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Matrix Differential Equations and Irreducible Realizations of Transfer Function Matrices

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ABSTRACT

In this report we consider the problem of the determination of the degree of a given transfer function matrix, and the association of a minimal set of state equations (an "irreducible realization"). In Part I a matrix is exhibited whose rank is equal to the Mac Millan degree of the transfer function matrix; this matrix is obtainable by inspection from the transfer function matrix. In Part II a matrix differential equation is obtained which facilitates the association of a set of state equations of minimal dimension with the transfer function matrix. This association does not require explicit knowledge of the poles of the system under consideration.

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INTRODUCTION

One of the central problems of linear, time-invariant systems is the relationship between the state variable and transfer function matrix characterizations. In this report, the properties of this relationship are studied by introducing a matrix differential equation as a characterization which is "intermediate" between the state equations and the transfer function matrix. It will be evident that this approach affords considerable advantages in determining the degree of the transfer function matrix, and the association of an irreducible set of state equations.

I: Determination of the Mac Millan Degree of a Transfer Function Matrix

The problem of associating a set of equations of the form:

$$\left.\begin{aligned} \dot{\tilde{x}}(t) &= F\tilde{x}(t) + G\tilde{u}(t) \\ \tilde{y}(t) &= H\tilde{x}(t) \end{aligned}\right\} \quad \cdots \qquad (*)$$

with a linear, time-invariant system described by the transfer function matrix:

$$\tilde{Y}(s) = Z(s)\tilde{U}(s)^{\dagger} \quad \cdots \qquad (**)$$

has received considerable attention [1] - [3]. A subsidiary problem is the determination of the minimal dimension of the vector $\tilde{x}$ for which the system (*) has the same zero-state behavior as the system described by (**). This minimal dimension is commonly called the Mac Millan degree of the matrix $Z(s)$. Of course, the degree is determinable by reducing $Z(s)$ to its Smith-Mac Millan form. In the following we describe a method for finding the degree which does not require this reduction.

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† Notation: The tilde denotes a vector; $\tilde{x}$, $\tilde{u}$, and $\tilde{y}$ are, respectively, n, m, and r vectors; F, G, and H are, respectively, $n \times n$, $n \times m$, and $r \times n$ matrices with real elements. $Z(s)$ is a $r \times m$ matrix of ratios of polynomials in the complex variable s. Throughout, it is assumed that $\lim_{s \to \infty} Z(s) = 0$.

We first focus our attention on the system (*). It is well-known [4-Chapt. 11] that for a system described by (*), number of controllable state variables = rank $\left[G,\ FG,\ \ldots\ F^{n-1}G\right]$; number of observable state variables = rank $\left[H^t,\ F^tH^t,\ \ldots\ F^{t^{n-1}}H^t\right]$. Consequently, if (*) is an irreducible realization of Z(s) (and thus both completely controllable and completely observable), then

$$\operatorname{rank}\left[G,\ FG,\ \ldots\ F^{n-1}G\right] = \operatorname{rank}\left[H^t,\ F^tH^t,\ \ldots,\ F^{t^{n-1}}H^t\right]$$
$$= n$$

Since each matrix has n rows, we conclude that for each the rows are linearly independent. Furthermore, if $q(\leq n)$ is the degree of the <u>minimal</u> polynomial of F, then for any $p > q - 1$ there exist numbers

α_i, $i = 0, \ldots q-1$, such that $F^p = \sum_{i=0}^{q-1} \alpha_i F^i$ (Cayley Hamilton theorem),

which means that the columns of F^pG or $F^{t^p}H^t$ for $p > q - 1$ are each linear combinations of some of the columns of $\left[G,\ FG,\ \ldots\ F^{q-1}G\right]$ or $\left[H^t,\ F^tH^t,\ \ldots,\ F^{t^{q-1}}H^t\right]$, respectively. Hence we need only consider powers of F up to and including $q - 1$.

Consider now the matrix product:

$$\begin{bmatrix} H \\ HF \\ \cdot \\ \cdot \\ \cdot \\ HF^{q-1} \end{bmatrix} \left[G\ FG \ldots\ F^{q-1}G\right] = \begin{bmatrix} HG & HFG & \ldots & HF^{q-1}G \\ HFG & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ HF^{q-1}G & \ldots & & HF^{2(q-1)}G \end{bmatrix}$$

which we write for convenience as :

$$H'G' = M'$$

It is easy to show that the linear independence of the columns of H' and rows of G' and the fact that H' and G' have common dimension n imply that the rank of M' is n. Consequently, if we can determine the matrices HF^pG, $p = 0, \ldots 2(q-1)$ from $Z(s)$, then the matrix M' above, and hence the degree $Z(s)$ may be determined. However, as will be seen by the following argument, these matrices may be found by inspection of the transfer function matrix.

Initially, consider the following method for determining the matrix $Z(s)$ from the state equations (*) : denote the minimal polynomial of F, $\psi(p)$, by : $\psi(p) = p^q + a_1p^{q-1} + \ldots + a_q \; (q \leq n)$.

Form:

$$\dot{\underset{\sim}{y}} = H\dot{\underset{\sim}{x}} = HF\underset{\sim}{x} + HG\underset{\sim}{u} \ldots \quad (1)$$

$$\ddot{\underset{\sim}{y}} = HF\dot{\underset{\sim}{x}} + HG\dot{\underset{\sim}{u}}$$

$$= HF^2\underset{\sim}{x} + HFG\underset{\sim}{u} + HG\dot{\underset{\sim}{u}} \ldots \quad (2)$$

$$\vdots$$

$$\underset{\sim}{y}^{(q)} = HF^q\underset{\sim}{x} + HF^{q-1}G\underset{\sim}{u} + HF^{q-2}G\dot{\underset{\sim}{u}} + \ldots + HG\underset{\sim}{u}^{(q-1)} \ldots \quad (q)$$

Multiply (1) by a_{q-1}, (2) by a_{q-2}, etc., and add these together. To this add $a_q\underset{\sim}{y} = a_qH\underset{\sim}{x}$. This gives:

$$\underset{\sim}{y}^{(q)} + \sum_{i=1}^{q} a_i \underset{\sim}{y}^{(q-i)} = H\left\{F^q + \sum_{i=1}^{q} a_i F^{q-i}\right\} \cdot \underset{\sim}{x}$$

$$+ \left\{HF^{q-1}G + a_1 HF^{q-2}G + \ldots + a_{q-1}HG\right\} \underset{\sim}{u}$$

$$+ \left\{HF^{q-2}G + a_1 HF^{q-3}G + \ldots + a_{q-2}HG\right\} \dot{\underset{\sim}{u}} + \ldots$$

$$+ \; HG\, \underset{\sim}{u}^{(q-1)} .$$

The coefficient of $\underset{\sim}{x}$ in the above is identically zero by the Cayley-Hamilton theorem. Thus, the transfer function matrix is found directly by Laplace transforming each side of the equation, and dividing by the minimal polynomial of F. However, we are primarily interested in the matrix differential equation exactly as written above. For convenience, rewrite it as :

$$\psi(p) \cdot \underset{\sim}{y} = N(p) \cdot \underset{\sim}{u}$$

$\psi(p)$ is again the minimal polynomial of F, and $N(p)$ is the matrix polynomial in p represented by the right-hand side of the above equation.

We now make the following observation: write the transfer function matrix $Z(s)$ as :

$$Z(s) = \frac{1}{h(s)} \cdot Z'(s) ,$$

where $h(s)$ is the least common denominator of the elements of $Z(s)$, and $Z'(s)$ is a matrix polynomial in s. Then $h(s)$ is the minimal polynomial of an irreducible realization of $Z(s)$, and the coefficients of the powers of s in Z' are equal to the respective coefficients of the powers of p in the matrix $N(p)$ defined previously. That this is true is seen as follows:

From (*), $Z(s) = H[sI - F]^{-1}G$,

$$\text{then } [sI - F]^{-1} = \frac{\text{adjoint}[sI - F]}{\det[sI - F]} = \frac{F'(s)}{\psi(s)} .$$

(adjoint$[sI - F]$ is the transposed cofactor matrix of $[sI - F]$.) $F'(s)$ and $\psi(s)$ are formed by cancelling all factors common to $\det[sI - F]$ and the elements of the adjoint matrix of $[sI - F]$.

Then the polynomial ψ is the minimal polynomial of F [4 - p. 594]. Furthermore, if we make the assumption that (*) is an irreducible realization of $Z(s)$, then forming $\frac{HF'(s)G}{\psi(s)}$ will not introduce any further cancellations between factors of $\psi(s)$ and those of the elements of $HF'G$, since any such cancelling factors would represent modes which are either uncontrollable or unobservable. Thus, the least common denominator of the elements of $Z(s)$ is the minimal polynomial of the F matrix of an irreducible realization, and the equality of the coefficients of powers of p in $N(p)$ and $Z'(p)$ follows directly.

The result of this observation is that we can determine by inspection of the least common denominator $\psi(s)$ and the matrix $Z'(s)$ the coefficients a_i of the minimal polynomial, and the matrices HG, HFG, ... $HF^{q-1}G$. Knowing these and using the Cayley-Hamilton theorem, we can find $HF^qG, \ldots, HF^{2(q-1)}$ so that all the elements of the matrix M' will be known.

To recapitulate briefly then, given a matrix $Z(s)$, a procedure for finding its degree is as follows:

(a) Ensure that the numerator and denominator polynomials of all elements of $Z(s)$ are relatively prime. (Euclid's algorithm may be employed for this task, and no roots need be determined.)

(b) Find the least common denominator of all elements, and form the matrix differential equation:

$$\psi(p) \cdot \underset{\sim}{y} = N(p) \cdot \underset{\sim}{u} .$$

(c) From the coefficients of the matrix differential equation (and use of the Cayley-Hamilton theorem), construct the matrix M'.

(d) Degree of $Z(s)$ = rank M'.

This result was obtained previously by a different approach [5].

<u>Example</u>

Consider the following transfer function matrix [3]:

$$Z(s) = \frac{1}{s+1} \cdot \begin{bmatrix} 4 & 7 \\ 5 & 5 \end{bmatrix} + \frac{1}{(s+1)^3} \cdot \begin{bmatrix} 7 & 21 \\ 2 & 6 \end{bmatrix}$$

$$= \frac{1}{s^3+3s^2+3s+1} \cdot \begin{bmatrix} 4s^2+8s+11 & 7s^2+14s+28 \\ 5s^2+10s+7 & 5s^2+10s+11 \end{bmatrix} .$$

We see that $a_1 = 3$, $a_2 = 3$, $a_3 = 1$, $HG = \begin{bmatrix} 4 & 7 \\ 5 & 5 \end{bmatrix}$.

From the remaining matrix coefficients we obtain:

$$HFG = - \begin{bmatrix} 4 & 7 \\ 5 & 5 \end{bmatrix} \; ; \quad HF^2G = \begin{bmatrix} 11 & 28 \\ 7 & 11 \end{bmatrix} .$$

And the Cayley-Hamilton theorem gives:

$$HF^3G = - \begin{bmatrix} 25 & 70 \\ 11 & 23 \end{bmatrix} \; ; \quad HF^4G = \begin{bmatrix} 46 & 133 \\ 17 & 41 \end{bmatrix} .$$

$$\text{Thus, } M' = \begin{bmatrix} 4 & 7 & -4 & -7 & 11 & 28 \\ 5 & 5 & -5 & -5 & 7 & 11 \\ -4 & -7 & 11 & 28 & -25 & -70 \\ -5 & -5 & 7 & 11 & -11 & -23 \\ 11 & 28 & -25 & -70 & 46 & 133 \\ 7 & 11 & -11 & -23 & 17 & 41 \end{bmatrix} .$$

M' is triangularized to:

$$\begin{bmatrix} 1 & 1 & -1 & 7/5 & -1 & 11/5 \\ 0 & 1 & 0 & 9/5 & -1 & 32/5 \\ 0 & 0 & 1 & -2 & 3 & -6 \\ 0 & 0 & 0 & -4 & 0 & -12 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So that Rank M' = degree Z(s) = 4.

Remark

Making use of the results presented above, we can approximate from the transfer-function matrix to any desired degree of accuracy the impulse response matrix $W(t - \tau)$, without having to find the poles of the elements of $Z(s)$.

Letting $t - \tau = \xi$, we observe that

$$W(\xi) = H \cdot \exp(F\xi) \cdot G,$$

Expand $\exp(F\xi)$ as:

$$\exp(F\xi) = I + F\xi + \frac{(F\xi)^2}{2!} + \dots$$

Powers of F greater than or equal to q may be expressed as linear combinations of the first $q-1$ powers of F, so the series may be written:

$$\exp(F\xi) = \alpha_0(\xi) \cdot I + \alpha_1(\xi)F + \dots + \alpha_{q-1}(\xi)F^{q-1}.$$

Then, $W(\xi) = \sum_{i=0}^{q-1} \alpha_i(\xi)HF^iG$, where the $\alpha_i(\xi)$ and the HF^iG are known.

Of course, the determination of $\alpha_i(\xi)$ will in general be a very laborious process. This approach would be particularly useful if the response to some input were desired only over a relatively short time interval.

II: A Minimal Realization Procedure

In the literature, most of the methods for determining minimal realizations [1] - [3] of transfer function matrices require that the poles of all the elements of the matrix be known explicitly. In this section, a method for obtaining a minimal realization is described which requires nothing more complicated than elementary row and column operations on polynomial matrices; in particular, the poles need not be explicitly determined. We first associate with the transfer function matrix an equivalent matrix differential equation (different from that obtained in the previous section), and from this obtain the state equations. Since the introduction of the matrix differential equation obviates the finding of the poles of the transfer function matrix, this additional step more than justifies itself.

We first put the matrix $Z(s)$ in its Smith-MacMillan form. As before, we write

$$
\begin{aligned}
Z(s) &= \frac{1}{\psi(s)} \cdot Z'(s) \\
&= \frac{1}{\psi(s)} \cdot M(s)\Gamma'(s)N(s) \, ,
\end{aligned}
$$

where M, Γ', and N are polynomial matrices, M and N have constant, nonzero determinants, and Γ' has the form:

$$\Gamma'(s) = \begin{bmatrix} \gamma_1'(s) & 0 & \cdots & \cdots & 0 \\ 0 & & & & \\ \vdots & & \gamma_R'(s) & & \vdots \\ & & & 0 & \\ 0 & \cdots & & & 0 \end{bmatrix}_{r \times m},$$

where R is equal to the rank of $Z'(s)$, and the γ_i' are polynomials with leading coefficient unity, called the invariant polynomials of $Z'(s)$. Also, degree $\gamma_i' >$ degree γ_{i-1}', $i = 2, \ldots, R$.

Divide each element of Γ' by $\psi(s)$ to give:

$$Z(s) = M(s) \cdot \begin{bmatrix} \gamma_1/\psi_1 & 0 & & & 0 \\ 0 & & & & \\ \vdots & & \gamma_R/\psi_R & & \vdots \\ & & & 0 & \\ 0 & \cdots & & \cdots & 0 \end{bmatrix} \cdot N(s)$$

(Factors common to the γ_i' and ψ are cancelled.)

Write this as:

$Z(s) = M(s) \cdot \Psi^{-1}(s) \cdot \Gamma(s) \cdot N(s)$, where

$$\Psi(s) = \begin{bmatrix} \psi_1 & & & & & 0 \\ & \ddots & & & & \\ & & \psi_R & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{bmatrix}_{r \times r},$$

$$\Gamma(s) = \begin{bmatrix} \gamma_1 & & & & & 0 \\ & \ddots & & & & \\ & & \gamma_R & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{bmatrix}_{r \times m}$$

or, $Z(s) = [\Psi(s)M^{-1}(s)]^{-1} \cdot \Gamma(s)N(s)$.

Since $\det M = \text{const.}$, $\Psi(s)M^{-1}(s)$ is a matrix polynomial in s. Therefore, a matrix differential equation which describes a system (zero-state) equivalent to that described by $Z(s)$ is given by:

$$\Psi(p)M^{-1}(p) \cdot \underset{\sim}{y} = \Gamma(p) \cdot N(p) \cdot \underset{\sim}{u} \ldots \quad (*).$$

We observe that no extraneous state variables have been introduced in the process of associating the matrix differential equation, for: the number of arbitrary initial conditions

$= \text{degree } \{\det \Psi(p) \cdot M^{-1}(p)\}$,

$= \text{degree } \{\det \Psi(p)\}$,

$= \sum_{i=1}^{R} \text{degree } \{\psi_i(p)\}$,

$=$ MacMillan degree of $Z(s)$,

$=$ dimension of an irreducible realization of $Z(s)$.

Also, the matrix differential equation has been obtained using only elementary row and column operations on polynomial matrices.

We now proceed to associate a minimal set of state equations with the matrix differential equation.† For simplicity, let $A(p) = \Psi(p) \cdot M^{-1}(p)$, $B(p) = \Gamma(p) \cdot N(p)$. We first triangularize the matrix $A(p)$ using elementary row operations; i.e., find a unimodular matrix $H(p)$ such that $A'(p) = H(p)A(p)$ has the form:

$$A'(p) = \begin{bmatrix} a_{11}(p) & \cdots & & a_{1r}(p) \\ 0 & a_{22}(p) & & \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & & a_{rr}(p) \end{bmatrix}$$

Write $B'(p) = H(p)B(p)$ as:

$$B'(p) = \begin{bmatrix} b_{11}(p) & \cdots & \cdots & b_{1m}(p) \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ b_{r1}(p) & \cdots & \cdots & b_{rm}(p) \end{bmatrix}$$

The triangularization may always be performed so that degree $b_{kj}(p) <$ degree $b_{kk}(p)$, $j = 1, \ldots, k-1$, $k = 1, \ldots, r$.

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† This algorithm is due to Prof. E. Polak, Dept. of Electrical Engineering, Univ. of California, Berkeley.

Let $\underset{\sim}{y}$ = column $(y_1 \ldots y_r)$, $\underset{\sim}{u}$ = column $(u_1 \ldots, u_m)$. Then the bottom row of the matrix equation may be written:

$$a_{rr}(p)y_r = \sum_{j=1}^{m} b_{rj}(p)u_j, \text{ where}$$

$$a_{rr}(p) = p^{k_{rr}} + a_{rr}^{1} p^{k_{rr}-1} + \ldots + a_{rr}^{k_{rr}}, \text{ and}$$

$$b_{rj}(p) = b_{rj}^{0} p^{k_{rr}-1} + b_{rj}^{1} p^{k_{rr}-2} + \ldots + b_{rj}^{k_{rr}-1}.$$

(k_{ji} is the degree of the $\underline{ji}$th element of $A'(p)$). The degree of the polynomials $b_{rj}(p)$ are strictly less than the degree of $a_{rr}(p)$ because of our assumption that $Z(s)$ has a zero at infinity.

We associate state variables as:

$$x_1 = y_r,$$

$$x_2 = \dot{x}_1 + a_{rr}^{1} x_1 - \sum_{j=1}^{m} b_{rj}^{0} u_j,$$

$$x_3 = \dot{x}_2 + a_{rr}^{2} x_1 - \sum_{j=1}^{m} b_{rj}^{1} u_j,$$

$$\vdots$$

$$x_{k_{rr}} = \dot{x}_{k_{rr}-1} + a_{rr}^{k_{rr}-1} x_1 - \sum_{j=1}^{m} b_{rj}^{k_{rr}-2} u_j,$$

and

$$\dot{x}_{k_{rr}} = -a_{rr}^{k_{rr}} x_1 + \sum_{j=1}^{m} b_{rj}^{k_{rr}-1} u_j .$$

Consider now the (r - 1)th row of the matrix equation. This has the form:

$$a_{r-1\, r-1}(p)\, y_{r-1} + a_{r-1\, r}(p)\, y_r = \sum_{j=1}^{m} b_{r-1\, j}(p) u_j .$$

We first express the polynomial $a_{r-1\, r}(p) y_r$ as a linear combination of the x_i $(i = 1, \ldots, k_{rr})$, the u_i $(i = 1, \ldots, m)$, and derivatives of the u_i up to order less than degree $\{a_{r-1\, r-1}(p)\}$. (Again it may be shown that our assumption that $Z(s)$ has a zero as $s \to \infty$ makes this always possible.) State variables are then associated with $a_{r-1\, r-1}(p) y_{r-1}$ in exactly the same manner as above; i.e., let $x_{k_{rr}+1} = y_{r-1}$

$$x_{k_{rr}+2} = \dot{x}_{k_{rr}+1} + a_{r-1\, r-1}^{1} x_{k_{rr}+1} - \sum_{j=1}^{m} b'^{0}_{r-1\, j} u_j .$$

(The coefficients of the u_j are primed here because they will include terms introduced when $a_{r-1\, r}(p)\, y_r$ is expressed in terms of the first k_{rr} state variables, the u_j, and derivatives of the u_j)

$\vdots$

$$\dot{x}_{k_{rr}+k_{r-1\,r-1}} = -a_{r-1\,r-1}^{k_{r-1\,r-1}} x_{k_{rr}+1}$$

$$+ \sum_{j=1}^{k_{rr}} \alpha_j^{r\;r-1} x_j + \sum_{j=1}^{m} b_{r-1\,j}^{'\,k_{r-1\,r-1}-1} u_j .$$

The term $\sum_{j=1}^{k_{rr}} \alpha_j^{r\;r-1} x_j$ represents the coupling between the

$x_i\,(i = k_{rr}+1, \ldots, k_{rr} + k_{r-1\,r-1})$ and the $x_i\,(i = 1, \ldots, k_{rr})$.

We proceed up the rows of the matrix $A'(p)$ in this fashion, associating new state variables with the diagonal elements. Clearly, the total number of state variables = degree $\{\det A'(p)\}$ = MacMillan degree of Z. Hence, this realization is minimal.

The matrices F, G, and H have the form shown on the following page.

$$
\left[\begin{array}{ccccc|ccccc|c|ccccc}
-a_{rr}^{1} & 1 & 0 & \cdots & 0 & & & & & & & & & & & \\
-a_{rr}^{2} & 0 & 1 & \cdots & 0 & & & 0 & & & \cdots & & & 0 & & \\
\vdots & \vdots & & & \vdots & & & & & & & & & & & \\
-a_{rr}^{k_{rr}} & 0 & 0 & \cdots & 0 & (k_{rr}\times k_{rr}) & & & & & & & & & & \\
\hline
0 & & \cdots & & 0 & -a_{r-1\,r-1}^{1} & 1 & 0 & \cdots & 0 & & & & & & \\
\vdots & & & & \vdots & -a_{r-1\,r-1}^{2} & 0 & 1 & \cdots & 0 & \cdots & & & 0 & & \\
0 & & & & & \vdots & \vdots & \vdots & & \vdots & & & & & & \\
\alpha_{1}^{r\,r-1} & & \cdots & & \alpha_{k_{rr}}^{r\,r-1} & -a_{r-1\,r-1}^{k_{r-1\,r-1}} & 0 & 0 & \cdots & 0 & (k_{r-1\,r-1}\times k_{r-1\,r-1}) & & & & & \\
\hline
 & & \vdots & & & & & & \vdots & & & & & \vdots & & \\
\hline
0 & & \cdots & & 0 & 0 & & \cdots & & 0 & & -a_{11}^{1} & 1 & 0 & \cdots & 0 \\
\vdots & & & & \vdots & \vdots & & & & \vdots & & -a_{11}^{2} & 0 & 1 & \cdots & 0 \\
0 & & & & 0 & 0 & & & & 0 & \cdots & \vdots & \vdots & \vdots & & \vdots \\
\alpha_{1}^{r1} & & \cdots & & \alpha_{k_{rr}}^{r1} & \alpha_{1}^{r2} & & \cdots & & \alpha_{k_{r-1\,r-1}}^{r2} & & -a_{11}^{k_{11}} & 0 & 0 & \cdots & 0
\end{array}\right]
(k_{11}\times k_{11})
$$

$$
G = \begin{bmatrix}
b_{r1}^{0} & \cdots & b_{rm}^{0} \\
b_{r1}^{1} & \cdots & b_{rm}^{1} \\
\vdots & & \vdots \\
b_{r1}^{k_{rr}-1} & \cdots & b_{rm}^{k_{rr}-1} \\
\vdots & & \vdots \\
b_{11}^{'\delta(Z)} & \cdots & b_{1m}^{'\delta(Z)}
\end{bmatrix} ;
$$

($\delta(Z)$ = MacMillan degree of Z)

$$
H = \begin{bmatrix}
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & & 1 & 0 & \cdots & 0 \\
\vdots & & & \vdots & & & & & 0 & & & \\
\vdots & & \vdots & & & & \vdots & \cdots & \vdots & & & \vdots \\
0 & & & 1 & 0 & \cdots & & & & & & \\
1 & \cdots & 0 & 0 & & \cdots & 0 & & 0 & & \cdots & 0
\end{bmatrix}
$$

Since the motivation for obtaining a minimal representation is frequently to effect an analog simulation, it would be useful to indicate how such a simulation could be constructed. The situation for two inputs and two outputs is shown in Fig. 1; the generalization is obvious.

The procedure will be clarified by considering the following example, which was also treated in Part I.

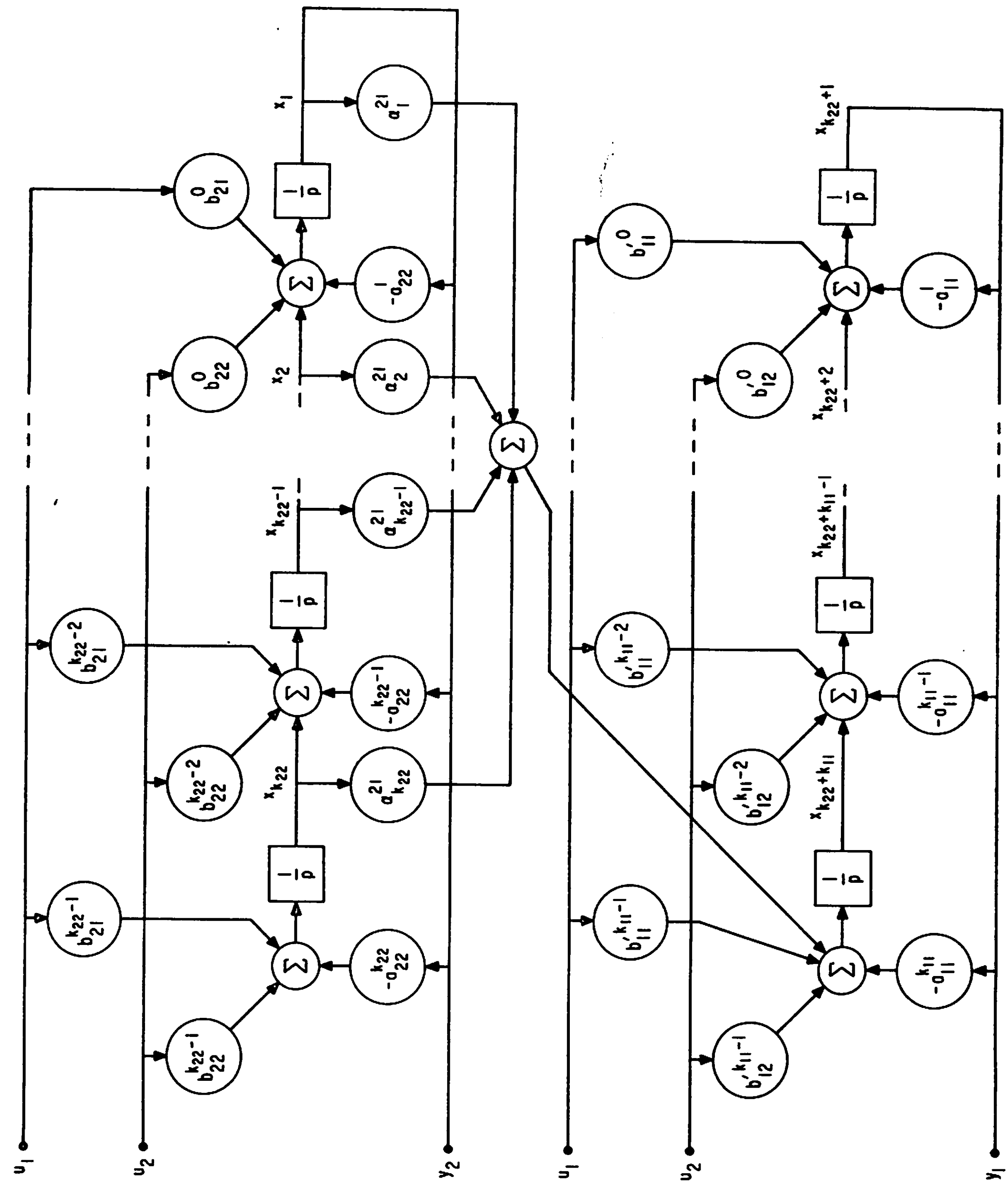

Figure 1.

Obtain an irreducible realization for:

$$Z(s) = \frac{1}{s^3 + 3s^2 + 3s + 1} \cdot \begin{bmatrix} 4s^2 + 8s + 11 & 7s^2 + 14s + 28 \\ 5s^2 + 10s + 7 & 5s^2 + 10s + 11 \end{bmatrix}$$

without explicit knowledge of the poles.

The Smith-Macmillan form for $Z(s)$ is:

$$Z(s) = \begin{bmatrix} 4s^2 + 8s + 11 & -\frac{60}{27} \\ -(5s^2 + 10s + 7) & -\frac{75}{27} \end{bmatrix} \begin{bmatrix} \frac{1}{s^3 + 3s^2 + 3s + 1} & 0 \\ 0 & \frac{s^2 + 2s + 5}{s + 1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{9}(5s^2 + 10s + 32 \\ 0 & 1 \end{bmatrix}$$

It is evident that degree $Z(s) = 3 + 1 = 4$. The equivalent minimal matrix differential equation is thus:

$$\begin{bmatrix} p^3 + 3p^2 + 3p + 1 & 0 \\ 0 & p + 1 \end{bmatrix} \begin{bmatrix} 5/27 & -4/27 \\ \frac{1}{15}(5p^2 + 10p + 2) & -\frac{1}{15}(4p^2 + 8p + 11) \end{bmatrix} \cdot \underset{\sim}{y}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & p^2 + 2p + 5 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{9}(5p^2 + 10p + 32) \\ 0 & p^2 + 2p + 5 \end{bmatrix} \cdot \underset{\sim}{u}.$$

Operating on the equation with $\begin{bmatrix} -27/2 & -21/2 \\ 5p^2 + 10p + 7 & 5p^2 + 10p + 11 \end{bmatrix}$

puts it in the triangular form:

$$\begin{bmatrix} p+1 & -\frac{7}{2}(p+1) \\ 0 & p^3 + 3p^2 + 3p + 1 \end{bmatrix} \cdot \underset{\sim}{y} = \begin{bmatrix} -27/2 & -21/2 \\ 5p^2 + 10p + 7 & 5p^2 + 10p + 11 \end{bmatrix} \cdot \underset{\sim}{u}.$$

Let $\underset{\sim}{y}$ = column (y_1, y_2), $\underset{\sim}{u}$ = column (u_1, u_2). The state variables are selected as:

$$x_1 = y_2,$$

$$x_2 = \dot{x}_1 + 3x_1 - 5u_1 - 5u_2,$$

$$x_3 = \dot{x}_2 + 3x_1 - 10u_1 - 10u_2,$$

and, $x_4 = y_1.$

The irreducible state equations are then:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ -7 & 7/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 5 & 5 \\ 10 & 10 \\ 7 & 11 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} ; \quad \underset{\sim}{y} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \underset{\sim}{x}.$$

An analog simulation is shown in Fig. 2.

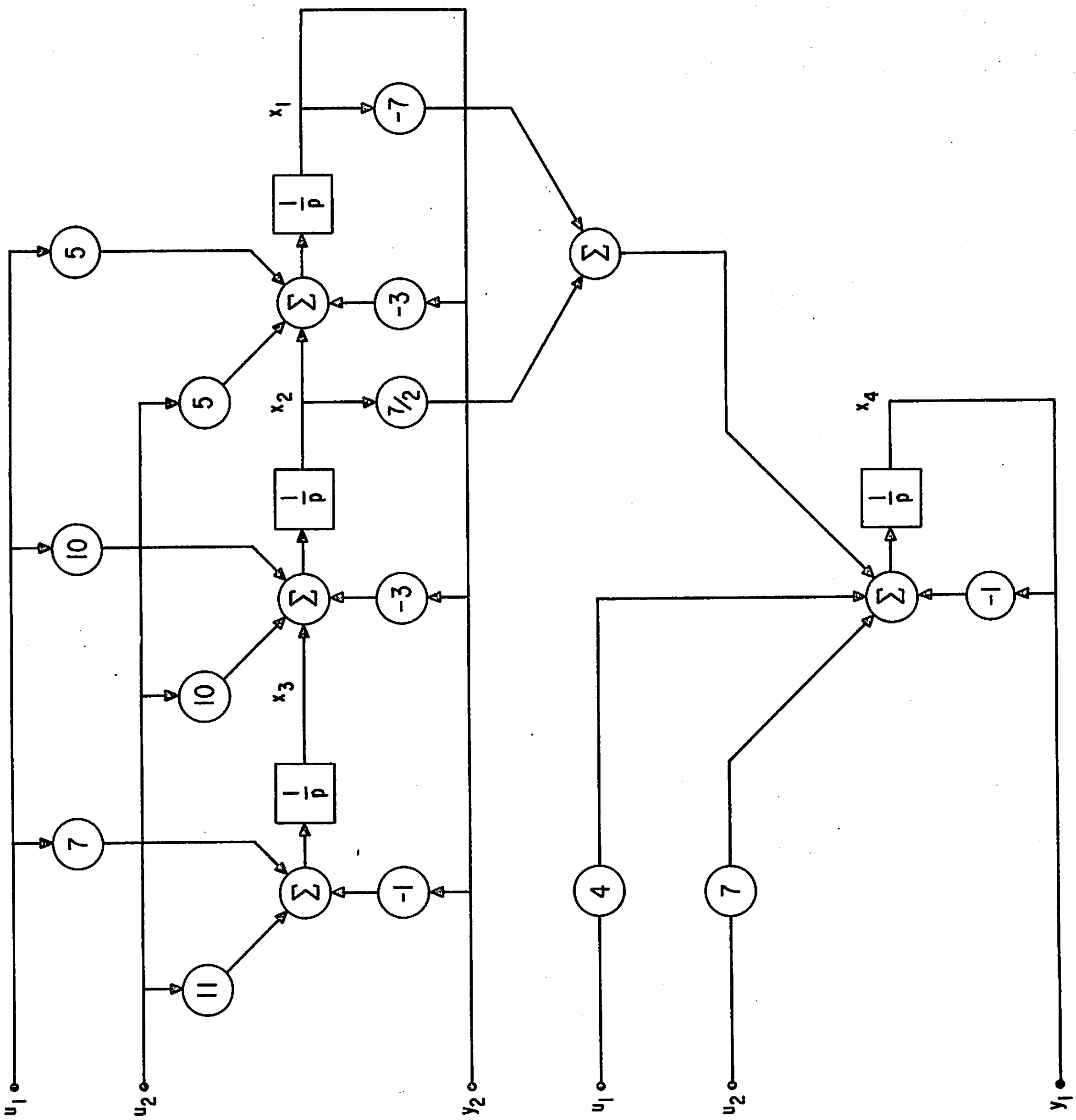

Figure 2.

CONCLUSION

An attempt has been made to illustrate the efficacy of matrix differential equations in problems associated with the determination of minimal realizations of transfer function matrices. More specifically, a procedure has been given which enables the straightforward determination of the degree of a given rational matrix. As mentioned before, this result was obtained previously [5]; it seems, however, that the use of the matrix differential equation is a more natural and straightforward approach. Also, an algorithm has been presented which enables the determination of an irreducible realization for a given rational matrix without the necessity of determining the poles of its elements. The computations required are quite straightforward and appear to be particularly adaptable to digital computer programming.

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