SYSTEM TRANSFORMATIONS AND THE POPOV STABILITY CRITERION

by

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ABSTRACT

A system transformation which can be used to extend Popov's criterion to sectors with non-zero lower bounds is discussed. Two tests are developed which are applied to the frequency response of the original system and establish sectors of stability without requiring that the transformations be actually carried out. A theorem is proved concerning the limitations of the transformation discussed.

INTRODUCTION

Consider the system Σ of Fig. 1, where G is a linear block with transfer function $G(s)$,[§] and N is a time-invariant, memoryless, piecewise-continuous, nonlinearity such that $\phi(0) = 0$. Using the terminology introduced by Aizerman and Gantmacher [1], we further describe the system as follows. If $a \leq \frac{\phi(\sigma)}{\sigma} \leq b$, $\sigma \neq 0$, a and b constants, then N (or ϕ) is said to be in the sector $[a, b]$. If $G(s)$ is strictly stable, then Σ is called a principal case, and if $G(s)$ is stable but not strictly stable, then Σ is called a particular case. If Σ is a particular case and there exists $\rho > 0$ such that the linear system obtained by putting $\phi(\sigma) = k\sigma$ is strictly stable for all $k \in (0, \rho)$, then Σ is said to satisfy the condition of stability-in-the-limit. If Σ is globally stable for all N in the sector $[a, b]$, then Σ is said to be absolutely stable in the sector $[a, b]$.

Popov [1] shows a sufficient condition that the system Σ be absolutely stable in the sector $[0, k]$ for the principal case and in the sector $[\epsilon, k]$ in the particular cases (where $\epsilon > 0$ is arbitrarily small) is that there exist a finite, real number q such that

$$\mathrm{Re} \left\{ (1+j\omega q)G(j\omega) + \frac{1}{k} \right\} > 0 \text{ for all } \omega \geq 0, \tag{1}$$

§For simplicity, assume $G(s)$ is rational in s. The infinite dimensional case may be treated with additional assumptions as shown by Desoer [2]. L^2 input sources may also be treated in the usual way.

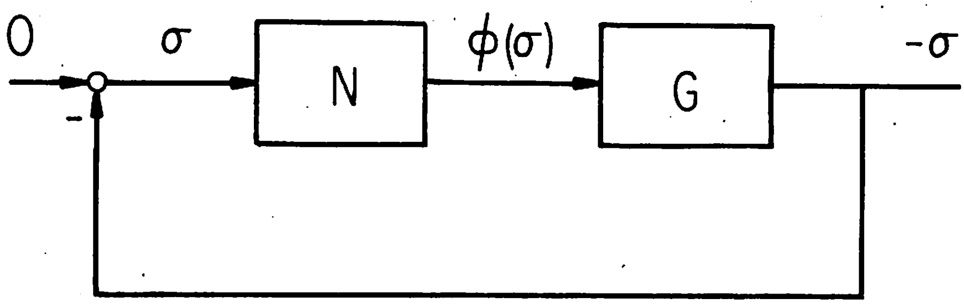

Fig. 1. The system Σ.

and, additionally, that stability-in-the-limit be satisfied for the particular cases.

This inequality condition has a simple graphical interpretation. If $G^*(j\omega)$ is defined by

$$G^*(j\omega) = X(j\omega) + j\omega Y(j\omega) = X^*(j\omega) + jY^*(j\omega), \tag{2}$$

where $X^*(j\omega) = X(j\omega) = \text{Re } G(j\omega)$, $Y(j\omega) = \text{Im } G(j\omega)$, and $Y^*(j\omega) = \omega Y(j\omega)$, then (1) becomes

$$X^*(j\omega) - qY^*(j\omega) + \frac{1}{k} > 0. \tag{3}$$

Thus, if $G^*(j\omega)$ is plotted in the (X^*, Y^*) plane with $\omega \geq 0$ as a parameter, the inequality (1) is satisfied everywhere to the right of the line $X^* - qY^* + \frac{1}{k} = 0$ which has slope $\frac{1}{q}$ and intersects the X^*-axis at the point $-\frac{1}{k}$. Therefore, Popov's condition is satisfied if and only if there exists a line in the (X^*, Y^*) plane with slope $\frac{1}{q} \neq 0$ and X^*-intercept $-\frac{1}{k}$ which lies always to the left of the $G^*(j\omega)$ plot. Such a line is called a Popov line.

It is clear that if $G^*(j\omega)$ is such that the point $(-\frac{1}{k}, 0)$ on the Popov line can be made arbitrarily close to the $G^*(j\omega)$ plot, then the Popov condition is both necessary and sufficient for the absolute stability of Σ in the sector $[0, k]$ ($[\epsilon, k]$ in the particular cases). On the other hand, if the point $(-\frac{1}{k}, 0)$ on the Popov line cannot be made arbitrarily close to the $G^*(j\omega)$ plot, then Popov's condition may not be a necessary condition for the absolute stability of Σ in the sector $[0, k]$, ($[\epsilon, k]$).

Other cases may arise where it is desirable to find a sector of absolute stability with a negative lower bound and still others where the system Σ is stable for linear gains in the sector (a, b), $a>0$, (i.e., a conditionally stable system) and a sector of absolute stability is desired within (a, b). There is, however, no way of dealing with these questions by applying Popov's condition to $G(j\omega)$ directly.

THE ROLE OF TRANSFORMATIONS

Popov's condition may be applied to these problems, however, after a suitable transformation. Consider the system $T_h(\Sigma)$ of Fig. 2, where h is any linear gain such that G' is stable. It is clear from Fig. 2 that the response of Σ and of $T_h(\Sigma)$ is completely determined by the initial conditions on G, and the response of the two systems is identical for identical initial condition on G. In particular, $T_h(\Sigma)$ is globally stable if and only if Σ is globally stable. Thus, it is clear that $T_h(\Sigma)$ is absolutely stable in the sector $[a, b]$ if and only if Σ is absolutely stable in the sector $[a+h, b+h]$ (since $N = N'+h$). In particular, if Popov's condition proves that $T_h(\Sigma)$ is absolutely stable in the sector $[0, k]$ (or $[\epsilon, k]$) it follows that Σ is absolutely stable in the sector $[h, k+h]$, (or $[h+\epsilon, k+h]$).

It is now clear how this transformation would be applied in the case of a sector with negative lower bound or a conditionally stable sector. It might be desired, however, to find a number of such sectors for a single system. This would require a number of transformations all with different values of h and each calling for a new frequency response to be calculated and a new plot to be made. It would be

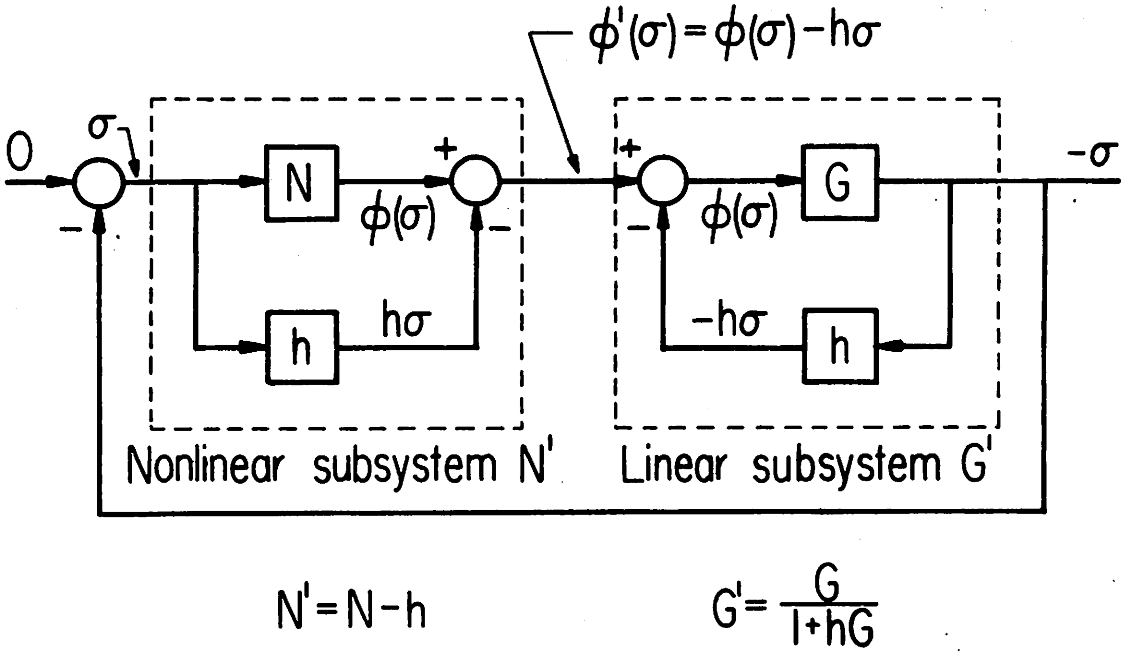

Fig. 2. The system $T_h(\Sigma)$.

helpful to have a method of examining the stability of $T_h(\Sigma)$ for arbitrary h with only a single plot. Such a method is given by Theorem 1.

Theorem 1: Popov's condition applied to the system $T_h(\Sigma)$ will prove the absolute stability of the system Σ in the sector $[h, k+h]$ if and only if:

(1) The linear part of $T_h(\Sigma)$ is strictly stable,

(2) There exists a finite, real number q such that for each $\omega \geq 0$ the circle

$$\left(X + \frac{k+2h}{2h(k+h)}\right)^2 + \left(Y - \frac{k\omega q}{2h(k+h)}\right)^2 = \frac{k^2(1+\omega^2 q^2)}{4h^2(k+h)^2} \tag{4}$$

in the Nyquist (i.e., the (X, Y) plane) encloses both the point $G(j\omega)$ and the origin or neither point. In the case h=0 or h=-k, the circle degenerates to the line $X - q\omega Y + \frac{1}{k} = 0$ or $X + q\omega Y - \frac{1}{k} = 0$, respectively.

The theorem is proved in the appendix.

At this point it should be emphasized that;

(1) The theorem defines a family of circles with ω as a parameter,

(2) The family of circles lies in the Nyquist plane for the original system Σ,

(3) The "if and only if" of the theorem refers only to the satisfaction of Popov's condition and in no way implies a necessary and sufficient condition for the absolute stability of Σ.

(4) In the case of $q=0$, the family degenerates to a single circle giving an extremely simple test [3], [4].

In order to apply the test of Theorem 1 it is helpful to specify the circles in more detail. It is easily checked that each circle intersects the X-axis at the two points $(-\frac{1}{h}, 0)$ and $(-\frac{1}{k+h}, 0)$ and that the slope of the tangents at these points is $\frac{-1}{\omega q}$ and $\frac{1}{\omega q}$, respectively. A few typical circles are shown in Fig. 3 for the case $h>0$, $q>0$. Notice that for a fixed q the Y component of the center of the circle is proportional to ω while the X component remains fixed; the circles therefore move up (or down) and get bigger as ω is increased while always intersecting the X-axis at the same two points.

The simplest application of the circle test (in the case $q \neq 0$) is illustrated in Fig. 4. The system is conditionally stable with 3 poles and 2 zero's; ω_1 is the frequency of the first crossing of the negative X-axis by the Nyquist plot and ω_2 is the frequency of the second crossing. $G(j\omega_2) = -a$. The largest possible circle which intersects the negative X-axis in the interval $(-a, 0)$ and does not intersect the Nyquist plot is drawn. This is defined as the ω_2 circle. Drawing this circle and assigning it parameter ω_2 determines h, k, and q. It is clear from Fig. 4 that all the circles for $\omega>\omega_1$ satisfy the condition of Theorem 1 and it is extremely unlikely that any difficulty will be encountered for $\omega<\omega_1$ since the lowest circle (for $\omega=0$) extends below the X-axis less than $\frac{a}{2}$.

In some of the following development it is more convenient to work in the (X^*, Y^*) plane. Therefore, the following corrolary is

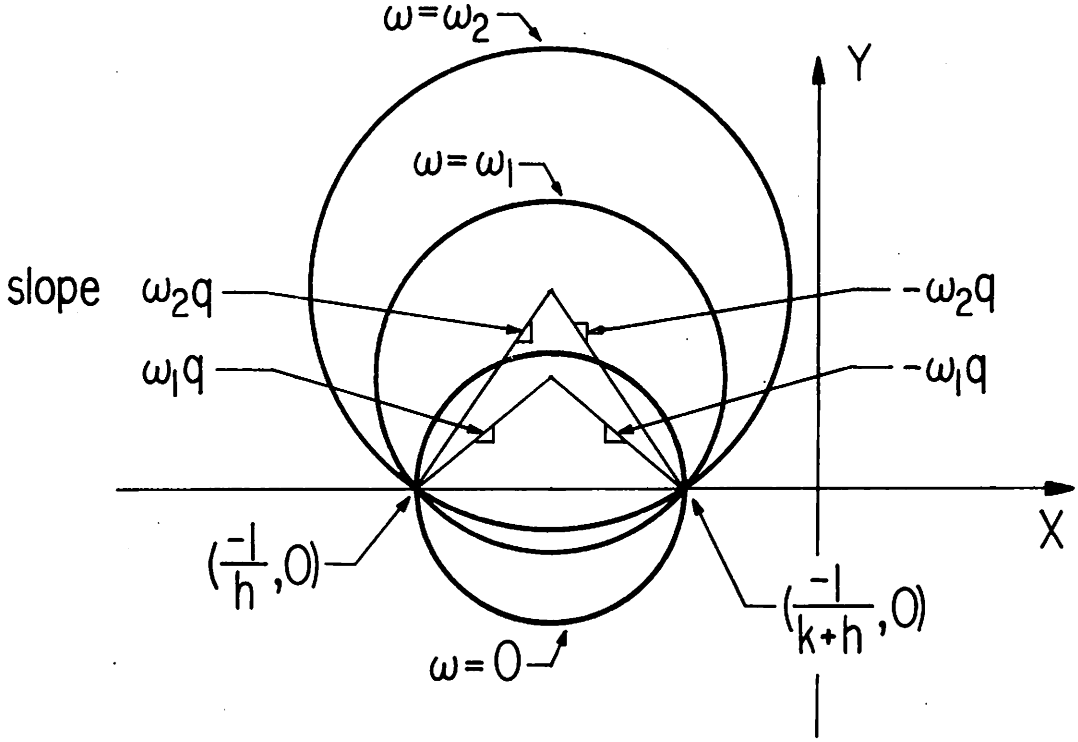

Fig. 3. A few typical circles in the case $h > 0$, $q > 0$.

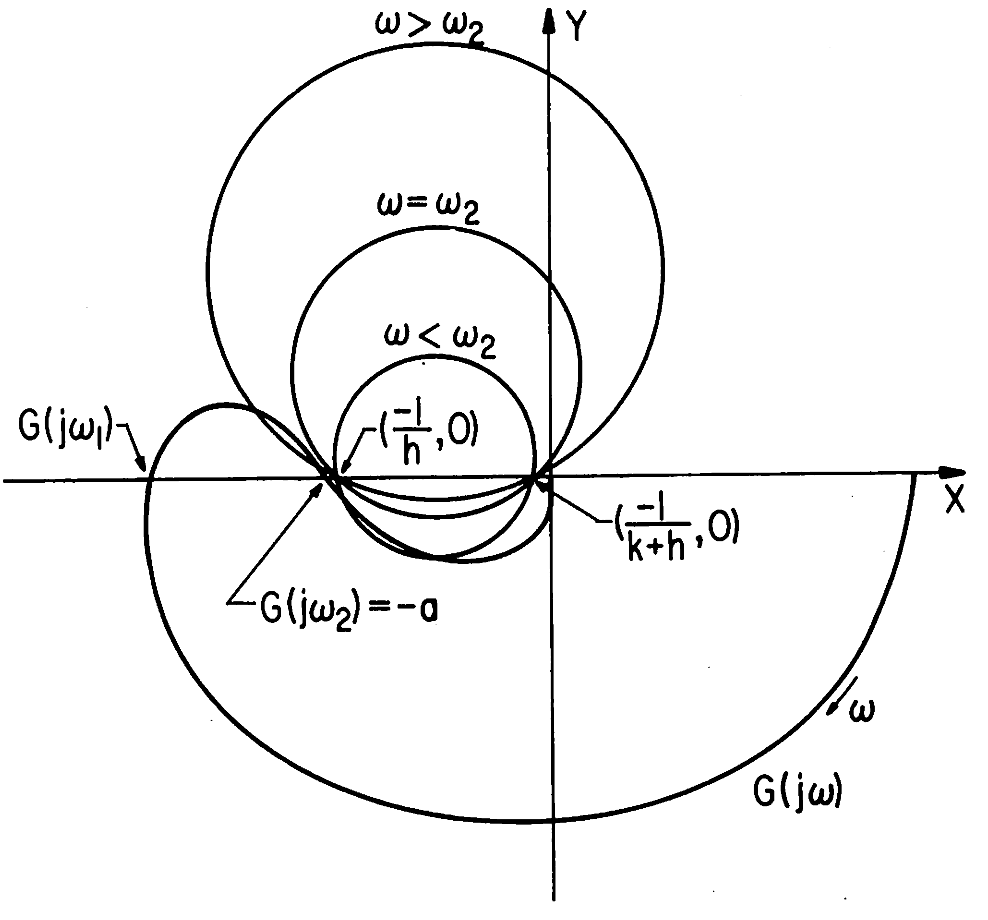

Fig. 4. Illustration of the circle test.

presented. The proof is immediate by mapping the (X, Y) plane onto the (X^*, Y^*) plane for each $\omega>0$.

Corollary 1: Popov's condition applied to the system $T_h(\Sigma)$ will prove the absolute stability of the system Σ in the sector $[h, k+h]$ if and only if:

(1) The linear part of the system $T_h(\Sigma)$ is strictly stable,

(2) There exists a finite, real number q such that for each $\omega>0$ the ellipse

$$\left(X^* + \frac{k+2h}{2h(k+h)}\right)^2 + \left(\frac{Y^*}{\omega} - \frac{k\omega q}{2h(k+h)}\right)^2 = \frac{k^2(1+\omega^2 q^2)}{4h^2(k+h)^2} \qquad (5)$$

in the (X^*, Y^*) plane encloses both the point $G^*(j\omega)$ and the origin or neither point. In the case of $h=0$ or $h=k$ the ellipse degenerates to the line $X^* - qY^* + \frac{1}{k} = 0$ or $X^* + qY^* - \frac{1}{k} = 0$, respectively.

Note that in the (X^*, Y^*) plane the degenerate cases yield a <u>single</u> line but the case $q=0$ yields a <u>family</u> of ellipses.

A second corrolary gives rise to a simple test which is useful for finding sectors of absolute stability which do not include the character $\phi(\sigma) \equiv 0$.

<u>Corollary 2</u>: If $\{E_\omega\}$, $\omega \epsilon (0, \infty)$ is a family of ellipses as defined in Corollary 1 then:

(1) $\omega_2 > \omega_1 \Rightarrow E_{\omega_1}$ lies inside E_{ω_2}.

(2) $\lim_{\omega \to \infty} E_\omega$ exists and is the parabola defined by

$$X^{*2} + \frac{k+2h}{h(k+h)} X^* - \frac{kq}{h(k+h)} Y^* + \frac{1}{h(k+h)} = 0 \qquad (6)$$

Proof: Expanding the defining equation for E_ω yields

$$X^{*2} + \frac{k+2h}{h(k+h)} X^* + \frac{Y^{*2}}{\omega^2} - \frac{kq}{h(k+h)} Y^* + \frac{1}{h(k+h)} = 0 \qquad (*)$$

let $\omega_2 > \omega_1$ and assume (X_o^*, Y_o^*) is a point on E_{ω_1}, i.e.,

$$X_o^{*2} + \frac{k+2h}{h(k+h)} X_o^* + \frac{Y_o^{*2}}{\omega_1^2} - \frac{kq}{h(k+h)} Y_o^* + \frac{1}{h(k+h)} = 0.$$

But, clearly,

$$X_o^{*2} + \frac{k+2h}{h(k+h)} X_o^* + \frac{Y_o^{*2}}{\omega_2^2} - \frac{kq}{h(k+h)} Y_o^* + \frac{1}{h(k+h)} \leq 0$$

i.e., (X_o^*, Y_o^*) lies inside E_{ω_2}.

Thus, (1) is proved and (2) is immediately evident from (*). By applying Corollary 2 to the ellipses of Corollary 1, the following test is obtained.

Parabola Test: If:

(1) The linear part of the system $T_h(\Sigma)$ is strictly stable,[§§] and

(2) There exist $k>0$, q, h, such that the parabola

$$X^{*2} + \frac{k+2h}{h(k+h)} X^* - \frac{kq}{h(k+h)} Y^* + \frac{1}{h(k+h)} = 0$$

does not contain either the origin or any part of the $G^*(j\omega)$ plot, then the system Σ is absolutely stable in the sector $[h, k+h]$.

It is immediate that the parabola passes through the two points $(-\frac{1}{h}, 0)$ and $(-\frac{1}{k+h}, 0)$ and that the slope of the tangents at these points is $-\frac{1}{q}$ and $\frac{1}{q}$, respectively. It is easily checked that if a perpendicular is constructed from the intersection of these two tangents to the X^*-axis, the parabola intersects this perpendicular at its mid-point and has slope zero at this point. While the parabola test is very simple to apply, it should be noted that it imposes somewhat stricter conditions than required by the Popov test. See Fig. 5 for an example.

Consider next the case mentioned previously where the X^*-intercept of the Popov line cannot be made arbitrarily close to the $G^*(j\omega)$ plot. It might be conjectured in this case that there may exist a transformation of the form $T_h(\Sigma)$ which, when tested with Popov's condition, would yield a sector of absolute stability for Σ of the form $[h, k+h]$ with $h<0$ and $k_1<k+h<k_2$, where k_1 is the upper bound of the Popov sector for Σ, and k_2 is the upper bound of the corresponding Hurwitz sector.

[§§] In the case where $G(s)$ is stable, the stability of the linear part of $T_h(\Sigma)$ may be insured if $G^*(j\omega)$ does not enclose the $-\frac{1}{h}$ point; this in turn is guaranteed by the satisfaction of the second condition.

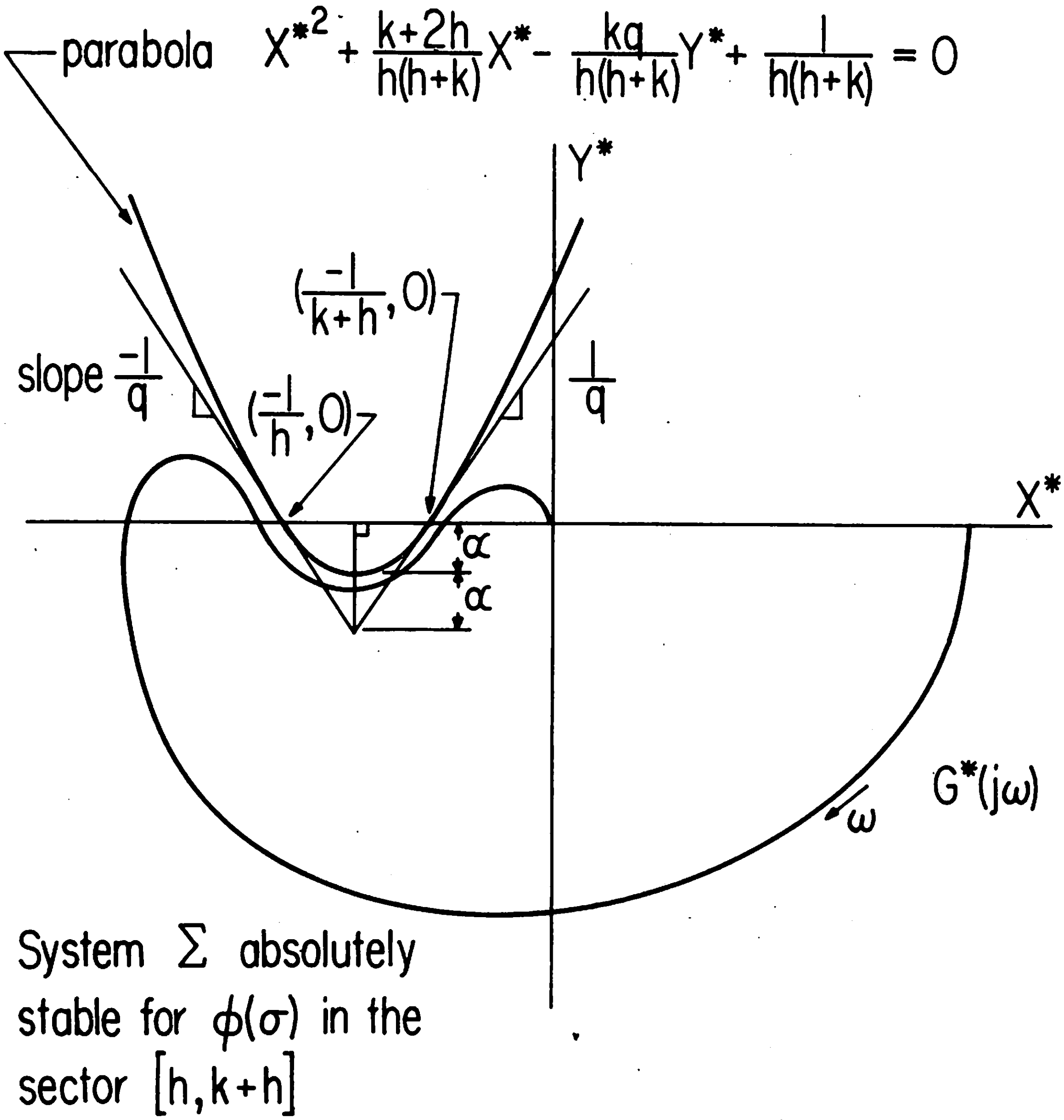

Fig. 5. Application of the parabola test.

Stated differently it might be hoped that the transformation $T_h(\Sigma)$ with $h<0$ might tend to "straighten out" the Popov locus so that the Popov line could be drawn "closer" to the locus and result in an improved upper bound. The following theorem shows that this is not possible.

Theorem 2: If the point $(-\frac{1}{k_1}, 0)$ on the Popov line for the system Σ cannot be made arbitrarily close to the $G^*(j\omega)$ plot, if the upper bound of the Popov sector for Σ is k_1, and if Popov's condition applied to $T_h(\Sigma)$ with $h<0$ proves the absolute stability of Σ in the sector $[h, k+h]$, then $k+h$ is strictly less than k_1. Stated another way, if in the case under consideration the lower bound of the Popov sector is lowered by means of a transformation $T_h(\Sigma)$, then the upper bound is also lowered.

Proof: Assume that there exists a system Σ_o for which the theorem is false. It is immediate that Σ_o is the principal case since a particular case can never be absolutely stable in a sector $[a, b]$ with $a<0<b$. Therefore, the $G^*(j\omega)$ plot for Σ_o lies entirely in the finite plane [1]. Hence, it is clear that the Popov line passing through the point $(-\frac{1}{k_1}, 0)$ approaches arbitrarily close to the $G^*(j\omega)$ plot at two points, one above and one below the X^*-axis, since if this were not the case the upper bound of the Popov sector could be increased by rotating or translating the Popov line. Let these two points be $G^*(j\omega_1)$ and $G^*(j\omega_2)$. Now by the assumptions of the theorem and by Corollary 1 of Theorem 1 there exists a family of ellipses defined by (5) with $h<0$ and $k+h>k_1>0$ such that each member of the family contains the corresponding point $G^*(j\omega)$ (since each ellipse obviously contains the origin). Hence, by Corollary 2 the parabola defined by (6) contains every ellipse of the family and,

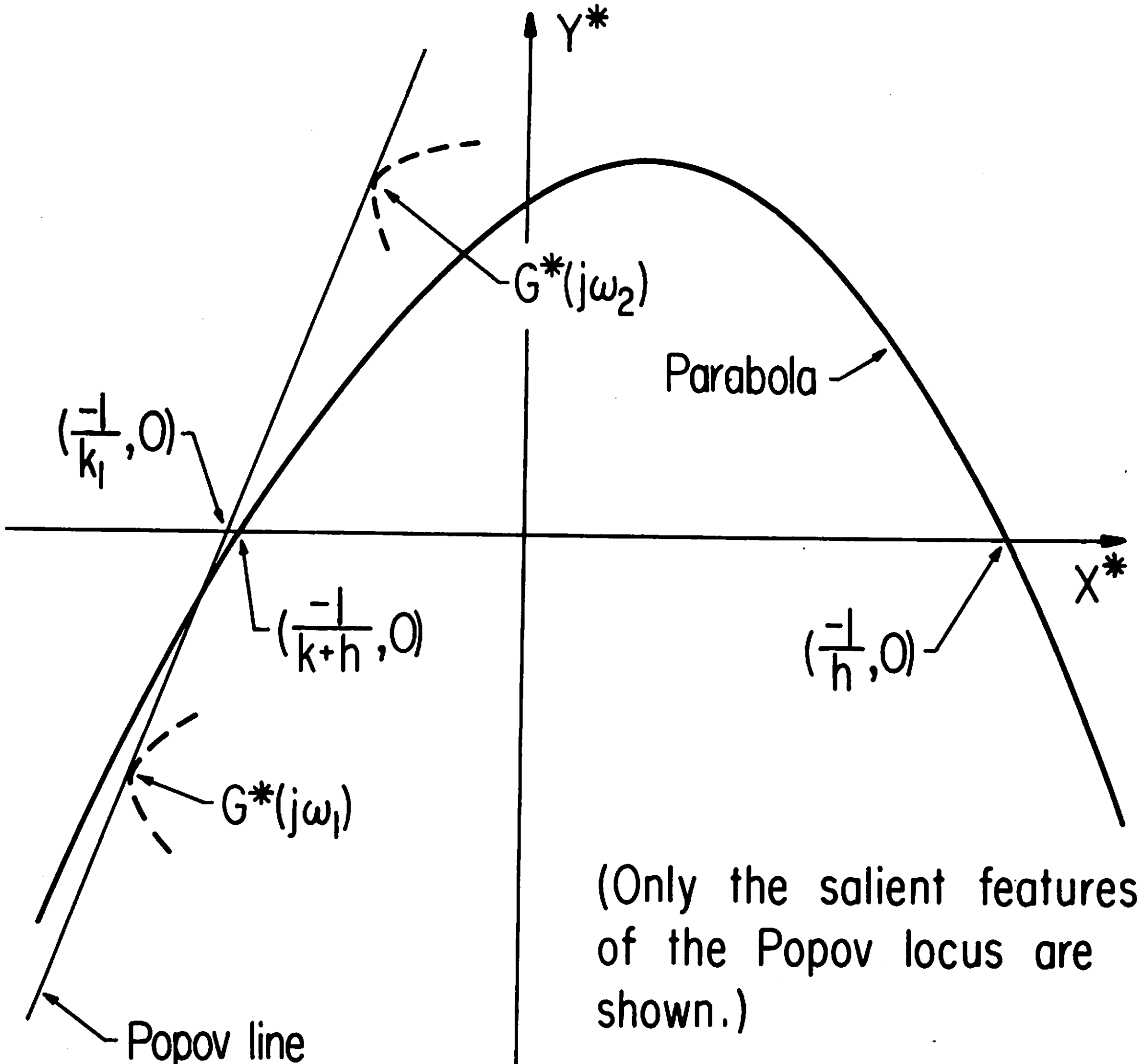

Fig. 6. Pertaining to proof of Theorem 2.

therefore, contains the entire $G^*(j\omega)$ plot. This parabola passes through the two points $(-\frac{1}{h}, 0)$ and $(-\frac{1}{k+h}, 0)$. But it is clear (from Fig. 6) that if $-\frac{1}{h}>0$ and $\frac{-1}{k+h} \geq \frac{-1}{-k_1}$, then the parabola cannot contain both points $G^*(j\omega_1)$ and $G^*(j\omega_2)$. This is a contradiction; therefore, the theorem is proved.

Of course, (in cases where the upper bound of the Hurwitz sector is not attained) the upper bound of the Popov sector can be improved by a transformation $T_h(\Sigma)$ with $h>0$, but this results in a sector of absolute stability with a positive lower bound and hence is of limited practical importance.

CONCLUSIONS

Two new stability tests of practical significance have been presented. The parabola test is very simple to apply and the circle test can save a significant amount of labor, especially where the frequency response of the linear part of the original system must be physically measured.

It has been conjectured that a transformation of the type discussed might increase the upper bound of the Popov sector without increasing the lower bound. Theorem 2 proves that this conjecture is false.

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APPENDIX

Proof of Theorem 1:

For the particular cases, the Popov theorem can be proved by transforming the particular cases into the principal case by a transformation of the form $T_\epsilon(\Sigma)$ [4]. Therefore, it is clear that it is sufficient to prove Theorem 1 for the case where $T_h(\Sigma)$ is the principal case which requires the linear part of $T_h(\Sigma)$ to be strictly stable. Hence, condition (1) of the theorem.

Now let $X(j\omega) = \text{Re } G(j\omega)$ and $Y(j\omega) = \text{Im } G(j\omega)$ as before and let $U(j\omega) = \text{Re } G'(j\omega)$ and $V(j\omega) = \text{Im } G'(j\omega)$, where $G'(j\omega) = \frac{G(j\omega)}{1+hG(j\omega)}$.

Then

$$U + jV = \frac{X+jY}{1+hX+jhY} = \frac{(X+jY)(1+hX-jhY)}{(1+hX)^2 + (hY)^2} = \frac{X(1+hX) + hY^2 + jY}{(1+hX)^2 + (hY)^2} \tag{7}$$

hence

$$U = \frac{X(1+hX) + hY^2}{(1+hX)^2 + (hY)^2} \qquad V = \frac{Y}{(1+hX)^2 + (hY)^2} \tag{8}$$

This defines a bijective, conformal map $T_h:(X, Y) \longrightarrow (U, V)$. Note that $T_h(0, 0) = (0, 0)$.

Assume Popov's condition applied to $T_h(\Sigma)$ proves the absolute stability of Σ in the sector $[h, k+h]$. (i.e., proves the absolute stability of $T_h(\Sigma)$ in the sector $[0, k]$. Then by (3),

$$U(j\omega) - q\omega V(j\omega) + \frac{1}{k} > 0 \text{ for all } \omega \geq 0.$$

i.e., for each $\omega \geq 0$ the point $(U(j\omega), V(j\omega))$ lies to the right of the line $U - q\omega V + \frac{1}{k} = 0$ as does the origin of the (U, V) plane.

Now map the (U, V) plane onto the (X, Y) plane by $(T_h)^{-1}$. Clearly $T_h^{-1}(0, 0) = (0, 0)$ and $T_h^{-1}(U(j\omega), V(j\omega)) = (X(j\omega), Y(j\omega))$. Substituting (8) in the equation for the line gives

$$\frac{X(1+hX) + hY^2 - q\omega Y}{(1+hX)^2 + (hY)^2} + \frac{1}{k} = 0$$

which is equivalent to (4). That is, the lines $U - q\omega V + \frac{1}{k} = 0$ map onto the circles defined by(4). Therefore, condition (2) of the theorem is satisfied.

Conversely, suppose that condition (2) is satisfied. Map the (X, Y) plane onto the (U, V) plane by T_h and, since the circles defined by (4) map onto the lines $U - q\omega V + \frac{1}{k} = 0$, it is clear that the points $(U(j\omega), V(j\omega))$ lie to the right (the origin side) of the lines $U - q\omega V + \frac{1}{k} = 0$. Thus, $T_h(\Sigma)$ is absolutely stable in the sector $[h, k+h]$. This concludes the proof of Theorem 1.