NORMAL FORM AND STABILITY OF A CLASS OF COUPLED NONLINEAR NETWORKS

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P. P. Varaiya and R-W Liu

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ELECTRONICS RESEARCH LABORATORY
University of California, Berkeley

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ABSTRACT

This paper is concerned with the properties of a class of nonlinear coupled R, L, C networks. Sufficient conditions are given which insure a unique response defined by a set of differential equations in the normal form. Next we study the stability of these networks and relate the property of asymptotic stability with the property of "weak" observability at the resistor terminals.

I. Introduction and Notation

This paper is concerned with the properties of a class of coupled nonlinear R, L, C networks. We first impose conditions on the network topology and on the network elements which insure a unique response defined by a set of differential equations in the normal form. The nature of the conditions and of the results is similar to that given by Desoer and Katzenelson[1] and by Holzmann and Liu.[2] A novel part of the proof is a "constructive" method of solving a class of nonlinear algebraic equations. Next we study the stability of this class of networks and relate the question of asymptotic stability to the notion of observability.[3] We show that for "passive" networks with linear inductors and linear capacitors, the notion of asymptotic stability coincides with that of observability at the resistor terminals. For nonlinear passive networks this equivalence is obtained if we suitably weaken the definition of observability.

As far as possible, the notation used here is that of Kuh and Rohrer.[4] Thus let $\mathcal{N}$ be a nonseparable connected network, and let $\mathcal{T}$ be a normal tree of $\mathcal{N}$. We assume that each tree branch is in parallel with a current source and each link contains a voltage source. The sources are assumed independent. We denote the link element voltages (currents) by $v_R(i_R)$, $v_S(i_S)$, $v_L(i_L)$; the voltage sources - in the links - by e_R, e_S, e_L; the tree branch element voltages (currents) by $v_G(i_G)$, $v_C(i_C)$, $v_\Gamma(i_\Gamma)$ and the current sources - across the tree branches - by j_G, j_C and j_Γ. Then the Kirchhoff voltage law is given by (see 4)

$$v_S + F_{SC} v_C = e_S$$

$$v_R + F_{RC} v_C + F_{RG} v_G = e_R \tag{1}$$

$$v_L + F_{LC} v_C + F_{LG} v_G + F_{L\Gamma} v_\Gamma = e_L$$

and the Kirchhoff current law is given by

$$i_C - F_{SC}^T i_S - F_{RC}^T i_R - F_{LC}^T i_L = j_C$$

$$i_G - F_{RG}^T i_R - F_{LG}^T i_L = j_G \tag{2}$$

$$i_\Gamma - F_{L\Gamma}^T i_L = j_\Gamma \ .$$

II. Normal Form

The following conditions are imposed throughout. Let μ stand for R, L or C. Then there is a normal tree $\mathcal{T}$ of $\mathcal{N}$ such that

C1. μ-elements in the links are coupled among themselves. Dually, the μ-elements in the tree branches are coupled among themselves.

C2. The link elements are either voltage-controlled or flux-controlled, whereas the elements in the tree branches are either current-controlled or charge-controlled.

More explicitly, C1 and C2 become,

$$i_R = \hat{i}_R(v_R) \quad i_L = \hat{i}_L(\phi_L) \quad \text{and} \quad q_S = \hat{q}_S(v_S)$$
$$\tag{3}^{\dagger}$$
$$v_G = \hat{v}_G(i_G) \quad \phi_\Gamma = \hat{\phi}_\Gamma(i_\Gamma) \quad \text{and} \quad v_C = \hat{v}_C(q_C) \,.$$

Equations (1) - (3) can be conveniently rewritten as

$$v_R + F_{RG}\,\hat{v}_G(i_G) = e_R^* \triangleq e_R - F_{RC}\,v_C$$
$$\tag{R}$$
$$-F_{RG}^T\,\hat{i}_R(v_R) + i_G = j_G^* \triangleq j_G + F_G^T\,i_L\,,$$

$$\phi_L + F_{L\Gamma}\,\hat{\phi}_\Gamma(i_\Gamma) \triangleq \phi$$
$$\tag{L}$$
$$-F_{L\Gamma}^T\,\hat{i}_L(\phi_L) + i_\Gamma \triangleq j_\Gamma\,,$$

$$v_S + F_{SC}\,\hat{v}_C(q_C) = e_S$$
$$\tag{C}$$
$$-F_{SC}^T\,\hat{q}_S(v_S) + q_C \triangleq q.$$

We also have the differential equation

† To avoid notational problems we assume that the network is time-invariant.

$$\dot{q} = F_{LC}^T \, i_L + F_{RC}^T \, i_R + j_C$$

$$\dot{\phi} = - F_{LC} \, v_C - F_{LG} \, v_G + e_L \, . \tag{D}$$

We remark that q is the vector of the fundamental cutset charges and ϕ is the vector of the fundamental loop fluxes. We also notice that the equations R, L and C are of the form:

$$x + A\, f(y) = u$$

$$- A^T g(x) + y = v \tag{*}$$

where $x, \; u \in R^n$; $y, \; v \in R^m$; $f: R^m \rightarrow R^n$; $g: R^n \rightarrow R^m$ and A is a fixed $n \times m$ matrix. We now state some conditions on f and g such that (*) has a unique solution in x and y for each value of u and v.

Theorem 1.1 If the functions f and g satisfy conditions H1 and H2 or they satisfy conditions H1 and H3, then (*) has a unique solution.

H1. f and g are differentiable and the Jacobian matrices $F(y) \triangleq \frac{\partial f}{\partial y}(y)$ and $G(x) \triangleq \frac{\partial f}{\partial x}(x)$ are positive semi-definite† for all x and y.

H2. Either $F(y)$ is a symmetric positive definite matrix for all y or $G(x)$ is a symmetric positive definite matrix for all x.

† An $n \times n$ matrix M is positive semi-definite if $\langle x, Mx \rangle > 0$ for all x. It is positive definite if $\langle x, Mx \rangle > 0$ for all $x \neq 0$.

H3. Either $F(y)$ is diagonal for all y or $G(x)$ is diagonal for all x.

Proof: We wish to determine the solutions (if any) to the set of equations

$$\begin{aligned} x + A\, f(y) &= u \\ -A^T g(x) + y &= v\,. \end{aligned} \qquad (*)$$

Define $\alpha(x, y) = \frac{1}{2}\{\|x + A\, f(y) - u\|^2 + \|-A^T g(x) + y - v\|^2\}$. Then x and y solve $(*)$ if and only if $\alpha(x, y) = 0$. Consider the differential equation

$$\frac{dx}{dt} = -\frac{\partial\alpha}{\partial x} = -\{(x + A\, f(y) - u) - [A^T G(x)]^T (-A^T g(x) + y - v)\}$$

$$\frac{dy}{dt} = -\frac{\partial\alpha}{\partial y} = -\{[A\, F(y)]^T (x + A\, f(y) - u) + (-A^T g(x) + y - v)\}\,.$$

Along a solution of the differential equation,

$$\frac{d\alpha}{dt} = \frac{\partial\alpha}{\partial x}\frac{dx}{dt} + \frac{\partial\alpha}{\partial y}\frac{dy}{dt} = -\left\|\frac{\partial\alpha}{\partial x}\right\|^2 - \left\|\frac{\partial\alpha}{\partial y}\right\|^2 \leq 0\,.$$

Define $\quad z = -A^T g(x) + u - v$

and $\quad w = x + A\, f(y) - u\,.$

Then $\frac{d\alpha}{dt} = 0$ if and only if

$$\begin{bmatrix} I & -(A^T G(x))^T \\ (A\,F(y))^T & I \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} \triangleq \begin{bmatrix} & M & \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = 0. \tag{3}$$

Now $\det(m) = \det(I + A\,F(y)\,A^T\,G(x)) = \det(I + A^T\,G(x)\,A\,F(y))$. It is easy to show that H2 implies $\det M \geq 1$ and H3 implies $\det M \geq 1$. Hence (3) holds if and only if $w = 0$ and $z = 0$, i.e., $\frac{d\alpha}{dt} = 0$ if and only if x and y solve (*). Also $\det M \geq 1$ implies that $\alpha(x, y) \to \infty$ as $\|x\| + \|y\| \to \infty$. A theorem of Liapunov[5] shows that (*) has at least one solution. We now prove uniqueness. Suppose (x_1, y_1) and (x_2, y_2) solve (*) for some fixed u and v. Then

$$\begin{aligned} (x_1 - x_2) + A\,(f(y_1) - f(y_2)) &= 0 \\ -A^T\,(g(x_1) - g(x_2)) + (y_1 - y_2) &= 0. \end{aligned} \tag{4}$$

Consider one-dimensional arcs $x(\theta)$ and $y(\theta)$, $0 \leq \theta \leq 1$ given by

$$x(\theta) = \theta x_1 + (1 - \theta)\,x_2$$

$$y(\theta) = \theta y_1 + (1 - \theta)\,y_2\,.$$

Then $\frac{dx}{d\theta} = x_1 - x_2$ and $\frac{dy}{d\theta} = y_1 - y_2$. Furthermore, (4) is equivalent to (5).

$$
\begin{aligned}
&(x_1 - x_2) + A \int_{\theta=0}^{1} F(y(\theta))\,(y_1 - y_2)\,d\theta = 0 \\
&-A^T \int_{\theta=0}^{1} G(x(\theta))\,(x_1 - x_2)\,d\theta + (y_1 - y_2) = 0\,.
\end{aligned}
\tag{5}
$$

But $\int_{\theta=0}^{1} F(y(\theta))\,d\theta$ and $\int_{\theta=0}^{1} G(x(\theta))\,d\theta$ have the same properties as $F(y)$ and $G(x)$ respectively so that (5) holds if and only if $x_1 = x_2$ and $y_1 = y_2$.

Q. E. D.

In the following we assume that f and g satisfy the hypotheses of Theorem 1.1.

Corollary 1.1: The solution of (*) can be obtained as the limit of the solution of an asymptotically stable differential equation.

Remark: If we suitably bound the norms of the matrices $F(*)$ and $G(y)$, then the differential equation can be replaced by a difference equation. See Katzenelson and Seitelman.[6]

Let $w \triangleq g(x)$ and $z \triangleq f(y)$. The proof of the next two corollaries are straightforward and hence omitted.

Corollary 1.2: (a) $\frac{\partial w}{\partial u}$ is positive semi-definite. It is positive definite if G(x) is positive definite for all x. Dually $\frac{\partial z}{\partial v}$ is positive semi-definite. It is positive definite if F(y) is positive definite for all y.

(b) If G(x) and F(y) are symmetric matrices, then $\frac{\partial w}{\partial u}$ and $\frac{\partial z}{\partial v}$ are also symmetric.

Corollary 1.3: (a) If $\|u\| + \|v\| \to \infty$ then $\|x\| + \|y\| \to \infty$.

(b) If f and g also satisfy

$$\langle x, g(x) \rangle > 0 \quad \text{for} \quad x \neq 0$$

$$\langle x, f(y) \rangle > 0 \quad \text{for} \quad y \neq 0$$

then

$$\langle u, w \rangle > 0 \quad \text{for} \quad u \neq 0$$

and

$$\langle v, z \rangle > 0 \quad \text{for} \quad v \neq 0 .$$

We now impose the conditions of Theorem 1.1 on the network characteristics.

Theorem 2.1: If each of the equations L, C, and R satisfies the hypotheses of Theorem 1, then the network response is unique and is defined by a differential equation in normal form.

Proof: Theorem 1.1 implies that the equations R, L, and C can be solved giving,

$$v_G = \tilde{v}_G(e_R^*, j_G^*), \quad i_R = \tilde{i}_R(e_R^*, j_G^*) \tag{6}$$

$$i_L = \tilde{i}_L(\phi, j_\Gamma), \quad v_C = \tilde{v}_C(e_S, q) \tag{7}$$

Furthermore since $e_R^* \triangleq e_R - F_{RC} v_C$ and $j_G^* \triangleq j_G + F_{LG}^T i_L$ we can obtain (using 7), v_G and i_R in terms of q, ϕ, e_R and j_G. Substituting these functions in (D) we obtain the right hand side of D as a function of q, ϕ, and the sources.

Q. E. D.

III. Stability

a) From now on we assume that all the sources are identically zero. We also suppose that the network satisfies all the hypotheses of Theorem 2.1 and in addition the following conditions

H4. The resistors are passive, i.e.,

$$\langle v_R, \tilde{i}_R(v_R) \rangle > 0 \quad \text{if} \quad v_R \neq 0$$

$$\langle i_G, \tilde{v}_G(i_G) \rangle > 0 \quad \text{if} \quad i_G \neq 0 .$$

H5. The inductors and capacitors are passive, i.e.,

$$\left\langle i_\Gamma, \hat{\phi}_\Gamma(i_\Gamma) \right\rangle > 0 \text{ if } i_\Gamma \neq 0$$

$$\left\langle \phi_L, \hat{i}_L(\phi_L) \right\rangle > 0 \text{ if } \phi_L \neq 0$$

$$\left\langle v_S, \hat{q}_S(v_S) \right\rangle > 0 \text{ if } v_S \neq 0$$

and $$\left\langle q_C, \hat{v}_C(q_C) \right\rangle > 0 \text{ if } q_C \neq 0 .$$

Furthermore, it will be assumed that the Jacobian matrices associated with the inductors and capacitors, i. e, the matrices $\frac{\partial \phi_\Gamma}{\partial i_\Gamma}$, $\frac{\partial i_L}{\partial \phi_L}$, $\frac{\partial q_S}{\partial v_S}$ and $\frac{\partial v_C}{\partial q_C}$ are symmetric.

By the well-known theorem on exact differential forms[7] we obtain

<u>Lemma 3.1</u>. The hypothesis H5 implies that there are real-valued functions p_Γ, p_L, p_S, and p_C of the variables i_Γ, ϕ_L, v_S and v_C such that,

$$\frac{\partial p_\Gamma}{\partial i_\Gamma} = \phi_\Gamma, \quad \frac{\partial p_L}{\partial \phi_L} = i_L, \quad \frac{\partial p_S}{\partial v_S} = q_S \quad \text{and} \quad \frac{\partial p_C}{\partial q_C} = v_S .$$

Furthermore each of the $p_i \geq 0$ and $p_i(x_i) = 0$ if and only if $x_i = 0$.

We will now make assumption

H6. The tree-branch capacitors and the link inductors are realistic, i.e.,

$$p_C(q_C) + p_L(\phi_L) \to \infty \quad \text{as}$$

$$\| q_C \| + \| \phi_L \| \to \infty \; .$$

Remark: H5 is equivalent to saying that the inductors and capacitors represent a conservative system. H6 is equivalent to saying that as the charge in the tree capacitors or the flux in the link inductors become unbounded, the energy stored also becomes unbounded.

b) A built-in Liapunov function: Let q and ϕ be fixed and consider,

$$p(q, \phi) \triangleq \int_0^q \langle v_C, dq' \rangle + \int_0^\phi \langle i_L, d\phi' \rangle \; . \tag{8}$$

We first remark that assumption H5 and Corollary 1.2b imply that $\frac{\partial v_C}{\partial q}$ and $\frac{\partial i_L}{\partial \phi}$ are symmetric matrices so that the integral in (8) is independent of the path of integration. Therefore $p(q, \phi)$ is a well defined number.

Lemma 3.2. (a) $p(q, \phi) \geq 0$ for all q, ϕ and $p(q, \phi)$ if and only if $q = 0$ and $\phi = 0$.

(b) Furthermore $p(q, \phi) \to \infty$ as $\| q \| + \| \phi \| \to \infty$.

Proof: (a) By Corollary 1.2a $\frac{\partial v_C}{\partial q}$ and $\frac{\partial i_L}{\partial \phi}$ are positive semi-definite matrices. This implies that $p(q, \phi) \geq 0$. By H5 and Corollary 1.3b $p(q, \phi) = 0$ if and only if $q = 0$ and $\phi = 0$.

(b) By Corollary 1.3a, $\|q\| + \|\phi\| \to \infty$ implies that $\|q_C\| + \|v_S\| \to \infty$. Now from (8) and the equations (L) and (C) we have

$$p(q, \phi) = \int_0^{q_C} \langle v_C, dq_C' \rangle + \int_0^{q_S} \langle v_S, dq_S' \rangle + \int_0^{\phi_L} \langle i_L, d\phi_L' \rangle + \int_0^{\phi_\Gamma} \langle i_\Gamma, d\phi_\Gamma' \rangle$$

$$p(q, \phi) = p_C(q_C) + p_L(\phi_L) + \int_0^{q_S} \langle v_S, dq_S' \rangle + \int_0^{q_\Gamma} \langle i_\Gamma, d\phi_\Gamma \rangle . \tag{9}$$

Now,

$$\int_0^{q_S} \langle v_S, dq_S' \rangle + \int_0^{\phi_\Gamma} \langle i_\Gamma, d\phi_\Gamma \rangle = \langle v_S, q_S \rangle + \langle i_\Gamma, \phi_\Gamma \rangle - p_S(v_S) - p_\Gamma(i_\Gamma) \geq 0$$

by H5. Also by H6, $p_C(q_C) + p_L(\phi_L) \to \infty$ as $\|q_C\| + \|\phi_L\| \to \infty$. Hence $p(q, \phi) \to \infty$ as $\|q\| + \|\phi\| \to \infty$.

Q. E. D.

Suppose $q(t)$ and $\phi(t)$ is the solution of the normal form starting in the initial condition $q(0)$ and $\phi(0)$. Let $p(t) \triangleq p(q(t), \phi(t))$. Then,

$$\frac{dp}{dt} = \frac{\partial p}{\partial q}\frac{dq}{dt} + \frac{\partial p}{\partial \phi}\frac{d\phi}{dt}$$

$$= -\langle v_R(t), i_R(t)\rangle - \langle v_G(t), i_G(t)\rangle \text{ by (D), (R), (L), (C).}$$

$$\leq 0 \text{ for all } t \text{ and}$$

$$\frac{dp}{dt} = 0 \text{ if and only if } v_R = 0 \text{ and } i_G = 0.$$

The last two statements follow from H4. We therefore have

__Theorem 3.1__: (a) The zero-input of a network satisfying the conditions of Theorem 2.1 and H4-H6 is bounded.

(b) The network is globally asymptotically stable[5] if and only if $v_R(t) \equiv 0$ and $i_G(t) \equiv 0$ for all t implies that $q(t) \equiv 0$ and $\phi(t) \equiv 0$ for all t.

__Proof__: (a) Let $q(0)$ and $\phi(0)$ be the initial state and $p(t) = p(q(t), \phi(t))$. Then since $\frac{dp}{dt} \leq 0$ we have $p(t) \leq p(0)$ for all t. Since $p(q, \phi) \to \infty$ as $\|q\| + \|\phi\| \to \infty$ we have $q(t)$ and $\phi(t)$ bounded for all t.

(b) Consider the set of solutions of the normal form for which p is a constant, i.e., $\frac{dp}{dt} \equiv 0$ for all t, or equivalently $v_R(t) \equiv 0$ and

$i_G(t) \equiv 0$. By (D) and (R) this set is identical to the solutions of the following pair of equations,

$$
\begin{aligned}
q &= F_{LC}^T \, i_L \\
\phi &= - F_{LC} \, v_C
\end{aligned}
\tag{9}
$$

and

$$
F_{RC} \, v_C(t) \equiv 0, \quad F_{LG}^T \, i_L \equiv 0 \, . \tag{10}
$$

It is well known that all the trajectories of the network state q and ϕ converge to the trajectories which satisfy (9) and (10). Hence the network is globally asymptotically stable if and only if (9) and (10) have the trivial solution $q(t) \equiv 0$ and $\phi(t) \equiv 0$.

Q. E. D.

Definition 3.1: (a) The network is observable at the resistor terminals if $v_G(t) = 0$, $i_R(t) = 0$ over a nonvanishing time interval implies that $q(t) \equiv 0$ and $\phi(t) \equiv 0$ for all t.

(b) The network is weakly observable at the resistor terminals if $v_G(t) \equiv 0$ and $i_R(t) \equiv 0$ for all t implies that $q(t) \equiv 0$ and $\phi(t) \equiv 0$.

Corollary 3.2: (a) The network is observable $\Longrightarrow$ the network is weakly observable.

(b) The network is globally asymptotically stable if and only if it is weakly observable at the resistor terminals.

(c) For networks with linear inductors and linear capacitors but non-linear resistors the network is asymptotically stable if and only if it is observable at the resistor terminals.

Proof: (a) Follows from the definition; (b) is equivalent to Theorem 3.1b and (c) is a well-known fact about time-invariant linear differential equations.

Q.E.D.

Corollary 3.2(c) gives a useful stability criterion for networks with linear inductors and capacitors. Suppose that the inductor and capacitor characteristics can be expressed as $i_L = \Gamma\phi_L$, $\phi_\Gamma = L i_\Gamma$, $q_S = C v_S$ where Γ, L, S and C are positive definite symmetric matrices. Then simple manipulations yield, $\phi = \mathcal{L} i_L$ and $q = \mathcal{C} v_C$ where $\mathcal{L} = \left[\Gamma^{-1} + F_{L\Gamma} L F_{L\Gamma}^T\right]$ and $\mathcal{C} = \left[S^{-1} + F_{SC}^T C F_{SC}\right]$.
Substituting in (9) and (10) yields

$$\begin{aligned} \mathcal{C}\dot{v}_C &= F_{LC}^T i_L \\ \mathcal{L}\dot{i}_L &= -F_{LC} v_C \end{aligned} \tag{11}$$

and

$$F_{RC} v_C = 0, \quad F_{LG}^T i_L = 0. \tag{12}$$

$$\text{Let} \quad A \triangleq \begin{bmatrix} 0 & \mathcal{C}^{-1} F_{LC}^{T} \\ -\mathcal{L}^{-1} F_{LC} & 0 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} F_{RC} & 0 \\ 0 & F_{LG}^{T} \end{bmatrix} \tag{13}$$

Then (11) and (12) are equivalent to

$$\dot{x} = Ax$$

$$Bx = 0 .$$

By the well-known conditions[3] for observability of a linear time-invariant system we get

<u>Corollary 3.3</u>: If the inductors and capacitors are <u>linear</u>, then the network is asymptotically stable if and only if the columns of the matrix

$$\left[B^T, (BA)^T, \ldots, (BA^{n-1})^T \right]$$

spans R^n where A and B are defined in (13) and A has dimension n x n.

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