourier transforms of f_1, f_2 respectively. Then

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(i\omega) F_2(i\omega) e^{i\omega t} dt \tag{A-1}$$

But $F_1(i\omega)$ and $F_2(i\omega)$ are elements of $L_2(0, \infty)$, it follows by Schwartz inequality that $F_1(i\omega) F_2(i\omega)$ is an element of $L_1(0, \infty)$. Thus by Riemann-Lebesque lemma,

$$\lim_{t \to \infty} h(t) = 0. \tag{A-2}$$

Q. E. D.

Lemma 2: Let the hypotheses of Theorem 1 be satisfied with $q \geq 0$. Then the same conclusions hold if the same hypotheses are satisfied with $q \leq 0$.

Proof: Let

$$\hat{\alpha}(t) = k\eta(t) - \alpha(t)$$

$$= \hat{c}(t) \eta(t) \tag{A-3}$$

where $\hat{c}(t) = k - c(t)$, which satisfies the same condition on $c(t)$, i.e.,

$$0 < \epsilon \leq \hat{c}(t) \leq k - \epsilon . \tag{A-4}$$

Then it is readily seen that (3) is equivalent to:

$$\eta(t) = -\hat{z}_1(t) - \int_0^t \hat{g}(t-\tau)\,\hat{\alpha}(\tau)\,d\tau \quad \text{for} \quad t \geq 0 \tag{A-5}$$

where

$$\hat{g}(t) = \mathcal{L}^{-1}\{\hat{G}(s)\} = \mathcal{L}^{-1}\left\{\frac{-G(s)}{1+kG(s)}\right\} \tag{A-6}$$

$$= -g(t) + k\int_0^t \hat{g}(t-\tau)\,g(\tau)\,d\tau \quad \text{for} \quad t \geq 0 \tag{A-7}$$

and

$$\hat{z}_1(t) = z_1(t) + k\int_0^t \hat{g}(t-\tau)\,z_1(\tau)\,d\tau \quad \text{for} \quad t \geq 0. \tag{A-8}$$

Next, the geometric interpretation of the condition (5),[3, 5] which is satisfied by G, implies that the Nyquist diagram of G does not encircle the critical point $(-\frac{1}{k}, 0)$ and $G(i\omega) \neq -\frac{1}{k}$ for all ω, i.e., $1 + kG(s) \neq 0$ for $\text{Re } s \geq 0$. Then $g \in L_1(0, \infty)$ implies that $\hat{g} \in L_1(0, \infty)$ by appealing to a theorem of Paley and Wiener.[8] By a theorem in Ref. 10, it follows that $\hat{g}$, $\hat{z}_1$, $\dot{\hat{g}}$ and $\dot{\hat{z}}_1$ are elements of $L_2(0, \infty)$. Moreover, from (A-8), $\hat{z}_1(0) = z_1(0)$ is a finite number, and as $t \to \infty$, $\lim_{t\to\infty} \hat{z}_1(t) = \lim_{t\to\infty} k\int_0^t \hat{g}(t-\tau)\,z_1(\tau)\,d\tau = 0$ by Lemma 1. Thus the equivalent system (A-5) satisfies the same conditions of the original system (3). A direct calculation[3] shows that for all ω

$$\text{Re}\,(1 - i\omega q)\,\hat{G}(i\omega) + \frac{1}{k} = \frac{1}{|1 + kG(i\omega)|^2}\left\{\text{Re}\,(1 + i\omega q)\,G(i\omega) + \frac{1}{k}\right\} \tag{A-9}$$

Since $G(i\omega)$ is bounded for all ω, $\mathrm{Re}(1 + i\omega q)\, G(i\omega) + \frac{1}{k} > 0$ implies that $\mathrm{Re}(1 - i\omega q)\, \hat{G}(i\omega) + \frac{1}{k} > 0$. Furthermore, we observe that

$$\hat{c}(t)\left[1 - \frac{\hat{c}(t)}{k}\right] - \frac{q}{2}\,\dot{\hat{c}}(t) = c(t)\left[1 - \frac{c(t)}{k}\right] - \frac{q}{2}\,\dot{c}(t)\,. \tag{A-10}$$

Therefore, if (4) and (5) are satisfied by G for some $q \leq 0$ and some k, then the same conditions (4) and (5) (with a different but still positive δ) are satisfied by $\hat{G}$ for $-q \geq 0$ and the same k.

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Lemma 3: Let f_1, f_2, f_3, h be real-valued functions and elements of $L_2(0, \infty)$. Let $F_1(i\omega)$, $F_2(i\omega)$, $F_3(i\omega)$, $H(i\omega)$ be their corresponding Fourier transforms. If

$$F_1(i\omega) = -H(i\omega)\, F_3(i\omega) + F_2(i\omega) \quad \text{for all } \omega \tag{A-11}$$

and if there exists a real number δ such that

$$\mathrm{Re}\, H(i\omega) \geq \delta > 0 \quad \text{for all } \omega \tag{A-12}$$

then

$$\int_0^\infty f_1(t)\, f_3(t)\, dt \leq \frac{1}{8\pi} \int_{-\infty}^{\infty} \frac{|F_2(i\omega)|^2}{\mathrm{Re}\, H(i\omega)}\, d\omega \leq C_1 \tag{A-13}$$

where

$$C_1 \triangleq \frac{1}{8\pi\delta} \int_{-\infty}^{\infty} |F_2(i\omega)|^2\, d\omega = \frac{1}{4\delta} \int_{-\infty}^{\infty} |f_2(t)|^2\, dt\,. \tag{A-14}$$

Proof: The proof is a straightforward application of the completing the square technique (see Refs. 3 and 5).

Lemma 4: The function,

$$h(t) = \frac{k}{4}\left[1 - a^2 \sin^2(\omega_0 t + \theta) - q a \omega_0 \cos(\omega_0 t + \theta)\right] \tag{A-15}$$

with $\omega_0 > 0$, $0 \leq a < 1$, is positive for all $t \geq 0$ if and only if

$$|q| < \frac{2\sqrt{1-a^2}}{\omega_0} \qquad \text{if} \quad \frac{1}{\sqrt{2}} < a < 1$$

and (A-16)

$$|q| < \frac{1}{a\omega_0} \qquad \text{if} \quad 0 \leq a < \frac{1}{\sqrt{2}}$$

Proof: Differentiating (A-15), we have

$$\frac{d\,h(t)}{dt} = \frac{k}{4}\sin(\omega_0 t + \theta)\left[-2a^2 \omega_0 \cos(\omega_0 t + \theta) + q a \omega_0^2\right]. \tag{A-17}$$

Let t_m be some time such that $\left.\frac{d\,h(t)}{dt}\right|_{t=t_m} = 0$. Then from (A-17) one obtains either (i) $\cos(\omega_0 t_m + \theta) = q\omega_0/2a$ or (ii) $\sin(\omega_0 t_m + \theta) = 0$.

For case (i) we have

$$\left|\frac{q\omega_0}{2a}\right| < 1 \qquad \text{or} \qquad |q| < \frac{2a}{\omega_0} \tag{A-18}$$

and from (A-15)

$$h(t_m) = \frac{k}{4}\left[1 - a^2\left(1 - \frac{q^2\omega_0^2}{4a^2}\right) - q a \omega_0 \frac{q\omega_0}{2a}\right]$$

$$= \frac{k}{4}\left[1 - a^2 - \frac{q^2\omega_0^2}{4}\right]. \tag{A-19}$$

If $1 - a^2 - \frac{q^2 \omega_0^2}{4} > 0$, i.e., $|q| < \frac{2\sqrt{1-a^2}}{\omega_0}$ (A-20)

then $h(t_m) > 0$. But

$$\left.\frac{d^2 h(t)}{dt^2}\right|_{t=t_m} = \frac{k a^2 \omega_0^2}{2} \sin^2(\omega_0 t_m + \theta) > 0, \tag{A-21}$$

we conclude that at t_m, $h(t_m)$ attains minimum. Hence for $|q| < \frac{2\sqrt{1-a^2}}{\omega_0}$ and $|q| < \frac{2a}{\omega_0}$, $h(t) > 0$ for all $t \geq 0$.

For case (ii), $\sin(\omega_0 t_m + \theta) = 0$, or $\cos(\omega_0 t_m + \theta) = \pm 1$, then from (A-15)

$$h(t_m) = \begin{cases} \frac{k}{4}[1 - q a \omega_0] & \text{for } \cos(\omega_0 t_m + \theta) = 1 & \text{(A-22-a)} \\ \frac{k}{4}[1 + q a \omega_0] & \text{for } \cos(\omega_0 t_m + \theta) = -1 & \text{(A-22-b)} \end{cases}$$

Taking the second derivative of h(t), one obtains

$$\left.\frac{d^2 h(t)}{dt^2}\right|_{t=t_m} = \frac{k}{4}\left[-2a^2 \omega_0^2 \cos 2(\omega_0 t_m + \theta) + q a \omega_0^3 \cos(\omega_0 t_m + \theta)\right]$$

$$= \begin{cases} \frac{k\omega_0^2}{4}(-2a + q\omega_0) & \text{for } \cos(\omega_0 t_m + \theta) = 1 & \text{(A-23-a)} \\ \frac{k\omega_0^2}{4}(-2a - q\omega_0) & \text{for } \cos(\omega_0 t_m + \theta) = -1. & \text{(A-23-b)} \end{cases}$$

Corresponding to $\cos(\omega_0 t_m + \theta) = 1$, if $q a \omega_0 < 1$ and $q\omega_0 \geq 2a$, then $h(t_m) > 0$ and $\left.\frac{d^2 h(t)}{dt^2}\right|_{t=t_m} \geq 0$. Hence $h(t) > 0$ for all $t \geq 0$.

Corresponding to $\cos(\omega_0 t_m + \theta) = -1$, if $-qa\omega_0 < 1$ and $-q\omega_0 \geq 2a$, then $h(t_m) > 0$ and $\left.\frac{d^2 h(t)}{dt^2}\right|_{t=t_m} \geq 0$. Hence $h(t) > 0$ for all $t \geq 0$. Combining the two, we see that if $|q|\, a\omega_0 < 1$ and $|q|\, \omega_0 \geq 2a$, then $h(t) > 0$ for all $t \geq 0$.

Thus we have shown that $h(t) > 0$ for all $t \geq 0$ if and only if either

$$\text{(i)} \qquad |q| < \frac{2\sqrt{1-a^2}}{\omega_0} \quad \text{and} \quad |q| < \frac{2a}{\omega_0} .$$

or (A-24)

$$\text{(ii)} \qquad |q| < \frac{1}{a\omega_0} \quad \text{and} \quad |q| \geq \frac{2a}{\omega_0} .$$

But (A-24) is equivalent to:

$$|q| < \frac{2\sqrt{1-a^2}}{\omega_0} \quad \text{if} \quad \frac{1}{\sqrt{2}} < a < 1 .$$

or (A-25)

$$|q| < \frac{1}{a\omega_0} \quad \text{if} \quad 0 \leq a < \frac{1}{\sqrt{2}} .$$

Hence, the lemma is proved. The family of q - a curves of (A-25) with equality sign is shown in Fig. 4.

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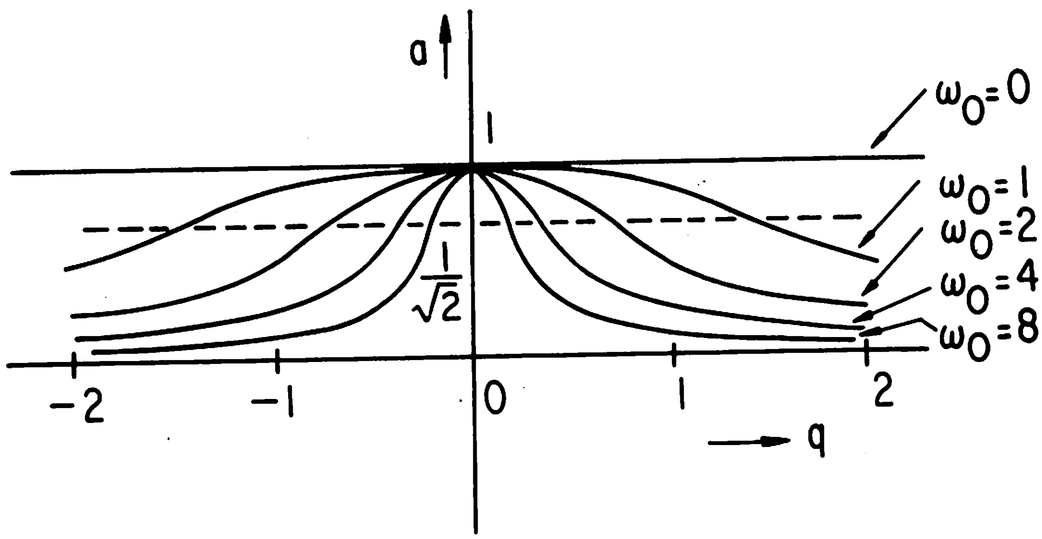

Fig. 4. Family of $\begin{cases} |q| = \dfrac{2\sqrt{1-a^2}}{\omega_0} & \text{for } \dfrac{1}{\sqrt{2}} < a < 1 \\ |q| = \dfrac{1}{a\,\omega_0} & \text{for } 0 \leq a \leq \dfrac{1}{\sqrt{2}} \end{cases}$

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